Topological dimension and sums of connectivity functions ☆

Krzysztof Ciesielski *, Jerzy Wojciechowski

Department of Mathematics, West Virginia University, P.O. Box 6310, Morgantown, WV 26506-6310, USA

Abstract

The main goal of this paper is to show that the inductive dimension of a σ-compact metric space X can be characterized in terms of algebraical sums of connectivity (or Darboux) functions $X \to \mathbb{R}$. As an intermediate step we show, using a result of Hayashi [Topology Appl. 37 (1990) 83], that for any dense G_δ-set $G \in \mathbb{R}^{2k+1}$ the union of G and some k homeomorphic images of G is universal for k-dimensional separable metric spaces. We will also discuss how our definition works with respect to other classes of Darboux-like functions. In particular, we show that for the class of peripherally continuous functions on an arbitrary separable metric space X our parameter is equal to either ind X or ind $X - 1$. Whether the latter is at all possible, is an open problem. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Inductive dimension; Connectivity functions; Darboux functions

AMS classification: Primary 54F45, Secondary 54C30; 26B40

1. Introduction

Our terminology and notation is standard and follows [1]. Let X be a non-empty set and $\mathcal{F}$ be a family of functions from X into $\mathbb{R}$. If m is a nonnegative integer, then let

$$m\mathcal{F} = \{f_1 + \cdots + f_m \colon f_1, \ldots, f_m \in \mathcal{F}\},$$

and let $\mathbb{R}^X$ be the family consisting of all functions from X into $\mathbb{R}$. Let $\mathrm{DIM}_{\mathcal{F}} X$ be defined by

$$\mathrm{DIM}_{\mathcal{F}} X = \min\big(\big\{m \in \mathbb{Z} \colon m \geqslant 0 \text{ and } (m+1)\mathcal{F} = \mathbb{R}^X\big\} \cup \{\infty\}\big).$$

Given a metric space X, a function $f : X \to \mathbb{R}$ is a *connectivity function* (*Darboux function*) if for every connected subset C of X the graph of the restriction $f \upharpoonright C$ is a connected subset of $X \times \mathbb{R}$ (the image $f[C]$ is connected in $\mathbb{R}$). The following theorem holds.

Theorem 1. *If n is a positive integer and $\mathcal{F}$ is the family of connectivity functions or the family of Darboux functions on $\mathbb{R}^n$, then*

$$\mathrm{DIM}_{\mathcal{F}} \, \mathbb{R}^n = n.$$

The proof of Theorem 1 is given by Ciesielski and Wojciechowski [4], except for the case $n = 1$ that has been proved by Ciesielski and Recław [2], and the inequality $\geqslant$ in the case of Darboux functions that has been demonstrated by Jordan [11,12].

Theorem 1 motivates the notation $\mathrm{DIM}_{\mathcal{F}} \, X$ and shows that (with suitably chosen family $\mathcal{F}$) $\mathrm{DIM}_{\mathcal{F}} \, X$ can be considered as a sort of dimension of X (*dimension relative to $\mathcal{F}$*). In this paper we are going to show that the dimension relative to the family of connectivity (Darboux) functions coincides with the inductive dimension ind on every σ-compact metric space.

Let X be a separable metric space. Given $A, B \subseteq X$, the *boundary* of $A \cap B$ in A will be denoted by $\mathrm{bd}_A B$. The *inductive dimension* ind A of a subset $A \subseteq X$ is defined inductively as follows. (See, for example, Engelking [5].)

  (i) ind $A = -1$ if and only if $A = \emptyset$.
 (ii) ind $A \leqslant m$ if for any $p \in A$ and any open neighborhood W of p there exists an open neighborhood $U \subseteq W$ of p such that ind $\mathrm{bd}_A U \leqslant m - 1$.
(iii) ind $A = m$ if ind $A \leqslant m$ and it is not true that ind $A \leqslant m - 1$.

Let $\mathcal{C}$ be the family of connectivity functions on X and $\mathcal{D}$ be the family of Darboux functions on X. Our main result is the following theorem.

