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CATEGORY ANALOGUE OF SUP-MEASURABILITY PROBLEM

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Abstract. A function $F\colon \mathbb{R}^2 \to \mathbb{R}$ is called *sup-measurable* if $F_f\colon \mathbb{R} \to \mathbb{R}$ given by $F_f(x) = F(x, f(x))$, $x \in \mathbb{R}$, is measurable for each measurable function $f\colon \mathbb{R} \to \mathbb{R}$. It is known that under different set theoretical assumptions, including CH, there are sup-measurable non-measurable functions, as well as their category analogues. In this paper we will show that the existence of the category analogues of sup-measurable non-measurable functions is independent of ZFC. A similar result for the original measurable case is the subject of a work in preparation by Rosłanowski and Shelah.

1. Introduction

Our terminology is standard and follows that from [3], [4], [10], or [12]. In particular, $\mathrm{pr}\colon X \times Y \to X$ will stand for the projection onto the first coordinate. A subset A of a Polish space X is *nowhere meager* provided $A \cap U$ is not meager for every non-empty open subset of X.

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The ternary Cantor subset of $\mathbb{R}$ will be identified with with its homeomorphic copy, 2^ω, which stands for the set of all function $x\colon \omega \to \{0,1\}$ considered with the product topology. In particular, the basic open subsets of 2^ω are in the form

$$[s] \stackrel{\text{def}}{=} \{x \in 2^\omega \colon s \subset f\},$$

where $s \in 2^{<\omega}$. Also, since $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $2^\omega \setminus E$ for some countable set E (the set of all eventually constant functions in 2^ω) in our more technical part of the paper we will be able replace $\mathbb{R}$ with 2^ω.

The study of sup-measurable functions[1] comes from the theory of differential equations. More precisely it comes from the question: For which functions $F\colon \mathbb{R}^2 \to \mathbb{R}$ does the Cauchy problem

$$y' = F(x, y), \quad y(x_0) = y_0 \tag{1}$$

have a (unique) *a.e.-solution* in the class of locally absolutely continuous functions on $\mathbb{R}$ in the sense that $y(x_0) = y_0$ and $y'(x) = F(x, y(x))$ for almost all $x \in \mathbb{R}$? (For more on this motivation see [8] or [2]. Compare also [9].) It is not hard to find measurable functions which are not sup-measurable. (See [13] or [1, Corollary 1.4].) Under the continuum hypothesis CH or some weaker set-theoretical assumptions nonmeasurable sup-measurable functions were constructed in [6], [7], [1], and [8]. The independence from ZFC of the existence of such an example is the subject of a work in prepartion by Rosłanowski and Shelah.

A function $F\colon \mathbb{R}^2 \to \mathbb{R}$ is a *category analogue of sup-measurable function* (or *Baire sup-measurable*) provided $F_f\colon \mathbb{R} \to \mathbb{R}$ given by $F_f(x) = F(x, f(x))$, $x \in \mathbb{R}$, has the Baire property for each function $f\colon \mathbb{R} \to \mathbb{R}$ with the Baire property. A Baire sup-measurable function without the Baire property has been constructed under CH in [5]. (See also [1] and [2].) The main goal of this paper is to show that the existence of such functions cannot be proved in ZFC. For this we need the following easy fact. (See [1, Proposition 1.5].)

Proposition 1. *The following conditions are equivalent.*

(i) *There is a Baire sup-measurable function $F\colon \mathbb{R}^2 \to \mathbb{R}$ without the Baire property.*

(ii) *There is a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ without the Baire property such that F_f has the Baire property for every Borel function $f\colon \mathbb{R} \to \mathbb{R}$.*

(iii) *There is a set $A \subset \mathbb{R}^2$ without the Baire property such that the projection* $\mathrm{pr}(A \cap f) = \{x \in \mathbb{R} \colon \langle x, f(x)\rangle \in A\}$ *has the Baire property for each Borel function $f\colon \mathbb{R} \to \mathbb{R}$.*

[1] This is abbreviation from *superposition-measurable function.*

(iv) *There is a Baire sup-measurable function* $F\colon \mathbb{R}^2 \to \{0,1\}$ *without the Baire property.*

The equivalence of (i) and (ii) follows from the fact that the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ is Baire sup-measurable if and only if F_f has a Baire property for every Borel function $f\colon \mathbb{R} \to \mathbb{R}$.[2]

The main theorem of the paper is the following.

Theorem 2. *It is consistent with the set theory ZFC that*

φ: *for every* $A \subset 2^\omega \times 2^\omega$ *for which the sets* A *and* $A^c = (2^\omega \times 2^\omega) \setminus A$ *are nowhere meager in* $2^\omega \times 2^\omega$ *there exists a homeomorphism* f *from* 2^ω *onto* 2^ω *such that the set* $\operatorname{pr}(A \cap f)$ *does not have the Baire property in* 2^ω.

Before proving this theorem let us notice that it implies easily the following corollary.

Corollary 3. *The existence of Baire sup-measurable function* $F\colon \mathbb{R}^2 \to \mathbb{R}$ *without the Baire property is independent from the set theory ZFC.*

Proof. As mentioned above under CH there exist Baire sup-measurable functions without the Baire property. So, it is enough to show that the property φ from Theorem 2 implies that there are no such functions.

So, take an arbitrary $A \subset \mathbb{R}^2$ without the Baire property. By (iii) of Proposition 1 it is enough to show there exists a Borel function $f\colon \mathbb{R} \to \mathbb{R}$ for which the set $\operatorname{pr}(A \cap f)$ does not have the Baire property.

