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Characterizing Topologies With Bounded Complete Computational Models

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Abstract

We give several characterizations of maximal point spaces of bounded continuous dcpos. Among them: they are regular second countable spaces which arise from a quasimetric whose dual gives rise to a compact space. In [2], we use one of these characterizations to show that the Polish spaces are the maximal point spaces of bounded continuous dcpos.

Key words and phrases: directed-continuous poset (dcpo), bounded continuous poset, ω-continuous poset; Scott, lower, Lawson topologies; maximal-point space; bitopological dual, quasiproximities, Urysohn sets, Polish space; cocompact and cocompactly quasimetrizable space; specialization order.

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1 Introduction

A *maximal point model* for a topological space X is consists of an ω-continuous dcpo P and an embedding $i : X \to \mathrm{Max}(P)$ such that:

- $i : X \to (\mathrm{Max}(P), \sigma|\mathrm{Max}(P))$ is a homeomorphism, and
- the relative Scott topology and the relative Lawson topology agree on $\mathrm{Max}(P)$.

A *bounded maximal point model* is one for which the poset has a supremum for each subset which is bounded above.

Here we give enough definitions to understand the above. A poset, $(P, \leq)$, is *directed-complete* if each of its directed subsets has a supremum, and *bounded complete* if it is directed-complete and each subset which is bounded above has a supremum. For $x, y \in P$, $(P, \leq)$ a directed-complete poset *x is way-below y* (written $x \ll y$) if whenever $y \leq \bigvee D$ and D is directed, then there is some $z \in D$ such that $x \leq z$. In a poset $(P, \leq)$, for $A \subseteq P$, $\uparrow A = \{x \mid \text{for some } a \leq x,\ a \in A\}$ and A is an *upper set* $A = \uparrow A$. Also, for $p \in P$, $\uparrow p = \uparrow \{p\}$. $\downarrow A$ and lower set are similarly defined, as are $\Downarrow A$, $\Uparrow A$, using the appropriate relation in place of $\leq$. A *directed-continuous poset (dcpo)* is a directed-complete poset so that for each $x \in P$, $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$.

The *Scott topology* on a poset is the topology whose open sets are those upper sets which meet a directed set whenever they contain its supremum. A function between two posets is continuous with respect to their Scott topologies if and only if it preserves directed suprema ([10], II, 2.1). On a dcpo, the Scott topology has as a base all sets of the form $\Uparrow x$. The *lower topology* of a poset is that generated by all sets of the form $X \setminus \uparrow x$, and its *Lawson topology* is the join of its Scott and lower topologies.

An element x, of a directed-complete poset, is *compact* if $x \ll x$, and for each $y \in P$, $K(y) = \{x \leq y \mid x \ll x\}$. An *algebraic poset (algebraic dcpo)* is a directed-complete poset so that for each $y \in P$, $K(y)$ is directed and $y = \bigvee K(y)$. Algebraic posets are so named, because lattices of subobjects of an algebra X (eg. subgroups, ideals) are usually algebraic. In particular, let D be any set of subsets of a set Y which is closed under directed unions and so that each set $S \subseteq Y$ is contained in a smallest element $J[S] \in D$ (so if $S \in D$, then $S = J[S]$). Then the poset $(D, \subseteq)$ is algebraic: its compact elements are the $J[F]$ for F a finite subset of Y, and each element is $S = \bigcup\{J[F] \mid F \text{ a finite subset of } S\}$, a supremum of elements of the directed set $\{J[F] \mid F \text{ a finite subset of } S\}$. We use below the special case that the set of filters of any lattice is algebraic.

A *basis* for a poset $(P, \leq)$ is a $B \subseteq P$ which meets each nonempty $\uparrow x \cap \Downarrow y$. An *$\omega$-continuous poset ($\omega$-dcpo)* is a dcpo with a countable basis.

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In [4], [5], Edalat initiated the study of classical mathematical structures via maximal point models. He noted that for a locally compact Hausdorff locally compact space X, $U(X) = (\{K \subseteq X \mid K \text{ compact}\}, \supseteq)$ (defined in [10]) was a maximal point model, and used such models in a wide variety of applications. These results focused attention on the question of which topological spaces have maximal point models. Lawson ([13]) settled this question by showing that a topological space has a maximal point model iff it is a Polish space. Edalat and Heckmann ([6]) provided a simple explicit construction of a maximal point model for a Polish space. But their models are not bounded complete.