Theorem 2. *If X is a σ-compact metric space, then*

$$\mathrm{DIM}_{\mathcal{C}} \, X = \mathrm{DIM}_{\mathcal{D}} \, X = \mathrm{ind} \, X.$$

Clearly

$$\mathrm{DIM}_{\mathcal{F}} \, X \geqslant \mathrm{DIM}_{\mathcal{G}} \, X \quad \text{for any } \mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R}^X. \tag{1}$$

Since $\mathcal{C} \subseteq \mathcal{D}$ for any space X, we have $\mathrm{DIM}_{\mathcal{C}} \, X \geqslant \mathrm{DIM}_{\mathcal{D}} \, X$, and so Theorem 2 follows immediately from the following two results.

Theorem 3. *If X is a separable metric space, then*

$$\mathrm{DIM}_{\mathcal{C}} \, X \leqslant \mathrm{ind} \, X.$$

Theorem 4. *If X is a σ-compact metric space, then*

$$\mathrm{DIM}_{\mathcal{D}} \, X \geqslant \mathrm{ind} \, X.$$

A natural question is whether Theorem 4 can be extended to all separable metric spaces or perhaps all that are complete. The answer is 'no' in both cases since Mazurkiewicz [13] has shown that for each positive integer n there exists a complete separable metric space X of inductive dimension n which is totally disconnected, that is, single points are its only connected subspaces. (See also [10, Example II 16].) Since for every totally disconnected space X we have

$$\mathrm{DIM}_{\mathcal{C}} X = \mathrm{DIM}_{\mathcal{D}} X = 0$$

(any function $f : X \to \mathbb{R}$ is a connectivity and Darboux), we get

$$\mathrm{DIM}_{\mathcal{C}} X = \mathrm{DIM}_{\mathcal{D}} X = 0 < n = \mathrm{ind}\, X, \tag{2}$$

for every space of Mazurkiewicz of inductive dimension $n > 0$. It might be interesting to answer the question whether the equation

$$\mathrm{DIM}_{\mathcal{C}} X = \mathrm{DIM}_{\mathcal{D}} X \tag{3}$$

holds for all separable metric spaces X or at least all that are complete.

To prove Theorem 3 we will prove the following result which seems to be of independent interest. We say that a separable metric space X is *m-dimensional* if $\mathrm{ind}\, X = m$. If Y is a metric space such that for every m-dimensional separable metric space X there is a subspace of Y homeomorphic to X, then we say that Y is *universal* for m-dimensional separable metric spaces.

Theorem 5. *If G is a dense G_δ-set in $\mathbb{R}^{2k+1}$, then there are homeomorphisms $h_j : \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$, for $j = 1, \ldots, k$, such that $G \cup \bigcup_{j=1}^{k} h_j[G]$ is universal for k-dimensional separable metric spaces.*

Theorem 5 will be used to prove the following fact, that easily implies Theorem 3.

Proposition 6. *For every positive integer k there exists a dense G_δ-set H in $\mathbb{R}^{2k+1}$ such that*

(i) *H is universal for k-dimensional separable metric spaces, and*
(ii) *for every $\varphi : \mathbb{R}^{2k+1} \to \mathbb{R}$ there are connectivity functions $g_0, \ldots, g_k : \mathbb{R}^{2k+1} \to \mathbb{R}$ such that $(g_0 + \cdots + g_k)(x) = \varphi(x)$ for every $x \in H$.*

The proof of Theorem 5 will be based on Lemma 9 and Theorem 11, that are proved in [4], and on Theorem 7, which is proved by Hayashi [9]. Theorem 5 is proved in Section 2, the proof of Theorem 3 is presented in Section 3, while Theorem 4 is proved in Section 4. The authors would like to thank Roman Pol for directing their attention to the results of Hayashi [9] and Mazurkiewicz [13].

2. A k-dimensional universal set

In this section we are going to present a proof of Theorem 5.

Let a *countable dense grid in* $\mathbb{R}^n$ be a product $B_1 \times \cdots \times B_n \subseteq \mathbb{R}^n$ where $B_1, \ldots, B_n$ are countable dense subsets of $\mathbb{R}$. If $B = B_1 \times \cdots \times B_n$ is a countable dense grid in $\mathbb{R}^n$ and $i \leqslant n$, then let $B^{(i)}$ consist of those points in $\mathbb{R}^n$ that differ from a point in B at most i coordinates, that is,

$$B^{(i)} = \big\{ \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n \colon |\{ j \colon x_j \notin B_j \}| \leqslant i \big\}.$$

Note that in particular $B^{(0)} = B$. Let $\mathbb{Q}$ be the set of rational numbers and I be the closed interval $[0, 1]$.