We will first show this under the additional assumption that the sets A and $\mathbb{R}^2 \setminus A$ are nowhere meager in $\mathbb{R}^2$. But then the set $A_0 = A \cap (\mathbb{R} \setminus \mathbb{Q})^2$ and its complement are nowhere meager in $(\mathbb{R} \setminus \mathbb{Q})^2$. Moreover, since $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $2^\omega \setminus E$ for some countable set E we can consider A_0 as a subset of $(2^\omega \setminus E)^2 \subset 2^\omega \times 2^\omega$. Then A_0 and its complement are still nowhere meager in $2^\omega \times 2^\omega$. Therefore, by φ, there exists an autohomeomorphism f of 2^ω such that the set $\operatorname{pr}(A_0 \cap f) = \{x \in 2^\omega \setminus E\colon \langle x, f(x)\rangle \in A_0\}$ does not have the Baire property in 2^ω. Now, as before, $f \restriction (2^\omega \setminus E)$ can be considered as defined on $\mathbb{R} \setminus \mathbb{Q}$. So if $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ is an extension of $f \restriction (2^\omega \setminus E)$ (under such identification) to $\mathbb{R}$ which is constant on $\mathbb{Q}$, then $\bar{f}$ is Borel and the set $\operatorname{pr}(A_0 \cap \bar{f})$ does not have the Baire property in $\mathbb{R}$.

[2] It is also true that $F\colon \mathbb{R}^2 \to \mathbb{R}$ is Baire sup-measurable provided F_f has the Baire property for every Baire class one function $f\colon \mathbb{R} \to \mathbb{R}$, and that $F\colon \mathbb{R}^2 \to \mathbb{R}$ is sup-measurable provided F_f is measurable for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$. See for example [2, Lemma 1 and Remark 1].

Now, if A is an arbitrary subset of $\mathbb{R}^2$ without the Baire property we can find non-empty open intervals U and W in $\mathbb{R}$ such that A and $(U \times W) \setminus A$ are nowhere meager in $U \times W$. Since U and W are homeomorphic with $\mathbb{R}$ the above case implies the existence of Borel function $f_0 \colon U \to W$ such that $\mathrm{pr}(A \cap f_0)$ does not have the Baire property in U. So any Borel extension $f \colon \mathbb{R} \to \mathbb{R}$ of f_0 works. $\square$

2. Reduction of the proof of Theorem 2 to the main lemma

The theorem will be proved by the method of iterated forcing, a knowledge of which is needed from this point on.

The idea of the proof is quite simple. For every nowhere meager subset A of $2^\omega \times 2^\omega$ for which $A^c = (2^\omega \times 2^\omega) \setminus A$ is also nowhere meager we will find a natural ccc forcing notion Q_A which adds the required homeomorphism f. Then we will start with the constructible universe $V = L$ and iterate with finite support these notions of forcing in such a way that every nowhere meager set $A^* \subset 2^\omega \times 2^\omega$, with $(2^\omega \times 2^\omega) \setminus A^*$ nowhere meager, will be taken care of by some Q_A at an appropriate step of iteration.

There are two technical problems with carrying through this idea. First is that we cannot possibly list in our iteration all nowhere meager subsets of $2^\omega \times 2^\omega$ with nowhere meager complements since the iteration can be of length at most continuum $\mathfrak{c}$ and there are $2^{\mathfrak{c}}$ such sets. This problem will be solved by defining our iteration as $P_{\omega_2} = \langle \langle P_\alpha, \dot{Q}_\alpha \rangle \colon \alpha < \omega_2 \rangle$ such that the generic extension $V[G]$ of V with respect to P_{ω_2} will satisfy $2^\omega = 2^{\omega_1} = \omega_2$ and have the property that

(m) every non-Baire subset A^* of 2^ω contains a non-Baire subset A of cardinality ω_1.

Thus in the iteration we will use only the forcing notions $Q_\alpha = Q_A$ for the sets A of cardinality ω_1, whose number is equal to ω_2, the length of iteration. Condition (m) will guarantee that this will give us enough control of all nowhere meager subsets A^* of $2^\omega \times 2^\omega$.

The second problem is that even if at some stage $\alpha < \omega_2$ of our iteration we will add a homeomorphism f appropriate for a given set $A \subset 2^\omega \times 2^\omega$, that is such that

$$V[G_\alpha] \models \text{“}\mathrm{pr}(A \cap f) \text{ is not Baire in } 2^\omega\text{,”}$$

where $G_\alpha = G \cap P_\alpha$, then in general there is no guarantee that the set $\mathrm{pr}(A \cap f)$ will remain non-Baire in the final model $V[G]$. The preservation of non-Baireness of each appropriate set $\mathrm{pr}(A \cap f)$ will be achieved by carefully crafting our iteration following a method known as the *oracle-cc* forcing iteration.

The theory of the oracle-cc forcings is described in details in [12, Chapter IV] (compare also [11, Chapter IV]) and here we will recall only the fragments that are relevant to our specific situation. In particular if

$$\Gamma \stackrel{\text{def}}{=} \{\lambda < \omega_1 \colon \lambda \text{ is a limit ordinal}\}$$

then

- an ω_1-*oracle* is any sequence $\mathcal{M} = \langle M_\delta \colon \delta \in \Gamma \rangle$ where M_δ is a countable transitive model of ZFC$^-$ that is, ZFC without the power set axiom) with a property that $\delta + 1 \subset M_\delta$, $M_\delta \models$ "δ is countable," and the set $\{\delta \in \Gamma \colon A \cap \delta \in M_\delta\}$ is stationary in ω_1 for every $A \subset \omega_1$.

The existence of an ω_1-oracle is equivalent to the diamond principle $\diamondsuit$.

With each ω_1-oracle $\mathcal{M} = \langle M_\delta \colon \delta \in \Gamma \rangle$ there is associated a filter $D_{\mathcal{M}}$ generated by the sets $I_{\mathcal{M}}(A) = \{\delta \in \Gamma \colon A \cap \delta \in M_\delta\}$ for $A \subset \omega_1$. It is proved in [12, Claim 1.4] that $D_{\mathcal{M}}$ is a proper normal filter containing every closed unbounded subset of Γ.

We will also need the following fact which, for our purposes, can be viewed as a definition of $\mathcal{M}$-cc property.