But, given maximal point models $i : X \to D$ and $j : Y \to E$ of the spaces X and Y, we model continuous maps $X \to Y$ by Scott continuous functions $D \to E$. Therefore, it is natural to demand (cf, Escardo [7]) that the continuous maps $D \to E$ capture the continuous maps $X \to Y$ in the sense that every continuous map $X \to Y$ extends to a Scott continuous function $D \to E$. Since continuous Scott domains are exactly the *densely injective spaces* (Exercise II.3.19, [10]) and the set of maximal points of a continuous domain is dense in the Scott topology, bounded complete maximal point models satisfy this requirement: For if D and E are bounded complete and $f : X \to Y$ is continuous, then $i : X \to D$ is a dense embedding, D is densely injective and $j \circ f : X \to D$ is continuous, so there exists a (*greatest*) continuous map $f^\sharp : D \to E$ such that $f^\sharp \circ i = j \circ f$.

We have found several characterizations of those topologies which have bounded complete maximal point models. Among them: They are the Polish (separable completely metrizable) spaces. The proof of this is given in [2], using the results shown below. This characterization answers positively a question asked of us by Lawson, and implicit in Kamimura and Tang [11].

Our other characterizations all involve the existence of a second topology which is compact and T_1, and well related to the first. The Polish space characterization is proved from one of these. Also, in the case of a locally compact space X, the traditional bounded complete maximal point model, $U(X) = (\{K \subseteq X \mid K \text{ compact}\}, \supseteq)$ is essentially the de Groot dual (cocompact topology) arising from the original, and is the smallest topology of the sort we find. In general, the de Groot dual is not compact, and there is no smallest such topology.

2 The characterizations

We take our notation on bitopological spaces from [12]; also, an excellent survey of these spaces, with another viewpoint and different notation is found in [14]. A bitopological space is a triple, (X, τ, τ^*) (often simply denoted X), where τ, τ^* are topologies on X, its *dual* is $X^* = (X, \tau^*, \tau)$; given two such, a map $f : X \to Y$ is *pairwise continuous* if it is continuous from τ_X to τ_Y and from τ^*_X to τ^*_Y. If Q is any property of bitopological spaces, X is *dually* Q if

X^* is Q, *pairwise* Q if X and X^* are both Q.

Let $\mathbb{I}$ denote: $[0,1]$, considered as a set,

- $([0,1],\sigma)$, as a topological space, where $\sigma = \{(a,1] \mid a \in [0,1]\} \cup \{[0,1]\}$,
- $([0,1],\sigma,\omega)$ as a bitopological space, where $\omega = \{[0,a) \mid a \in [0,1]\} \cup \{[0,1]\}$,
- $([0,1],d_{\scriptscriptstyle 1})$ as a quasimetric space, with $d_{\scriptscriptstyle 1}(x,y) = \max\{x-y,0\}$.

For any bitopological space $X = (X,\tau,\tau*)$, the *symmetrization topology*, τ^S, is the join of τ and τ^* (that is, the smallest topology containing both τ and τ^*).

We now cover the separation axioms for bitopological spaces. These are analogous to those for a topological space, except that the T_1 axiom (that points are closed) is broken into two parts: T_0 and weak symmetry. The reason is that the T_1 axiom does both these jobs, but they must be kept apart in our reasoning about bitopological spaces. Thus, a bitopological space $(X,\tau,\tau*)$ is:

normal if whenever $C^* \subseteq T$, C^* τ^*-closed, T τ-open, then there are an τ-open U and a τ^*-closed D^* such that $C^* \subseteq U \subseteq D^* \subseteq T$,[2]

completely regular if whenever $x \in T$, T τ-open, then there is a pairwise continuous $f : X \to \mathbb{I}$ such that $f(x) = 1$ and $f(y) = 0$ whenever $y \notin T$,

regular if whenever $x \in T$, T τ-open, then there are an τ-open U and a τ^*-closed D^* such that $x \in U \subseteq D^* \subseteq T$,

pseudoHausdorff (pH) if whenever $x \notin \mathrm{cl}(y)$ then there are disjoint τ-open T and τ^*-open T^* such that $x \in T$ and $y \in T^*$,

weakly symmetric (ws) if $x \notin \mathrm{cl}(y) \Rightarrow\ y \notin \mathrm{cl}^*(x)$,

$\boldsymbol{T_0}$ if: τ^S is a T_0 topology.