Our proof of Theorem 5 uses the following result of Hayashi [9]. (See also [7] for similar results.)

Theorem 7. *If G is a G_δ-set in I^{2k+1} containing $(\mathbb{Q}^{2k+1})^{(k)} \cap I^{2k+1}$, then G is universal for k-dimensional separable metric spaces.*

First notice that Theorem 7 implies immediately the following corollary.

Corollary 8. *If B is a countable dense grid in $\mathbb{R}^{2k+1}$ and G is a G_δ-set in $\mathbb{R}^{2k+1}$ containing $B^{(k)}$, then G is universal for k-dimensional separable metric spaces.*

Proof. Let $B = B_1 \times \cdots \times B_{2k+1}$. Let $g_1, \ldots, g_{2k+1} \colon \mathbb{R} \to \mathbb{R}$ be increasing homeomorphisms such that $B_i = g_i[\mathbb{Q}]$ and

$$g = g_1 \times \cdots \times g_{2k+1} \colon \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}.$$

Then

$$\big(\mathbb{Q}^{2k+1}\big)^{(k)} \cap I^{2k+1} \subseteq g^{-1}[G] \cap I^{2k+1}.$$

Let X be a k-dimensional separable metric space. It follows from Theorem 7 that there is a subspace Y of $g^{-1}[G] \cap I^{2k+1}$ that is homeomorphic to X. Then $g[Y]$ is a subspace of G that is homeomorphic to X. $\square$

To prove Theorem 5 we will also need a result proved implicitly in [4]. We will first introduce the notation used there. If $\langle B_i \colon i \in n \rangle$ is a family of subsets of $\mathbb{R}$ and f is a function from $\{1, \ldots, n\}$ into $\{0, 1\}$, then let

$$\prod_{i=1}^{n} (B_i \vee_f \mathbb{R}) = B_1' \times \cdots \times B_n',$$

where

$$B_i' = \begin{cases} B_i & \text{if } f(i) = 0, \\ \mathbb{R} & \text{if } f(i) = 1. \end{cases}$$

The following lemma is stated implicitly and proved in [4] (the inductive condition (8) in the proof of Proposition 2.4, p. 419).

Lemma 9. *If G is a dense G_δ-set in $\mathbb{R}^n$, then there are countable dense sets $B_i \subseteq \mathbb{R}$ and homeomorphisms $h_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, \ldots, n$, such that*

$$\prod_{i=1}^{n} (B_i \vee_f \mathbb{R}) \subseteq G \cup \bigcup_{i=1}^{k} h_i[G]$$

for every $k \in \{0, 1, \ldots, n\}$ and every function $f : \{1, \ldots, n\} \to \{0, 1\}$ such that $|f^{-1}(1)| = k$.

Lemma 9 implies immediately the following result.

Theorem 10. *If G is a dense G_δ-set in $\mathbb{R}^n$ and $k \leqslant n$, then there is a countable dense grid B in $\mathbb{R}^n$ and homeomorphisms $h_1, \ldots, h_k : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$B^{(k)} \subseteq G \cup \bigcup_{j=1}^{k} h_j[G].$$

Proof. Let G be a dense G_δ-set in $\mathbb{R}^n$. For $i = 1, \ldots, n$, let $B_i \subseteq \mathbb{R}$ be countable dense sets and $h_i : \mathbb{R}^n \to \mathbb{R}^n$ be homeomorphisms as in Lemma 9. Then

$$B = B_1 \times \cdots \times B_n$$

is a countable dense grid in $\mathbb{R}^n$ and

$$B^{(k)} = \bigcup \left\{ \prod_{i=1}^{n} (B_i \vee_f \mathbb{R}): \left| f^{-1}(1) \right| = k \right\}.$$

It follows from Lemma 9 that

$$B^{(k)} \subseteq G \cup \bigcup_{j=1}^{k} h_j[G]. \qquad \square$$

Proof of Theorem 5. Let G be a dense G_δ-set in $\mathbb{R}^{2k+1}$. By Theorem 10, there is a countable dense grid B in $\mathbb{R}^{2k+1}$ and homeomorphisms $h_1, \ldots, h_k : \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ such that $B^{(k)} \subseteq G \cup \bigcup_{j=1}^{k} h_j[G]$. By Corollary 8, $G \cup \bigcup_{j=1}^{k} h_j[G]$ is universal for k-dimensional separable metric spaces. $\quad \square$

3. Inductive dimension as the upper bound

Now we shall prove Theorem 3. Beside Theorem 5 we will need the following result. (See [4, Proposition 2.3].)