Fact 4. *Let P be a forcing notion of cardinality $\leq \omega_1$, $e \colon P \to \omega_1$ be one-to-one, and $\mathcal{M} = \langle M_\delta \colon \delta \in \Gamma \rangle$ be an ω_1-oracle. If there exists a $C \in D_{\mathcal{M}}$ such that for every $\delta \in \Gamma \cap C$*

> *$e^{-1}(E)$ is predense in P for every set $E \in M_\delta \cap \mathcal{P}(\delta)$, for which $e^{-1}(E)$ is predense in $e^{-1}(\{\gamma \colon \gamma < \delta\})$,*

then P has the $\mathcal{M}$-cc property.

This follows immediately from the definition of $\mathcal{M}$-cc property [12, Definition 1.5, p. 150].

Our proof will rely on the following main lemma.

Lemma 5. *For every $A \subset 2^\omega \times 2^\omega$ for which A and $A^c = (2^\omega \times 2^\omega) \setminus A$ are nowhere meager in $2^\omega \times 2^\omega$ and for every ω_1-oracle $\mathcal{M}$ there exists an $\mathcal{M}$-cc forcing notion Q_A of cardinality ω_1 such that Q_A forces*

> *there exists an autohomeomorphism f of 2^ω such that the sets $\mathrm{pr}(f \cap A)$ and $\mathrm{pr}(f \setminus A)$ are nowhere meager in 2^ω.*

The proof of Lemma 5 represents the core of our argument and will be presented in the next section. In the remainder of this section we will sketch how Lemma 5 implies Theorem 2. Since this follows the standard path, as described in [12, Chapter IV], the readers familiar with this treatment may proceed directly to the next section.

First of all, to define an appropriate iteration we will treat forcings Q_A from Lemma 5 as defined on ω_1. More precisely, in the iteration we will always replace Q_A with its order isomorphic copy $\langle \omega_1, \leq_A \rangle$. So, we can treat any finite support iteration $P_\alpha = \langle \langle P_\beta, \dot{Q}_\beta \rangle : \beta < \alpha \rangle$ of Q_A forcing notions as having an absolute and fixed universe, say $U_\alpha = \{g \in (\omega_1)^{\omega_2} \colon g^{-1}(\omega_1 \setminus \{0\}) \in [\alpha]^{<\omega}\}$. This will allow us to treat the $\diamondsuit_{\omega_2}$-sequence $\langle X_\alpha \colon \alpha < \omega_2 \rangle$ as a sequence of P_α-names of subsets of $2^\omega \times 2^\omega$. (After appropriate coding.)

We will also need the following variant of [12, Example 2.2].

Lemma 6. *Assume that $\diamondsuit_{\omega_1}$ holds and that $S \subset 2^\omega$ is such that S and S^c are nowhere meager in 2^ω. Then there exists an ω_1-oracle $\mathcal{M}$ such that if P is an arbitrary $\mathcal{M}$-cc forcing then P forces that*

S and S^c are nowhere meager in 2^ω.

Proof. By [12, Example 2.2] for any non-empty basic open set W of 2^ω there are oracles $\mathcal{M}^0_W$ and $\mathcal{M}^1_W$ such any $\mathcal{M}^0_W$-cc forcing forces that $S \cap W$ is not meager, and any $\mathcal{M}^1_W$-cc forcing forces that $S^c \cap W$ is not meager. So, by [12, Claim 3.1], there is a single ω_1-oracle $\mathcal{M}$ which "extends" all oracles $\mathcal{M}^i_W$, and it clearly does the job. $\square$

Now, the iteration P_{ω_2} is defined by choosing by induction the sequence $\langle \langle P_\alpha, \dot{A}_\alpha, \dot{\mathcal{M}}_\alpha, \dot{Q}_\alpha, \dot{f}_\alpha \rangle \colon \alpha < \omega_2 \rangle$ such that for every $\alpha < \omega_2$

(a) $P_\alpha = \langle \langle P_\beta, \dot{Q}_\beta \rangle \colon \beta < \alpha \rangle$ is a finite support iteration,

(b) $\dot{A}_\alpha$ is a P_α-name for which P_α forces that

$\dot{A}_\alpha$ and $(\dot{A}_\alpha)^c$ are nowhere meager subsets of $2^\omega \times 2^\omega$,

(c) $\dot{\mathcal{M}}_\alpha$ is a P_α-name for which P_α forces that

$\dot{\mathcal{M}}_\alpha$ is an ω_1-oracle and for every $\dot{Q}$ satisfying $\dot{\mathcal{M}}_\alpha$-cc we have

(i) for every $\beta < \alpha$ if $P_\alpha = P_\beta * \dot{P}_{\beta,\alpha}$ then

$$P_\beta \Vdash \text{“}\dot{P}_{\beta,\alpha} * \dot{Q} \text{ is } \dot{\mathcal{M}}_\beta\text{-cc,”}$$

(ii) if $\alpha = \gamma + 1$ then

$$\dot{Q} \Vdash \text{“}\mathrm{pr}(\dot{f}_\gamma \cap \dot{A}_\gamma), \mathrm{pr}(\dot{f}_\gamma \setminus \dot{A}_\gamma) \subset 2^\omega \text{ are nowhere meager in } 2^\omega\text{”},$$

(d) $\dot{Q}_\alpha$ is a P_α-name for a forcing such that P_α forces

$\dot{Q}_\alpha$ is an $\dot{\mathcal{M}}_\alpha$-cc forcing $Q_{\dot{A}_\alpha}$ from Lemma 5,

(e) $\dot{f}_\alpha$ is a $P_{\alpha+1}$-name for which $P_{\alpha+1}$ forces that

$\dot{f}_\alpha$ is a $\dot{Q}_\alpha$-name for the function f from Lemma 5.