Essentially the usual argument shows that if (X,τ,τ^*) is normal and ws, then it is pairwise completely regular ([12], 2.4 and 2.8). As in the one topology case, the following implications also hold ([12], 2.4): complete regularity $\Rightarrow$ regularity $\Rightarrow$ pH $\Rightarrow$ ws.

Further, let Q denote any of the bitopological separation properties except for normality. Then, if a bitopological space (X,τ,τ^*) is Q, then so is each subspace, $(Y,\tau|Y,\tau^*|Y)$. Also, if (X,τ,τ^*) is pairwise Q then the topological space (X,τ^S) satisfies the topological separation axiom Q.

In [9], it is shown that for any continuous poset, (P,σ,ω) is pairwise completely regular

The next proposition gives some useful bitopological equivalences. Recall that for a topology τ, its *weight*, $w(\tau)$, is the smallest cardinality of a base for

[2] Certainly, by taking C to be the complement of T, U^* to be that of D^*, this is equivalent to the statement: whenever $C^* \cap C = \emptyset$, C^* τ^*-closed, C τ-closed, then there are disjoint U, U^* such that U is τ-open, U^* is τ^*-open, $C^* \subseteq U$ and $C \subseteq U^*$.

it (where a *base* is a collection of open sets so that each open set is a union of some of its members). We also need the following concept: a collection of subsets of X has the *finite intersection property (fip)* if finite subsets of it always have nonempty intersection. By the Alexander subbase theorem, for the smallest topology ν in which a given collection of sets is closed, ν is compact if and only if each subset of the collection with the fip has nonempty intersection.

Proposition 2.1 *The following are equivalent for any topology τ of infinite weight, $w(\tau)$:*

(i) *There is a topology τ^*, so that (X, τ, τ^*) is completely regular and τ^* is compact.*

(ii) *There is a topology τ^+ for which (X, τ, τ^+) is pairwise completely regular, $w(\tau^+) \leq w(\tau)$, and τ^+ is compact.*

(iii) *There is a set $\mathcal{G}$ of cardinality $w(\tau)$, of maps from (X, τ) to $([0,1], \sigma)$, such that:*
   - *(W) τ is the weakest topology for which each element of $\mathcal{F}$ is continuous, and*
   - *(C) if each finite subset of a set of inequalities of the form $\{f(x) \geq a \mid f \in \mathcal{G}, a \in [0,1]\}$, can be solved then the set can be solved.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{B}$ be a base for τ of minimal cardinality. For each $(A, B) \in \mathcal{B} \times B$ so that there is a pairwise continuous $f : X \to \mathbb{I}$ such that $A \subseteq f^{-1}[(.5, 1]]$ and $f^{-1}[(0, 1]] \subseteq B$, choose one such map $f_{(A,B)}$. The set, $\mathcal{F}$ of functions so chosen then has the same cardinality as $\mathcal{B}$. Now let τ^+ be the weakest topology such that each $f \in \mathcal{F}$ is continuous from $(X, \tau^+) \to ([0,1], \omega)$. Then $\tau^+ \subseteq \tau^*$ and is thus compact; and $\{f^{-1}[[0, q)] \mid q \in (0,1] \cap \mathbb{Q}\}$ is a set of cardinality at most $w(\tau) \times \aleph_0 = w(\tau)$ generating τ^+, so $w(\tau^+) \leq w(\tau^*)$. The proof is completed by checking that (X, τ, τ^+) and (X, τ^+, τ) are both completely regular. The converse, (ii) $\Rightarrow$ (i) is clear.

The proof that (i) $\Rightarrow$ (iii), proceeds by noticing that τ is the weakest topology for which each $f \in \mathcal{F}$ ($\mathcal{F}$ from the previous paragraph), is continuous to $\mathbb{I}$. Also, each inequality of the form $f(x) \geq a$ where $a \in [0,1], f \in \mathcal{F}$, has $f^{-1}[[a, 1]]$ as its solution set, a closed set in the compact τ^*, so (C) results.