Theorem 11. *For every $n > 1$, there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ and a dense G_δ-subset G of $\mathbb{R}^n$ such that any function $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) = f(x)$ for $x \notin G$ is a connectivity function.*

Let us now introduce some notation. If $f, g : \mathbb{R}^n \to \mathbb{R}$ and $A \subseteq \mathbb{R}^n$, then we will write $g \equiv_A f$ if and only if $g(x) = f(x)$ for every $x \in \mathbb{R}^n \setminus A$. Notice that if $g \equiv_A f$ and $A \subseteq A'$, then $g \equiv_{A'} f$. Also $g \equiv_\emptyset f$ if and only if $g = f$, and $g \equiv_{\mathbb{R}^n} f$ for any $f, g : \mathbb{R}^n \to \mathbb{R}$. The following two lemmas are easy observations.

Lemma 12. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and $A \subseteq \mathbb{R}^n$. If $h : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection, then $g \equiv_{h[A]} (f \circ h^{-1})$ if and only if $(g \circ h) \equiv_A f$.*

Proof. Assume $g \equiv_{h[A]} (f \circ h^{-1})$. Then $g(x) = f(h^{-1}(x))$ for every $x \in \mathbb{R}^n \setminus h[A]$. If $y \in \mathbb{R}^n \setminus A$, then $h(y) \in \mathbb{R}^n \setminus h[A]$ so

$$(g \circ h)(y) = g\big(h(y)\big) = f\big(h^{-1}\big(h(y)\big)\big) = f(y),$$

implying that $(g \circ h) \equiv_A f$.

The opposite implication is proved similarly. □

Lemma 13. *Let $g'_0, \dots, g'_k : \mathbb{R}^n \to \mathbb{R}$. If $A \subseteq \mathbb{R}^n$, and $\{A_0, \dots, A_k\}$ is a partition of A, then for any $\varphi : A \to \mathbb{R}$ there are $g_0, \dots, g_k : \mathbb{R}^n \to \mathbb{R}$ such that*

$$g_i \equiv_{A_i} g'_i, \quad i = 0, \dots, k,$$

and the restriction of $g_0 + \cdots + g_k$ to A is equal to φ.

Proof. Define $g_i : \mathbb{R}^n \to \mathbb{R}$ by

$$g_i(x) = \begin{cases} \varphi(x) - \sum_{j \neq i} g'_j(x) & \text{if } x \in A_i, \\ g'_i(x) & \text{if } x \notin A_i. \end{cases}$$

Then $\varphi(x) = g_0(x) + \cdots + g_k(x)$ for every $x \in A$. □

Proof of Proposition 6. Let $n = 2k + 1$. By Theorem 11 there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ and a dense G_δ-subset G of $\mathbb{R}^n$ such that any function $g : \mathbb{R}^n \to \mathbb{R}$ with $g \equiv_G f$ is a connectivity function. By Theorem 5, there are homeomorphisms $h_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, \dots, k$, such that the G_δ-set $H = G \cup \bigcup_{j=1}^{k} h_j[G]$ is universal for k-dimensional separable metric spaces. Let $\{A_0, \dots, A_k\}$ be the partition of H defined inductively by

$$A_0 = G, \quad A_j = h_j[G] \setminus (A_0 \cup \cdots \cup A_{j-1}), \quad j = 1, \dots, k.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary function, and $h_0 : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function. It follows from Lemma 13, that there are functions $g_0, \dots, g_k : \mathbb{R}^n \to \mathbb{R}$ such that

$$g_i \equiv_{A_i} (f \circ h_i^{-1}), \quad i = 0, \dots, k,$$

and the restriction of $g_0 + \cdots + g_k$ to H is equal to $\varphi \restriction H$. It remains to prove that g_i's are connectivity functions.