If for some $\alpha < \omega_2$ the sequence $\langle \langle P_\beta, \dot{A}_\beta, \dot{\mathcal{M}}_\beta, \dot{Q}_\beta, \dot{f}_\beta \rangle \colon \beta < \alpha \rangle$ has been defined then we proceed as follows. Forcing P_α is already determined by (a). We choose $\dot{A}_\alpha$ as X_α from the $\diamondsuit_{\omega_2}$-sequence if it satisfies (b) and arbitrarily, still maintaining (b), otherwise. Since steps (d) and (e) are facilitated by Lemma 5, it is enough to construct $\dot{\mathcal{M}}_\alpha$ satisfying (c). For this we will consider two cases.

Case 1: α is a limit ordinal.

For a moment fix a $\beta < \alpha$ and work in V^{P_β}. Let $\mathcal{M}_\beta$ and $P_{\beta,\alpha}$ be the interpretations of $\dot{\mathcal{M}}_\beta$ and $\dot{P}_{\beta,\alpha}$, respectively. By the inductive assumption for every $\beta < \gamma < \alpha$ forcing $P_{\beta,\gamma}$ is $\mathcal{M}_\beta$-cc. So, by [12, Claim 3.2], $P_{\beta,\alpha}$ is $\mathcal{M}_\beta$-cc. Thus, by [12, Claim 3.3], in $(V^{P_\beta})^{P_{\beta,\alpha}} = V^{P_\alpha}$ there is an ω_1-oracle $\mathcal{M}^*_\beta$ such that if Q is $\mathcal{M}^*_\beta$-cc then $P_{\beta,\alpha} * Q$ is $\mathcal{M}_\beta$-cc.

So, in V^{P_α}, we have ω_1-oracles $\mathcal{M}^*_\beta$ for every $\beta < \alpha$. Thus, by [12, Claim 3.1], in V^{P_α} there exists an ω_1-oracle $\mathcal{M}_\alpha$ which is stronger than all $\mathcal{M}^*_\beta$'s in a sense that if Q is $\mathcal{M}_\alpha$-cc then Q is also $\mathcal{M}^*_\beta$-cc. So, there is a P_α-name $\dot{\mathcal{M}}_\alpha$ for $\mathcal{M}_\alpha$ for which (c) holds.

Case 2: α is a successor ordinal, $\alpha = \gamma + 1$. Then $P_\alpha = P_\gamma * \dot{Q}_\gamma$.

Since, by (d), P_γ forces that $\dot{Q}_\gamma$ is $\dot{\mathcal{M}}_\gamma$-cc, using (c) for $\alpha = \gamma$ we conclude that

$$P_\beta \Vdash \text{“}\dot{P}_{\beta,\alpha} \text{ is } \dot{\mathcal{M}}_\beta\text{-cc”}$$

for every $\beta < \gamma$. So, proceeding as in Case 1, in V^{P_α} we can find ω_1-oracles $\dot{\mathcal{M}}^*_\beta$ such that

$$P_\beta \Vdash \text{“}\dot{P}_{\beta,\alpha} * \dot{Q} \text{ is } \dot{\mathcal{M}}_\beta\text{-cc”}$$

for every Q which is $\dot{\mathcal{M}}^*_\beta$-cc. Let $\mathcal{M}$ be an ω_1-oracle from Lemma 6 used with $S = \mathrm{pr}(\dot{f}_\gamma \cap \dot{A}_\gamma)$. As above we can find, in V^{P_α}, an ω_1-oracle $\mathcal{M}_\alpha$ which is stronger than all $\mathcal{M}^*_\beta$'s and $\mathcal{M}$. Then, there is a P_α-name $\dot{\mathcal{M}}_\alpha$ for $\mathcal{M}_\alpha$ for which (c) holds. This finishes the construction of the iteration.

To finish the argument first note that the interpretations of $\mathrm{pr}(\dot{f}_\alpha \cap \dot{A}_\alpha)$ and $\mathrm{pr}(\dot{f}_\alpha \setminus \dot{A}_\alpha)$ in the final model $V[G]$ remain nowhere meager in 2^ω. This is the case since, by (e), $P_{\alpha+1}$ forces that

$$\mathrm{pr}(\dot{f}_\alpha \cap \dot{A}_\alpha) \text{ and } \mathrm{pr}(\dot{f}_\alpha \setminus \dot{A}_\alpha) \text{ are nowhere meager in } 2^\omega,$$

and, by (c)(i), that

$$\text{every } \dot{P}_{\alpha+1,\gamma} \text{ is } \dot{\mathcal{M}}_{\alpha+1}\text{-cc}$$

while, by condition (c)(ii), every $\dot{\mathcal{M}}_{\alpha+1}$-cc forcing preserves nowhere meagerness of $\mathrm{pr}(\dot{f}_\alpha \cap \dot{A}_\alpha)$ and $\mathrm{pr}(\dot{f}_\alpha \setminus \dot{A}_\alpha)$. To finish this part of the argument it is enough to note that $P_{\alpha+1}$ forces that “$\dot{P}_{\alpha+1,\omega_2}$ is $\dot{\mathcal{M}}_{\alpha+1}$-cc” which follows from [12, Claim 3.2].

To complete the argument it is enough to show that each nowhere meager subset A^* of $2^\omega \times 2^\omega$ from $V[G]$ with nowhere meager complement contains an interpretation of some $\dot{A}_\alpha$. However, P_{ω_2} is ccc. So, if $\dot{A}$ is a P_{ω_2}-name for A^* then the set

$$\left\{\alpha \in \Gamma \colon P_\alpha \Vdash \dot{A} \cap V^{P_\alpha} \text{ is nowhere meager in } 2^\omega \times 2^\omega\right\}$$

contains a closed unbounded subset of Γ. Thus $\diamondsuit_{\omega_2}$ guarantees that A^* contains an interpretation of some $\dot{A}_\alpha$.

3. Proof of Lemma 5

Let $\mathcal{K}$ be the family of all sequences $\bar{h} = \langle h_\xi \colon \xi \in \Gamma\rangle$ such that each h_ξ is a function from a countable set $D_\xi \subset 2^\omega$ onto $R_\xi \subset 2^\omega$ and that

$$D_\xi \cap D_\eta = R_\xi \cap R_\eta = \emptyset \ \text{ for every distinct } \ \xi, \eta \in \Gamma.$$

For each $\bar{h} \in \mathcal{K}$ we will define a forcing notion $Q_{\bar{h}}$. Forcing Q_A satisfying Lemma 5 will be chosen as $Q_{\bar{h}}$ for some $\bar{h} \in \mathcal{K}$.