Finally, to see (iii) $\Rightarrow$ (i), suppose $\mathcal{G}$ is a set of functions satisfying (W) and (C). Then the weakest topology, τ^+ so that $f : (X, \tau^+) \to ([0,1], \omega)$ is continuous for each $f \in \mathcal{G}$, is that generated by $\{f^{-1}[[0, r)] \mid r \in [0,1],\ f \in \mathcal{F}\}$, which is compact by the Alexander subbase theorem applied to (C). Also (W) requires (X, τ, τ^+) to be completely regular since if $x \in T \in \tau$, then there are $f_1, \ldots, f_n \in \mathcal{F}$, $r_1, \ldots, r_n \in (0,1)$, so that $x \in \bigcap_1^n f_i^{-1}[(r_i, 1]] \subseteq T$. Thus $h = \min\{\frac{f_1}{f_1(x) - r_1}, \ldots, \frac{f_n}{f_n(x) - r_n}\} : (X, \tau, \tau^+) \to \mathbb{I}$ is pairwise continuous, $h(x) = 1$, and $h^{-1}[(0, 1]] \subseteq T$. $\square$

Structures considered in the preceding proof are closely related to the idea of *quasiproximity:* a relation $\lhd$ on the subsets of X such that:

(qi) whenever $A \lhd B$, then $A \subseteq B$,

(qii) if $A \lhd B$ then for some C, $A \lhd C$ and $C \lhd B$,

(qiii) if $A \lhd B$ and $E \subseteq A, B \subseteq F$, then $E \lhd F$.

(qiv) $\emptyset \lhd \emptyset$ and $X \lhd X$,

(qv) if $A \lhd B$ and $A \lhd C$ then $A \lhd B \cap C$, and

(qvi) $A \lhd B$ and $C \lhd B$ then $A \cup C \lhd B$.

For any normal bitopological space (X, τ, τ^*), a quasi-proximity, $\lhd_X$, is defined by $A \lhd_X B \iff \mathrm{cl}_{\tau^*}(A) \subseteq \mathrm{int}_\tau(B)$ (ii above is immediate from normality, while the others are clear for any bitopological space).

Each quasiproximity gives rise to a topology, $\tau[\lhd]$, in which a set is open if and only if for each $x \in T$, $\{x\} \lhd T$, and has a dual, defined by $A \lhd^* B \Leftrightarrow X \setminus B \lhd X \setminus A$. Clearly, for a normal bitopological space (X, τ, τ^*), $\tau = \tau[\lhd_X]$ and $\tau^* = \tau[\lhd_X^*]$.

Spaces in which computation is considered are *second countable:* they have a countable base. They are also T_1. Thus for the main result, it is useful to notice a few facts about these properties:

In the countable case of proposition 1, index the set $\mathcal{F}$ of (iii) by $\mathbb{N}$, and for each $n \in \mathbb{N}$, define d_n, by $d_n(x, y) = \max\{(f_n(x) - f_n(y)), 0\}$. Then $d = \Sigma_{n=1}^{\infty} \frac{d_n}{2^n}$, is a quasimetric such that $\tau = \tau_d$ (the topology generated by the open balls $B_r(x) = \{y \mid d(x, y) < r\}$ for $x \in X,\ r > 0$). Also, τ_{d^*} is the weakest topology from which all the f_n are continuous into $([0, 1], \omega)$. This latter topology is compact by the Alexander subbase theorem, as in the proof of (iii) $\Rightarrow$ (i) above. So in the countable case, if (iii) of proposition 1 holds, then there is a quasimetric d such that $\tau = \tau_d$ and τ_{d^*} is compact; we call such a space *cocompactly quasimetrizable.* Conversely, for any quasimetric d, the bitopological space (X, τ_d, τ_{d^*}) is pairwise completely regular by the usual proof (consider functions of the form f_x, where $f_x(y) = \max\{1 - d(x, y), 0\}$). The resulting equivalent statement is (3) of the theorem below.

Finally, by the usual proof, if (X, τ, τ^*) is pairwise regular and (X, τ^S) is second countable, then (X, τ, τ^*) is normal, so it is pairwise completely regular. For convenience, we include this proof:

First notice that by the classical Lindelöf theorem, if (X, τ^S) is second countable, then each open cover of it has a countable subcover (first consider the countable set of basic open sets contained in an element of the cover, and then for each of these select one element of the original cover that contains it); this also holds for each closed subset, C, of such a space, since if $\mathcal{C}$ is a cover of C, then $\{X \setminus C\} \cup \mathcal{C}$ is one of X, so it contains a countable subcover $\mathcal{D}$, and then $\mathcal{D} \setminus \{X \setminus C\}$ is a countable subcover of C.