Let $i \in \{0, \dots, k\}$. Since $A_i \subseteq h_i[G]$, we have

$$g_i \equiv_{h_i[G]} (f \circ h_i^{-1}),$$

and so Lemma 12 implies that

$$g_i \circ h_i \equiv_G f.$$

Thus $g_i \circ h_i$ (and hence g_i) is a connectivity function on $\mathbb{R}^n$. $\square$

Proof of Theorem 3. Let X be a k-dimensional separable metric space. If $k = 0$, then any function $X \to \mathbb{R}$ is a connectivity function, so we can assume that $k \geqslant 1$. Let H be a G_δ-set from Proposition 6. Then there is a subspace A of H homeomorphic to X. Take an arbitrary $\varphi_0 : A \to \mathbb{R}$. We have to show that φ_0 is a sum of $k + 1$ connectivity functions on A.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary extension of φ_0 and let $g_0, \ldots, g_k : \mathbb{R}^n \to \mathbb{R}$ be connectivity functions such that $(g_0 + \cdots + g_k)(x) = \varphi(x)$ for all $x \in H$. Then the functions $g_i \restriction A$ are connectivity and $(g_0 \restriction A) + \cdots + (g_k \restriction A) = \varphi_0$. $\square$

4. Inductive dimension as the lower bound

In this section we are going to prove Theorem 4. In the proof that follows we will need some additional definitions and results from dimension theory. (See, for example, [10].)

Lemma 14. *If X is a separable metric space and*

$$X = \bigcup_{i=1}^{\infty} X_i,$$

where X_i is closed in X and $\operatorname{ind} X_i \leqslant m$, for $i = 1, 2, \ldots$, then $\operatorname{ind} X \leqslant m$.

Given $X \subseteq \mathbb{R}^n$ and an integer $m \geqslant 1$, we say that X is an *m-dimensional Cantor-manifold* if X is compact, $\operatorname{ind} X = m$, and for every $Y \subseteq X$ with $\operatorname{ind} Y \leqslant m - 2$, the set $X \setminus Y$ is connected.

The following lemma is proved in [10].

Lemma 15. *For any compact $Y \subseteq \mathbb{R}^n$ with $\operatorname{ind} Y \geqslant m$ there exists an m-dimensional Cantor manifold $X \subseteq Y$.*

We will also need the following result of Francis Jordan. (See [11, Lemma 3.3.8] or [12, Lemma 3.8].) A *perfect set* is a non-empty closed set without isolated points.

Lemma 16. *Let $n > 1$ and M be an n-dimensional Cantor manifold. If $n \geqslant k \geqslant 1$ and $f \in k\mathcal{D}$, where $\mathcal{D}$ is the family of Darboux functions $M \to \mathbb{R}$, then there is a connected perfect set $P \subseteq M$ such that the restriction of f to P is Darboux.*

A *Bernstein set*, is a set $B \subseteq \mathbb{R}^n$ such that $B \cap P \neq \emptyset$ and $B \setminus P \neq \emptyset$ for every perfect set $P \subseteq \mathbb{R}^n$. Note that the characteristic function of a Bernstein set is not Darboux on any perfect set.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Suppose, by way of contradiction, that there exists a k-dimensional σ-compact metric space X such that

$$\mathrm{DIM}_{\mathcal{D}} X < \mathrm{ind}\, X = k,$$

where $\mathcal{D}$ is the family of Darboux functions on X. We can assume that $X \subseteq \mathbb{R}^m$ for some positive integer m. Then

$$X = \bigcup_{i=1}^{\infty} X_i,$$

with X_i compact, $i = 1, 2 \ldots$ and it follows from Lemma 14 that there is a positive integer j with $\mathrm{ind}\, X_j \geqslant k$. By Lemma 15 there is a k-dimensional Cantor manifold $M \subseteq X_j$.