So fix an $\bar{h} \in \mathcal{K}$. Then $Q_{\bar{h}}$ is defined as the set of all triples $p = \langle n, \pi, h\rangle$ for which

(A) h is a function from a finite subset D of $\bigcup_{\xi\in\Gamma} D_\xi$ into 2^ω;
(B) $n < \omega$ and π is a permutation of 2^n;
(C) $|D \cap D_\xi| \le 1$ for every $\xi \in \Gamma$;
(D) if $x \in D \cap D_\xi$ then $h(x) = h_\xi(x)$ and $h(x) \restriction n = \pi(x \restriction n)$.

Forcing $Q_{\bar{h}}$ is ordered as follows. Condition $p' = \langle n', \pi', h'\rangle$ is stronger than $p = \langle n, \pi, h\rangle$, $p' \le p$, provided

$$n \le n', \ \ h \subset h', \ \text{ and } \ \pi'(s) \restriction n = \pi(s \restriction n) \ \text{ for every } s \in 2^{n'}. \tag{2}$$

Note that the second part of (D) says that for every $x \in D$ and $s \in 2^n$

$$x \in [s] \quad \text{if and only if} \quad h(x) \in [\pi(s)]. \tag{3}$$

Also, if $n < \omega$ we will write $[s] \restriction 2^n$ for $\{x \restriction 2^n \colon x \in [s]\}$. Note that in this notation the part of (2) concerning permutations says that π' extends π in a sense that π' maps $[t] \restriction 2^{n'}$ onto $[\pi(t)] \restriction 2^{n'}$ for every $t \in 2^n$.

In what follows we will use the following basic property of $Q_{\bar{h}}$.

(∗) For every $q = \langle n, \pi, h\rangle \in Q_{\bar{h}}$ and $m < \omega$ there exist an $n' \ge m$ and a permutation π' of $2^{n'}$ such that $q' = \langle n', \pi', h\rangle \in Q_{\bar{h}}$ and q' extends q.

The choice of such n' and π' is easy. First pick $n' \ge \max\{m, n\}$ such that $x \restriction n' \ne y \restriction n'$ for every different x and y from either domain D or range $R = h[D]$ of h. This implies that for every $t \in 2^n$ the set $D_t = \{x \restriction n' \colon x \in D \cap [t]\} \subset [t] \restriction 2^{n'}$ has the same cardinality as $D \cap [t]$ and $H_t = \{x \restriction n' \colon x \in h[D] \cap [\pi(t)]\} \subset [\pi(t)] \restriction 2^{n'}$ has the same cardinality as $h[D] \cap [\pi(t)]$. Since, by (3), we have also $|D \cap [t]| = |h[D] \cap [\pi(t)]|$ we see that $|D_t| = |H_t|$. Define π' on D_t by $\pi'(x \restriction n') = h(x) \restriction n'$ for every $x \in D_t$. Then π' is a bijection from D_t onto H_t and this definition ensures that an appropriate part of the condition (D) for h and π' is satisfied. Also, if for each $t \in 2^n$ we extend π' onto $[t] \restriction 2^{n'}$ as a bijection from $([t] \restriction 2^{n'}) \setminus D_t$ onto $([\pi(t)] \restriction 2^{n'}) \setminus H_t$, then the condition (2) will be satisfied. Thus such defined $q' = \langle n', \pi', h \rangle$ belongs to $Q_{\bar{h}}$ and extends q.

Next note that forcing $Q_{\bar{h}}$ has the following property, described in Fact 7, needed to prove Lemma 5. In what follows we will consider 2^ω with the standard distance:

$$d(r_0, r_1) = 2^{-\min\{n<\omega\colon\, r_0(n) \neq r_1(n)\}}$$

for different $r_0, r_1 \in 2^\omega$.

Fact 7. *Let $\bar{h} = \langle h_\xi \colon \xi \in \Gamma \rangle \in \mathcal{K}$ and $f = \bigcup\{h \colon \langle n, \pi, h \rangle \in H\}$, where H is a V-generic filter over $Q_{\bar{h}}$. Then f is a uniformly continuous one-to-one function from a subset D of 2^ω into 2^ω. Moreover, if for every $\xi \in \Gamma$ the graph of h_ξ is dense in $2^\omega \times 2^\omega$, then D and $f[D]$ are dense in 2^ω and f can be uniquely extended to an autohomeomorphism $\tilde{f}$ of 2^ω.*

Proof. Clearly f is a one-to-one function from a subset D of 2^ω into 2^ω. To see that it is uniformly continuous choose an $\varepsilon > 0$. We will find $\delta > 0$ such that $r_0, r_1 \in D$ and $d(r_0, r_1) < \delta$ imply $d(f(r_0), f(r_1)) < \varepsilon$. For this note that, by $(*)$, the set

$$S = \{q = \langle n, \pi, h \rangle \in Q_{\bar{h}} \colon 2^{-n} < \varepsilon\}$$

is dense in $Q_{\bar{h}}$. So take a $q = \langle n, \pi, h \rangle \in H \cap S$ and put $\delta = 2^{-n}$. We claim that this δ works.

Indeed, take $r_0, r_1 \in D$ such that $d(r_0, r_1) < \delta$. Then there exists a $q' = \langle n', \pi', h' \rangle \in H$ stronger than q such that r_0 and r_1 are in the domain of h'. Therefore, $n \le n'$ and for $j < 2$

$$f(r_j) \restriction n = h'(r_j) \restriction n = (h'(r_j) \restriction n') \restriction n = \pi'(r_j \restriction n') \restriction n = \pi(r_j \restriction n)$$

by the conditions (D) and (2). Since $d(r_0, r_1) < \delta = 2^{-n}$ implies that $r_0 \restriction n = r_1 \restriction n$ we obtain

$$f(r_0) \restriction n = \pi(r_0 \restriction n) = \pi(r_1 \restriction n) = f(r_1) \restriction n$$

that is, $d(f(r_0), f(r_1)) \le 2^{-n} < \varepsilon$. So f is uniformly continuous.