Now, suppose C and C^* are disjoint, C τ-closed and C^* τ^*-closed. For each $x \in C$ there is a τ^*-open T_x^* such that $x \in T_x^*$ and $\mathrm{cl}(T_x^*) \cap C^* = \emptyset$, and we can take a countable subcover $T_n^* = T_{x_n}^*$ of C. Similarly, there is a countable τ-open cover T_n of C such that each $\mathrm{cl}^*(T_n) \cap C = \emptyset$. Now, for each

$n \in \mathbb{N}$ let $U_n^* = T_n^* \setminus \bigcup_{m \leq n} \text{cl}^*(T_m) \in \tau^*,\ U_n = T_n \setminus \bigcup_{m \leq n} \text{cl}(T_m^*) \in \tau$. Then if $m \leq n$, we have $U_n^* \cap U_m \subseteq (X \setminus T_m) \cap T_m = \emptyset$, and similarly if $n \leq m$ then $U_n^* \cap U_m = \emptyset$; as a result, $V^* = \bigcup_{n \in \mathbb{N}} U_n^* \in \tau^*,\ V = \bigcup_{n \in \mathbb{N}} U_n \in \tau$ are disjoint and if $x \in C$ then for some $n \in \mathbb{N}$, $x \in T_n^* \cap C \subseteq U_n^* \subseteq V^*$, so $C \subseteq V^*$ and similarly $C^* \subseteq V$.

For any topological space (X, τ), its *de Groot dual* (also called its *cocompact topology*) is the weakest topology, τ^G, in which each compact saturated set is closed. If (X, τ, τ^*) is pH, then it is easy to see that $\tau^G \subseteq \tau^*$ (by a slight variant of the proof that each compact set in a Hausdorff space is closed – cf. (3.1, [12]). Thus if (X, τ^*, τ) is pH, then $(\tau^*)^G \subseteq \tau$. If (X, τ^*) is also T_1 and compact then τ^*-closed subsets of X are compact and saturated, thus $(\tau^*)^G$-closed, so τ-closed. That is:

if (X, τ^*, τ) is pH and (X, τ^*) is T_1 and compact then $\tau^* \subseteq \tau$.

To finish, notice that the Scott and Lawson topologies are equal on the of maximal point space for each bounded continuous dcpo, E. By the last paragraph of the proof that (1) $\Rightarrow$ (3), $\omega|\text{Max}(E)$ is compact and since the specialization of $\omega|\text{Max}(E)$ is $\geq |\text{Max}(E)$, that is, equality, $\omega|\text{Max}(E)$ is also T_1. (E, σ, ω) is pairwise completely regular, thus so is its subspace $(\text{Max}(E), \sigma|\text{Max}(E), \omega|\text{Max}(E))$, and thus $\omega|\text{Max}(E) \subseteq \sigma|\text{Max}(E)$, so their join, $\lambda|\text{Max}(E) = \sigma|\text{Max}(E)$. Thus, the relative Scott topology and the relative Lawson topology automatically agree on $\text{Max}(P)$, if P is a bounded continuous dcpo, so the requirement that these topologies agree is redundant for bounded continuous maximal point models.

As a result, each pairwise continuous $g : (X, \tau, \tau^*) \to \mathbb{I}$ is continuous from (X, τ) to $\mathbb{I}^S$ – the unit interval with the usual topology.

Since Lawson [13] has shown that each space with a (bounded complete) maximal point model is complete metric, thus regular and T_1, we shall assume this much separation below. With these concepts in place, we state the following result:

Theorem 2.2 *The following are equivalent for a topological space (X, τ):*

(1) *It has a bounded complete maximal point model.*

(2) *It has a countable set of closed sets, $\Gamma = \{C_n \mid n \in \mathbb{N}\}$ (which may be assumed closed under finite unions and intersections), such that:*
   - (br) *if $x \in T \in \tau$ there is an n such that $x \in \text{int}C_n$ and $C_n \subseteq T$,*
   - (r*) *if $x \notin C_n$ then for some $m \in \mathbb{N}$, $x \notin C_m$ and $C_n \subseteq \text{int}C_m$, and*
   - (cp) *each subset of Γ with the fip has nonempty intersection.*

(3) *it is second countable and T_1, and there is a quasi-metric, d, on X such that $\tau = \tau_d$ and τ_{d^*} is compact.*

Before proceeding to prove these equivalences, note that we have already shown that (3) is equivalent to each of the conditions (i) – (iii) of proposition 1. Further, in these results we may require that $\tau^*, \tau^+ \subseteq \tau$, and are second countable, and that $\mathcal{F}$ is countable. In these circumstances, the following equivalent to (ii) has also been shown:

(iv) *There is a compact and second countable topology* $\tau^+ \subseteq \tau$ *such that* (X, τ, τ^+) *is pairwise regular.*

Proof. To see that (2) $\Rightarrow$ (3), it will do to show that (2) $\Rightarrow$ (iv). But by (cp), the set $\{X \setminus C_n \mid n \in \mathbb{N}\}$ generates a compact topology which we call τ^*, and by (rb), $\{\text{int}C_n \mid n \in \mathbb{N}\}$ is a countable base for τ. (X, τ, τ^*) is regular by (rb) as well, and by (r*), (X, τ^*, τ) is also regular.

To see that (3) $\Rightarrow$ (2), we similarly show $(iii) \Rightarrow (2)$. Let $(C_n)_{n\in\mathbb{N}}$ be an indexing of the countable set of finite unions of finite intersections of sets of the form $f^{-1}[[q, 1]]$, where $q \in (0, 1) \cap \mathbb{Q}$, $f \in \mathcal{F}$. Their complements form a base for a subtopology of the compact τ^* (which is thus necessarily compact), showing (cp). To see (br) let $x \in T \in \tau$; then by (W) of (iii), there are $f_1, \ldots, f_n \in \mathcal{F}$, $r_1, \ldots, r_n \in (0, 1)$, such that $x \in \bigcap_1^n f_i^{-1}[(r_i, 1]] \subseteq T$. In particular, for each $i \le n$, $r_i < f_i(x)$, so we can find $q_i \in (r_i, f_i(x)) \cap \mathbb{Q}$. Let $C = \bigcap_1^n f_i^{-1}[[q_i, 1]]$; then $x \in \bigcap_1^n f_i^{-1}[(q_i, 1]] \subseteq \text{int}C$ and $C \subseteq T$. For (r*) first notice that each C_n can be written in the form $\bigcap_{h=1}^j (\bigcup_{i=1}^k f_{hi}^{-1}[[r_{hi}, 1]])$; if $x \notin \bigcap_{h=1}^j (\bigcup_{i=1}^k f_{hi}^{-1}[[r_{hi}, 1]])$ then for some h, $f_{hi}(x) < r_{hi}$ for each $i \le k$, so let $q_k \in (f_{hi}(x), r_{hi})$. But then for $C_m = \bigcup_{i=1}^k f_{hi}^{-1}[[q_{hi}, 1]]$, $x \notin C_m$ and $C_n \subseteq \bigcup_{i=1}^k f_{hi}^{-1}[(q_{hi}, 1]] \subseteq \text{int}C_m$.

To see (1) $\Rightarrow$ (3), we show (1) $\Rightarrow$ (i): Suppose E is an ω-bounded complete continuous cpo and let $X = \text{Max}(E)$, $\tau = \sigma|X$ and $\tau^* = \omega|X$. For any continuous cpo, by [[9], 1.1, 1.4 and 1.6], (E, σ, ω) is pairwise completely regular with specialization $\le$, thus its subspace $(X, \sigma|X, \omega|X)$ is also pairwise completely regular, with specialization $\le |X \times X$, which is equality, so in particular, (X, τ) is T_1, so $\tau^* \subseteq \tau$. Since E has a countable basis, σ and ω are second countable and so τ and τ^* are second countable.

Suppose for all finite subsets F of I, $X \ne \bigcup_{i\in F} X\setminus \uparrow e_i$. Then for all such F, the set $\{e_i | i \in F\}$ is bounded and so $\bigvee_{i\in F} e_i$ exists. The collection $\{\bigvee_{i\in F} e_i | F$ a finite subset of $I\}$ is then directed, thus has a supremum. Let x be a maximal element above this supremum. Then $x \in X \setminus (\bigcup_{i\in I} X\setminus \uparrow e_i)$, so $\{X\setminus \uparrow e_i \mid i \in I\}$ is not a cover. Thus (X, τ^*) is compact, showing (i).