Let $B \in \mathbb{R}^m$ be a Bernstein set and $f : X \to \mathbb{R}$ be the characteristic function of $B \cap X$. Since $\mathrm{DIM}_{\mathcal{D}} X < k$, we have $f \in k\mathcal{D}$. Hence the restriction of f to M is in $k\mathcal{D}'$ where $\mathcal{D}'$ is the family of Darboux functions on M. It follows from Lemma 16 that the restriction of f to some perfect set in $\mathbb{R}^m$ is Darboux. Since no restriction of the characteristic function of a Bernstein set to a connected perfect set can be Darboux, we got a contradiction proving that $\mathrm{DIM}_{\mathcal{D}} X \geqslant \mathrm{ind}\, X$.   □

5. Dimension relative to other classes of Darboux-like functions

In this section we will consider how our definition of dimension works with some other classes of Darboux-like functions. (See [6] or [3].) Given a topological space X, a function $f : X \to \mathbb{R}$ is:

- *almost continuous* (in sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of f contains also the graph of a continuous function from X to $\mathbb{R}$;
- *extendable* provided there exists a connectivity function $F : X \times [0, 1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$.

The classes that are defined above are denoted by $\mathrm{AC}(X)$, $\mathrm{Ext}(X)$, and $\mathrm{PC}(X)$, respectively. The following inclusion relations hold when $X = \mathbb{R}^n$. (See [6] or [3].)

$$\mathrm{Ext}(\mathbb{R}) \longrightarrow \mathrm{AC}(\mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R}) \longrightarrow \mathcal{D}(\mathbb{R}) \longrightarrow \mathrm{PC}(\mathbb{R})$$

and, for $n > 1$,

$$\mathrm{Ext}(\mathbb{R}^n) = \mathcal{C}(\mathbb{R}^n) = \mathrm{PC}(\mathbb{R}^n) \longrightarrow \mathrm{AC}(\mathbb{R}^n) \cap \mathcal{D}(\mathbb{R}^n) \begin{array}{c} \nearrow \mathrm{AC}(\mathbb{R}^n) \\ \searrow \mathcal{D}(\mathbb{R}^n) \end{array}$$

where $\longrightarrow$ denotes a strict inclusion.

Natkaniec [14, Proposition 1.7.1] proved that every function $f : \mathbb{R}^n \to \mathbb{R}$ is a sum of two almost continuous functions. This implies that

$$\mathrm{DIM}_{\mathrm{AC}} \mathbb{R}^n = 1 \quad \text{for every } n = 1, 2, 3, \ldots \tag{4}$$

making the class AC useless in our definition of dimension. The situation is different for the remaining two classes.

Let X be a separable metric space. Since

$$\mathrm{Ext}\big(\mathbb{R}^{2k+1}\big) = \mathcal{C}\big(\mathbb{R}^{2k+1}\big) = \mathrm{PC}\big(\mathbb{R}^{2k+1}\big)$$

for $k \geqslant 1$, and since any function $X \to \mathbb{R}$ is both peripherally continuous and extendable when $\mathrm{ind}\, X = 0$, the inequalities

$$\mathrm{DIM}_{\mathrm{Ext}}\, X \leqslant \mathrm{ind}\, X \quad \text{and} \quad \mathrm{DIM}_{\mathrm{PC}}\, X \leqslant \mathrm{ind}\, X \tag{5}$$

follow from Proposition 6 in precisely the same way as Theorem 3 does. Moreover, it is immediate to see that the analog of Theorem 2 for the class Ext is also true.

Theorem 17. *If X is a σ-compact metric space, then*

$$\mathrm{DIM}_{\mathrm{Ext}}\, X = \mathrm{ind}\, X.$$

Proof. The inequality $\mathrm{DIM}_{\mathrm{Ext}}\, X \leqslant \mathrm{ind}\, X$ is a restatement of (5). The other inequality holds since for every σ-compact metric space X we have $\mathrm{DIM}_{\mathcal{C}}\, X = \mathrm{ind}\, X$ and the inequality $\mathrm{DIM}_{\mathrm{Ext}}\, X \geqslant \mathrm{DIM}_{\mathcal{C}}\, X$ is implied by $\mathrm{Ext}(X) \subseteq \mathcal{C}(X)$ and (1). $\square$

In the case of the class PC the situation is quite different. Unlike for the classes $\mathcal{C}$, $\mathcal{D}$, and Ext (see (2) which holds also for $\mathrm{DIM}_{\mathrm{Ext}}\, X$) the dimension relative to the class PC is very close to the inductive dimension for every separable metric space. However, it is not clear whether we have equality even for all compact metric spaces.