Essentially the same argument (with the same values of ε and δ) shows that $f^{-1} \colon f[D] \to D$ is uniformly continuous. Thus, if $\tilde{f}$ is the unique continuous extension of f into $\mathrm{cl}(D)$, then $\tilde{f}$ is a homeomorphism from $\mathrm{cl}(D)$ onto $\mathrm{cl}(f[D])$.

To finish the argument assume that all functions h_ξ have dense graphs, take a $t \in 2^m$ for some $m < \omega$, and notice that the set

$$S_t = \{q = \langle n, \pi, h\rangle \in Q_{\bar h} \colon \text{ the domain } D' \text{ of } h \text{ intersects } [t]\}$$

is dense in $Q_{\bar h}$. Indeed, if $q = \langle n, \pi, h\rangle \in Q_{\bar h}$ then, by $(*)$, strengthening q if necessary, we can assume that $m \le n$. Then, refining t if necessary, we can also assume that $m = n$, that is, that t is in the domain of π. Now, if $[t]$ intersects the domain of h, then already q belongs to S_t. Otherwise take $\xi \in \Gamma$ with $D' \cap D_\xi = \emptyset$ and pick $\langle x, h_\xi(x)\rangle \in [t] \times [\pi(t)]$, which exists by the density of the graph of h_ξ. Then $\langle n, \pi, h \cup \{\langle x, h_\xi(x)\rangle\}\rangle$ belongs to S_t and extends q.

This shows that $D \cap [t] \ne \emptyset$ for every $t \in 2^{<\omega}$, that is, D is dense in 2^ω.

A similar argument shows that for every $t \in 2^{<\omega}$ the set

$$S^t = \{q = \langle n, \pi, h\rangle \in Q_{\bar h} \colon \text{ the range of } h \text{ intersects } [t]\}$$

is dense in $Q_{\bar h}$, which implies that $h[D]$ is dense in 2^ω. Thus $\tilde f$ is a homeomorphism from $\mathrm{cl}(D) = 2^\omega$ onto $\mathrm{cl}(h[D]) = 2^\omega$. $\square$

Now take $A \subset 2^\omega \times 2^\omega$ for which A and $A^c = (2^\omega \times 2^\omega) \setminus A$ are nowhere meager in $2^\omega \times 2^\omega$ and fix an ω_1-oracle $\mathcal{M} = \langle M_\delta \colon \delta \in \Gamma\rangle$. By Fact 7 in order to prove Lemma 5 it is enough to find an $\bar h = \langle h_\xi \colon \xi \in \Gamma\rangle \in \mathcal{K}$ such that

$$Q_A = Q_{\bar h} \text{ is } \mathcal{M}\text{-cc} \tag{4}$$

and $Q_{\bar h}$ forces that, in $V[H]$,

$$\text{the sets } \mathrm{pr}(f \cap A) \text{ and } \mathrm{pr}(f \setminus A) \text{ are nowhere meager in } 2^\omega. \tag{5}$$

(In (5) function f is defined as in Fact 7.)

To define $\bar h$ we will construct a sequence $\langle\langle x_\alpha, y_\alpha\rangle \in 2^\omega \times 2^\omega \colon \alpha < \omega_1\rangle$ aiming at $h_\xi = \{\langle x_{\xi+n}, y_{\xi+n}\rangle \colon n < \omega\}$, where $\xi \in \Gamma$.

Let $\{\langle s_n, t_n\rangle \colon n < \omega\}$ be an enumeration of $2^{<\omega} \times 2^{<\omega}$ with each pair $\langle s, t\rangle$ appearing for an odd n and for an even n. Points $\langle x_{\xi+n}, y_{\xi+n}\rangle$ are chosen inductively in such a way that

(i) $\langle x_{\xi+n}, y_{\xi+n}\rangle$ is a Cohen real over $M_\delta[\langle\langle x_\alpha, y_\alpha\rangle \colon \alpha < \xi + n\rangle]$ for every $\delta \le \xi$, $\delta \in \Gamma$, that is, $\langle x_{\xi+n}, y_{\xi+n}\rangle$ is outside all meager subsets of $2^\omega \times 2^\omega$ which are coded in $M_\delta[\langle\langle x_\alpha, y_\alpha\rangle \colon \alpha < \xi + n\rangle]$;

(ii) $\langle x_{\xi+n}, y_{\xi+n}\rangle \in A$ if n is even, and $\langle x_{\xi+n}, y_{\xi+n}\rangle \in A^c$ otherwise.

(iii) $\langle x_{\xi+n}, y_{\xi+n}\rangle \in [s_n] \times [t_n]$.

The choice of $\langle x_{\xi+n}, y_{\xi+n}\rangle$ is possible since both sets A and A^c are nowhere meager, and we consider each time only countably many meager sets. Condition (iii) guarantees that the graph of each of h_ξ will be dense in $2^\omega \times 2^\omega$.

Note that if $\Gamma \ni \delta \le \alpha_0 < \cdots < \alpha_{k-1}$, where $k < \omega$, then (by the product lemma in M_δ)

$$\langle\langle x_{\alpha_i}, y_{\alpha_i}\rangle\colon i<k\rangle \text{ is an } M_\delta\text{-generic Cohen real in } (2^\omega \times 2^\omega)^k. \tag{6}$$

For $q = \langle n, \pi, h\rangle \in Q_{\bar h}$ define

$$\hat q = \bigcup_{\langle s,t\rangle \in \pi} [s] \times [t].$$

Clearly $\hat q$ is an open subset of $2^\omega \times 2^\omega$ and condition (2) implies that for every $q, r \in Q_{\bar h}$ with $r = \langle n', \pi', h'\rangle$

$$\text{if } q \le r \text{ then } \hat q \subset \hat r \text{ and } \hat q \cap ([s]\times[t]) \ne \emptyset \text{ for every } \langle s,t\rangle \in \pi'. \tag{7}$$

Also for $\delta \in \Gamma$ let $(Q_{\bar h})^\delta = \left\{\langle n,\pi,h\rangle \in Q_{\bar h} \colon h \subset \bigcup_{\zeta<\delta} h_\zeta\right\}$. To prove (4) and (5) we will use also the following fact.