To show (2) $\Rightarrow$ (1), assume that Γ is closed under finite intersections and unions; it is also partially ordered by reverse inclusion. Let Δ be the poset of filters on X with a base of (nonempty) elements of Γ, partially ordered by inclusion. By the proof in the third paragraph of the introduction, Δ is an algebraic dcpo whose compact elements are the finitely generated (that is, principal) filters. $\{X\}$ is clearly the bottom element of Δ. If $\mathcal{F}$ and $\mathcal{G}$ are in Δ and bounded by $\mathcal{H} \in \Delta$, then $\mathcal{F} \vee G = \{A \cap B | A \in \mathcal{F},\ B \in \mathcal{G}\} \in \Delta$. Thus Δ is a bounded complete algebraic cpo and which is a Scott domain since Γ is countable. The rest of the proof consists of checking the following four assertions; details may be found in [3].

We define ϕ on X, by $\phi(x) = \{C \in \Gamma | x \in C\}$. In fact, $\phi : X \to \text{Max}(\Delta)$ is a bijection with inverse, $\psi : \text{Max}(\Delta) \to X$, given by $\psi(\mathcal{F}) = \bigcap \mathcal{F}$, $\mathcal{F} \in$

Max(Δ).

Next, for $\mathcal{F} \in \Delta$, define $\mathrm{rd}(\mathcal{F}) = \{B \in \Gamma \mid \exists A \in \mathcal{F},\ A \lhd_X B\}$. Then $\mathrm{rd} : \Delta \to \Delta$ is a Scott-continuous projection.

Let $\Lambda = \mathrm{rd}(\Delta)$. In [1], Proposition 4.1.3, it is shown that the image of an algebraic dcpo under a Scott-continuous projection is a continuous dcpo. So from the previous paragraph, it follows that Λ is a bounded complete ω-continuous cpo.

Now for $x \in X$, define $\phi_{rd}(x) = rd(\phi(x))$. The proof is completed by noting that $\phi_{rd} : (X, \tau) \to (\mathrm{Max}(\Lambda), \sigma \mid \mathrm{Max}(\Lambda))$ is a homeomorphism (with inverse $\psi_{rd} : \mathrm{Max}(\Lambda) \to X$, given by $\psi_{rd}(\mathcal{F}) = \bigcap \mathcal{F}, \mathcal{F} \in \mathrm{Max}(\Lambda)$.) □

We use these equivalents in the proof in [2] that each Polish space has a bounded complete maximal point model. The converse, indeed that any space with a maximal point model is Polish, is shown in [13].

3 Locally compact and ultrametric spaces

Of course, each Hausdorff locally compact second countable space has its traditional bounded complete computational model, $U(X) = (\{K \subseteq X \mid K$ compact$\}, \supseteq)$. But it is a special case of the theorem, given (a) and (b) below:

(a) A subset of a locally compact Hausdorff space (X, τ) is closed in τ^G if and only it is compact or equals X; thus the only difference between UX and the poset of nonempty closed sets of τ^G ordered by $\supseteq$ is the bottom element of the latter.

(b) For each Hausdorff locally compact space (X, τ), the bitopological space (X, τ, τ^G) is pairwise regular and τ^G is compact. This bitopological space is ws (since both topologies are T_1) and normal, since if $C \subseteq T$, C is τ^G-closed, and T is open, cover C by compact neighborhoods of its elements $\{D_x \mid x \in C\}$; this has a finite subcover $\{D_x \mid x \in F\}$, and $U = \bigcup\{\mathrm{int}(D_x) \mid x \in F\}$ is open, $D = \bigcup\{D_x \mid x \in F\}$ is compact (and like every set in a T_1-space, saturated) so D is τ^G-closed, and $C \subseteq U \subseteq D \subseteq T$. Thus it is pairwise regular, and so a bounded complete computational model for the original space by (iv) above (or completely regular, so apply (i) of the proposition). That this is the *smallest* such computational model follows from the minimality property of τ^G which was mentioned immediately after its definition, and this accounts for the particularly straightforward manner in which continuous maps are extended to this computational model.

A similar analysis applies to the bounded complete computational model of a complete separable ultrametric space discussed in [8]: let Γ be the set of closed balls of radius 2^{-m} for some positive integer m, and with centers in a fixed countable dense set D. Since these sets are clopen, (X, τ, τ^*) is pairwise regular. To see that Γ is compact-generating, first note that in an ultrametric space, if $N_r(x)$ and $N_s(y)$ meet and $r \leq s$ then $N_r(x) \subseteq N_s(y)$ and if this inclusion is proper in an ultrametric space, then $r < s$. Thus a subset of Γ with the finite intersection property must be a chain under inclusion, thus must contain sets of arbitrarily small diameters, and so has nonempty intersection by completeness.

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