Theorem 18. *If X is a separable metric space, then*

$$\mathrm{ind}\, X - 1 \leqslant \mathrm{DIM}_{\mathrm{PC}}\, X \leqslant \mathrm{ind}\, X. \tag{6}$$

Proof. Let $k = \mathrm{ind}\, X$. The inequality $\mathrm{DIM}_{\mathrm{PC}}\, X \leqslant k$ is a restatement of (5). To prove the other inequality we will show that

($*$) for every $g_1, \ldots, g_{k-1} \in \mathrm{PC}(X)$ and $\varepsilon > 0$ there exist a closed subset Y of X of cardinality continuum such that

$$\big|g_i(x) - g_i(y)\big| < \varepsilon$$

for every $x, y \in Y$ and $i = 1, 2, \ldots, k-1$.

We prove ($*$) by induction on $k \geqslant 1$. If $k = 1$, take $Y = X$. The cardinality of X cannot be smaller than continuum since for some $x \in X$ and $r > 0$ the boundaries of the open balls in X with center x and radius smaller than r are non-empty and pairwise disjoint.

Assume that $k \geqslant 2$. Let $g_1, \ldots, g_{k-1} \in \mathrm{PC}(X)$ and $\varepsilon > 0$. There is $p \in X$ and an open neighborhood W of p such that $\mathrm{ind}\, \mathrm{bd}(U) = k - 1$ for any open U with $p \in U \subseteq W$. Since g_1 is peripherally continuous, there is an open neighborhood U of p such that $U \subseteq W$ and the image $g_1[\mathrm{bd}(U)]$ is contained in the open interval $(g_1(p) - \varepsilon/2, g_1(p) + \varepsilon/2)$. Since

$\operatorname{ind} \operatorname{bd}(U) = k - 1$, it follows from the inductive hypothesis that there is a closed subset Y of $\operatorname{bd}(U)$ of cardinality continuum such that

$$\left| g_i(x) - g_i(y) \right| < \varepsilon \tag{7}$$

for every $x, y \in Y$ and $i = 2, 3, \ldots, k - 1$. Then Y is closed in X and it follows from the choice of U, that (7) holds also for $i = 1$ completing the proof of $(*)$.

Now we show that $(*)$ implies that

$$\operatorname{DIM}_{\mathrm{PC}} X \geqslant k - 1.$$

Let Z be a subset of X such that $A \cap Z \neq \emptyset$ and $A \setminus Z \neq \emptyset$ for every closed $A \subseteq X$ of cardinality continuum. The existence of such Z can be proved by listing all closed subsets of X of cardinality continuum in a sequence $\langle A_\alpha \rangle_{\alpha < \mathfrak{c}}$ of length continuum, defining two sequences $\langle a_\alpha \rangle_{\alpha < \mathfrak{c}}$ and $\langle b_\alpha \rangle_{\alpha < \mathfrak{c}}$ of points in X by transfinite induction so that

$$a_\alpha \in A_\alpha \setminus \left(\{ a_\beta : \beta < \alpha \} \cup \{ b_\beta : \beta < \alpha \} \right),$$

and

$$b_\alpha \in A_\alpha \setminus \left(\{ a_\beta : \beta \leqslant \alpha \} \cup \{ b_\beta : \beta < \alpha \} \right),$$

for every $\alpha < \mathfrak{c}$, and putting

$$Z = \{ a_\alpha : \alpha < \mathfrak{c} \}.$$

Let $f : X \to \mathbb{R}$ be the characteristic function of the set Z. The proof will be complete when we show that

$$f \notin (k - 1)\mathrm{PC}(X).$$

Suppose, by way of contradiction, that

$$f = g_1 + \cdots + g_{k-1}$$

for some $g_1, \ldots, g_{k-1} \in \mathrm{PC}(X)$. By $(*)$ there is a closed subset Y of X of cardinality continuum such that

$$\left| g_i(x) - g_i(y) \right| < \frac{1}{k - 1}$$

for every $x, y \in Y$ and $i = 1, 2, \ldots, k - 1$. Therefore

$$\left| f(x) - f(y) \right| < 1$$

for every $x, y \in Y$. Since $Y \cap Z$ and $Y \setminus Z$ are both non-empty, there are $x, y \in Y$ with $f(x) = 0$ and $f(y) = 1$ and we get a contradiction. Thus the proof is complete. $\square$