Fact 8. *Let $\delta \in \Gamma$ and let $E \in M_\delta$ be a predense subset of $(Q_{\bar h})^\delta$. Then for every $k<\omega$ and $p = \langle n,\pi,h\rangle \in (Q_{\bar h})^\delta$ the set*

$$B^k_p = \bigcup \left\{(\hat q)^k \colon \ q \text{ extends } p \text{ and some } q_0 \in E\right\} \tag{8}$$

is dense in $(\hat p)^k \subset (2^\omega \times 2^\omega)^k$.

Proof. By way of contradiction assume that B^k_p is not dense in $(\hat p)^k$. Then there exist $m < \omega$ and $s_0, t_0, \ldots, s_{k-1}, t_{k-1} \in 2^m$ with the property that $P = \prod_{i<k}([s_i]\times[t_i]) \subset (\hat p)^k$ is disjoint from B^k_p. Increasing m and refining the s_i's and t_j's, if necessary, we may assume that $m \ge n$, all s_i's and t_j's are different, $\bigcup_{i<k}[s_i]$ is disjoint from the domain D of h, and $h[D] \cap \bigcup_{i<k}[t_i] = \emptyset$. We can also assume that $x \restriction m \ne y \restriction m$ for every different x and y from D and from $h[D]$. Now, refining slightly the argument for $(*)$ we can find $r = \langle m, \pi', h\rangle \in (Q_{\bar h})^\delta$ extending p such that $\pi'(s_i) = t_i$ for every $i<k$. (Note that $P \subset (\hat p)^k$.) We will obtain a contradiction with the predensity of E in $(Q_{\bar h})^\delta$ by showing that r is incompatible with every element of E.

Indeed if q were an extension of $r \le p$ and an element q_0 of E, then we would have $(\hat q)^k \subset B^k_p$. But then, by (7) and the fact that $\langle s_i, t_i\rangle \in \pi'$ for $i<k$, we would also have $(\hat q)^k \cap P \ne \emptyset$, contradicting $P \cap B^k_p = \emptyset$. This finishes the proof of Fact 8. □

Now we are ready to prove (4), that is, that $Q_{\bar h}$ is $\mathcal{M}$-cc. So, fix a bijection $e\colon Q_{\bar h} \to \omega_1$ and let

$$C = \left\{\delta \in \Gamma \colon (Q_{\bar h})^\delta = e^{-1}(\delta) \in M_\delta\right\}.$$

Then $C \in D_{\mathcal{M}}$. (Just use a suitable nice codding or [12, Claim 1.4(4)].) Take a $\delta \in C$ and fix an $E \subset \delta$, $E \in M_\delta$, for which $e^{-1}(E)$ is predense in $(Q_{\bar{h}})^\delta$. By Fact 4 it is enough to show that

$$e^{-1}(E) \text{ is predense in } Q_{\bar{h}}.$$

Take $p_0 = \langle n, \pi, h_0 \rangle$ from $Q_{\bar{h}}$, let $h = h_0 \restriction \bigcup_{\eta<\delta} D_\eta$ and $h_1 = h_0 \setminus h$, and notice that the condition $p = \langle n, \pi, h \rangle$ belongs to $(Q_{\bar{h}})^\delta$. Assume that $h_1 = \{\langle x_i, y_i \rangle : i < k\}$. Since $s(h_1) = \langle \langle x_i, y_i \rangle : i < k \rangle \in (\hat{p})^k$, $B^k_p \in M_\delta$ (as defined from $(Q_{\bar{h}})^\delta \in M_\delta$) and, by Fact 8, B^k_p is dense in $(\hat{p})^k$ condition (6) implies that $s(h_1) \in B^k_p$. So there are $q = \langle n_0, \pi_0, g \rangle \in (Q_{\bar{h}})^\delta$ extending p and some $q_0 \in e^{-1}(E)$ for which $s(h_1) \in \hat{q}^k$. But then $p' = \langle n_0, \pi_0, g \cup h_1 \rangle$ belongs to $Q_{\bar{h}}$ and extends q. This finishes the proof of (4).

The proof of (5) is similar. We will prove only that $\mathrm{pr}(f \setminus A) = \mathrm{pr}(f \cap A^c)$ is nowhere meager in 2^ω, the argument for $\mathrm{pr}(f \cap A)$ being essentially the same.

By way of contradiction assume that $\mathrm{pr}(f \setminus A)$ is not nowhere meager in 2^ω. So there exists an $s^* \in 2^{<\omega}$ such that $\mathrm{pr}(f \setminus A)$ is meager in $[s^*]$. Let a condition $p^* \in Q_{\bar{h}}$ and $Q_{\bar{h}}$-names $\dot{U}_m$, for $m < \omega$, be such

$$p^* \Vdash_{Q_{\bar{h}}} \text{ each } \dot{U}_m \text{ is an open dense subset of } [s^*] \text{ and } \mathrm{pr}(f \setminus A) \cap \bigcap_{m<\omega} \dot{U}_m = \emptyset.$$

For each $m < \omega$, since p^* forces that $\dot{U}_m$ is an open dense subset of $[s^*]$, for every $t \in 2^{<\omega}$ extending s^* there is a maximal antichain $\langle p^m_{s,k} : k < \kappa^m_s \rangle$ in $Q_{\bar{h}}$ forcing that $\dot{U}_m \cap [t]$ contains some basic open subset $[s]$.