Corollary 19. *If X is a space of Mazurkiewicz of dimension $k \geqslant 2$, then the class $\mathrm{PC}(X)$ is not equal to either $\mathcal{C}(X)$, $\mathcal{D}(X)$, or $\mathrm{Ext}(X)$.*

Proof. If X is a space of Mazurkiewicz of dimension $k \geqslant 2$, then

$$\operatorname{DIM}_{\mathcal{C}} X = \operatorname{DIM}_{\mathcal{D}} X = \operatorname{DIM}_{\mathrm{Ext}} X = 0 < k - 1 \leqslant \operatorname{DIM}_{\mathrm{PC}} X. \qquad \square$$

Problem 1. Does there exist a separable (complete separable, σ-compact, compact) metric space X such that

$$\mathrm{DIM}_{\mathrm{PC}}\, X = \mathrm{ind}\, X - 1? \tag{8}$$

It is clear that if X satisfies (8), then X cannot be a finite-dimensional manifold since $\mathrm{DIM}_{\mathrm{PC}}\, \mathbb{R}^n = \mathrm{ind}\, \mathbb{R}^n$. Moreover, such a space must be at least two-dimensional. Indeed, if $\mathrm{ind}\, X = 0$, then $X \neq \emptyset$ so $\mathrm{DIM}_{\mathcal{F}}\, X \geqslant 0$ for every $\mathcal{F} \subseteq \mathbb{R}^X$. If $\mathrm{ind}\, X = 1$, then there is an $x \in X$ and an open neighborhood W of x such that $\mathrm{bd}\, U \neq \emptyset$ for every open U with $x \in U \subseteq W$. If $f : X \to \mathbb{R}$ is the characteristic function of the singleton $\{x\}$, then f is not peripherally continuous implying that $\mathrm{DIM}_{\mathrm{PC}}\, X \geqslant 1$.

References

[1] K. Ciesielski, Set Theory for the Working Mathematician, London Math. Soc. Student Texts, Vol. 39, Cambridge Univ. Press, 1997.

[2] K. Ciesielski, I. Recław, Cardinal invariants concerning extendable and peripherally continuous functions, Real Anal. Exchange 21 (1995–96) 459–472 (preprint* available [1]).

[3] K. Ciesielski, J. Jastrzębski, Darboux-like functions within the classes of Baire one, Baire two, and additive functions, Topology Appl. 103 (2000) 203–219 (preprint* available).

[4] K. Ciesielski, J. Wojciechowski, Sums of connectivity functions on $\mathbb{R}^n$, Proc. London Math. Soc. 76 (2) (1998) 406–426 (preprint* available).

[5] R. Engelking, General Topology, Rev. compl. edn., Sigma Series in Pure Mathematics, Vol. 6, Heldermann, Berlin, 1989.

[6] R.G. Gibson, T. Natkaniec, Darboux like functions, Real Anal. Exchange 22 (2) (1996–97) 492–533 (preprint* available).

[7] R. Geoghegan, R.R. Summerhill, Pseudo-boundaries and pseudo-interiors in Euclidean spaces and topological manifolds, Trans. Amer. Math. Soc. 194 (1974) 141–165.

[8] M. Hagan, Equivalence of connectivity maps and peripherally continuous transformations, Proc. Amer. Math. Soc. 17 (1966) 175–177.

[9] Y. Hayashi, An axiomatic characterization of the dimension of subsets of Euclidean spaces, Topology Appl. 37 (1990) 83–92.

[10] W. Hurewicz, H. Wallman, Dimension Theory, Princeton University Press, Princeton, NJ, 1948.

[11] F. Jordan, Cardinal numbers connected with adding Darboux-like functions, Ph.D. Dissertation, 1998.

[12] F. Jordan, Sums of Darboux-like functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, Real Anal. Exchange 24 (2) (1998–99) 729–759 (preprint* available).

[13] S. Mazurkiewicz, Sur les problèmes κ et λ de Urysohn, Fund. Math. 10 (1927) 311–319.

[14] T. Natkaniec, Almost continuity, Real Anal. Exchange 17 (1991–92) 462–520.

[15] J.R. Stallings, Fixed point theorems for connectivity maps, Fund. Math. 47 (1959) 249–263.

---

[1] Preprints marked by * are available in electronic form from the Set Theoretic Analysis Web Page http://www.\math.wvu.edu/homepages/kcies/STA/STA.html.