Note that each of these antichains must be countable, since the forcing notion $Q_{\bar{h}}$ is $\mathcal{M}$-cc and therefore ccc. Combining all these antichains we get a sequence $\langle p^m_{s,k} \in Q_{\bar{h}} : m < \omega, s \in 2^{<\omega}, k < \kappa^m_s \rangle$ such that

- $\kappa^m_s \le \omega$,
- $p^m_{s,k} \Vdash_{Q_{\bar{h}}} [s] \subseteq \dot{U}_m$,
- for every $q \in Q_{\bar{h}}$ extending p^* and $t \in 2^{<\omega}$ extending s^* there are $s \in 2^{<\omega}$ and $k < \kappa^m_s$ such that the conditions q and $p^m_{s,k}$ are compatible and $t \subset s$.

Note that for sufficiently large $\delta \in \Gamma$ we have $p^m_{s,k} \in (Q_{\bar{h}})^\delta$ for all $m < \omega$, $s \in 2^{<\omega}$, and $k < \kappa^m_s$.

Now, by the definition of ω_1-oracle, the set B_0 of all $\delta \in \Gamma$ for which

$$\langle p^m_{s,k} \in Q_{\bar{h}} : m < \omega, s \in 2^{<\omega}, k < \kappa^m_s \rangle \in M_\delta \quad \text{and} \quad (Q_{\bar{h}})^\delta \in M_\delta$$

is stationary in ω_1. (Just use a suitable nice coding, or see [12, Chapter IV, Claim 1.4(4)]). Thus, using clause (iii) of the choice of x_ξ's, we may find a $\delta \in B_0$, an odd $j < \omega$, and a condition $p_0 = \langle n_0, \pi_0, h_0 \rangle \in Q_{\bar{h}}$ such that

- $p_0 \leq p^*$, $s^* \subset x_{\delta+j}$, and
- $x_{\delta+j}$ belongs to the domain of h_0.

Then $p_0 \Vdash$ "$x_{\delta+j} \in [s^*] \cap \mathrm{pr}(f \setminus A)$" (remember j is odd so $\langle x_{\delta+j}, y_{\delta+j}\rangle \in A^c$). We will show that

$$p_0 \Vdash x_{\delta+j} \in \bigcap_{m<\omega} \dot{U}_m,$$

which will finish the proof.

So, assume that this is not the case. Then there exist an $i < \omega$ and a $p_1 = \langle n, \pi, h_1\rangle \in Q_{\bar{h}}$ stronger than p_0 such that $p_1 \Vdash$ "$x_{\delta+j} \notin \dot{U}_i$." Let us define $h = h_1 \restriction \{x_\alpha : \alpha < \delta\}$ and $h_1 \setminus h = \{\langle a_l, b_l\rangle \colon l < m\}$. Notice that the condition $p = \langle n, \pi, h\rangle$ belongs to $(Q_{\bar{h}})^\delta$. We can also assume that $\langle x_{\delta+j}, y_{\delta+j}\rangle = \langle a_0, b_0\rangle$.

Now consider the set Z of all $\langle z_0, z'_0, \ldots, z_{m-1}, z'_{m-1}\rangle \in (2^\omega \times 2^\omega)^m$ for which

- there exist $s \in 2^{<\omega}$, $k < \kappa^i_s$, and $q \in (Q_{\bar{h}})^\delta$ such that $s \subset z_0$, q extends p and $p^i_{s,k}$, and $\langle z_0, z'_0, \ldots, z_{m-1}, z'_{m-1}\rangle \in (\hat{q})^m$.

Claim. The set Z belongs to the model M_δ and it is an open dense subset of $(\hat{p})^m$.

Proof. It should be clear that Z is (coded) in M_δ. (Remember the choice of δ.) To show that it is dense in $(\hat{p})^m$ we proceed like in the proof of Fact 8. We choose $s_0, t_0, \ldots, s_{m-1}, t_{m-1}$ and r exactly as there. Next pick a condition $q \in Q_{\bar{h}}$, a sequence $s \in 2^{<\omega}$, and $k < \kappa^m_s$ such that

$$s_0 \subset s \quad \text{and} \quad q \ \text{ extends } \ p^i_{s,k} \ \text{ and } \ r.$$

(Remember the choice of the $p^i_{s,k}$'s.) Clearly we can demand that $q \in (Q_{\bar{h}})^\delta$. Now note that it is possible to choose a $\bar{z} = \langle z_0, z'_0, \ldots, z_{m-1}, z'_{m-1}\rangle \in (\hat{q})^m$ such that $s \subset z_0$, $s_i \subset z_i$, $t_i \subset z'_i$. Then $\bar{z} \in Z \cap \prod_{i<k}([s_i] \times [t_i])$.

Since Z is clearly open, this completes the proof of Claim.

Now, by (6) and the Claim above, $\langle\langle a_l, b_l\rangle \colon l < m\rangle$ belongs to Z since $\langle\langle a_l, b_l\rangle \colon l < m\rangle$ belongs to $(\hat{p}_1)^m = (\hat{p})^m$. But this means that there exist $q = \langle n^q, \pi^q, h^q\rangle \in (Q_{\bar{h}})^\delta$ and $s \in 2^{<\omega}$ such that:

- $q \leq p$, $q \Vdash$ "$[s] \subseteq \dot{U}_i$", and
- $\langle\langle a_l, b_l\rangle \colon l < m\rangle \in (\hat{q})^m$, and $x_{\delta+j} = a_0 \in [s]$.

But then $p_2 = \langle n^q, \pi^q, h^q \cup \{\langle a_l, b_l\rangle \colon l < m\}\rangle$ belongs to $Q_{\bar{h}}$ and extends both q and p_1. So, p_2 forces that $x_{\delta+j} = a_0 \in [s] \subseteq \dot{U}_i$, contradicting our assumption that $p_1 \Vdash$ "$x_{\delta+j} \notin \dot{U}_i$."

This finishes the proof of (5) and of Lemma 5.

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