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Between continuous and uniformly continuous functions on $\mathbb{R}^n$ ☆

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Abstract

We study classes of continuous functions on $\mathbb{R}^n$ that can be approximated in various degree by uniformly continuous ones (uniformly approachable functions). It was proved by Berarducci et al. [Topology Appl. 121 (2002)] that no polynomial function can distinguish between them. We construct examples that distinguish these classes (answering a question by Berarducci et al. [Topology Appl. 121 (2002)]) and we offer appropriate forms of uniform approachability that enable us to obtain a general theorem on coincidence in the class of *all* continuous functions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Our set theoretical and topological notations are standard and follow [7] and [13], respectively. Given a metric space X we denote by $C(X)$ (or simply C) the set of continuous functions $f:X\to\mathbb{R}$. We use the abbreviation "u.c." for "uniformly continuous". The class of uniformly continuous functions (from currently considered space X into $\mathbb{R}$) will be denoted by UC. The main classes studied in this paper are the following.

Definition 1.1 [1]. Let X be a metric (or, more generally, uniform) space, $f:X\to\mathbb{R}$, $K\subseteq X$, and $M\subseteq X$.

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1. $g : X \to \mathbb{R}$ is a $\langle K, M\rangle$-*approximation* of f if g is u.c., $g[M] \subseteq f[M]$, and $g(x) = f(x)$ for each $x \in K$.
2. f is *uniformly approachable* (briefly, *UA*) if f has a $\langle K, M\rangle$-approximation for each compact $K \subseteq X$ and each $M \subseteq X$.
3. f is *weakly uniformly approachable* (briefly, *WUA*) if f has an $\langle x, M\rangle$-approximation (that is, more formally, $\langle \{x\}, M\rangle$-approximation) for each $x \in X$ and for each $M \subseteq X$.

Clearly every u.c. function is *UA*, and *WUA* is a special case of *UA* when the compact set K reduces to a point x. It is also not difficult to check that every *WUA* function is continuous [1, Fact 2.2]. Thus $UC \to UA \to WUA \to C$. This justifies the title of the paper.

Is should be also mentioned here that for the functions from $\mathbb{R}$ to $\mathbb{R}$ three of the above notions coincide, that is, $UA \leftrightarrow WUA \leftrightarrow C$. (See [1, Proposition 3.5].) However Maxim R. Burke noticed [1, Example 3.3] that on $\mathbb{R}^2$ there are continuous non-*WUA* functions. (In fact, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = xy$, is such a function.) Let us recall that *WUA* functions were introduced in [11] under the name "uniformly approachable functions" (see also [4]). They provided an easy and elegant solution of the problem of whether the uniform continuity can be characterized (in appropriate sense) by means of closure operators in the sense of [12] (since *WUA* functions are easily seen to be continuous with respect to every closure operator).

It is easy to see that if the set M is empty then $\langle K, M\rangle$-approximations always exist and the notion is uninteresting. (For $K \neq \emptyset$ any u.c. extension g of $f|K$ to a u.c. function, which exists by Katětov extension theorem, is a $\langle K, \emptyset\rangle$-approximation of f.) However, if M is properly chosen, then the condition $g[M] \subseteq f[M]$ is much stronger than it could be expected. In fact, it has been proved in [1, Theorem 8.5] that, under the continuum hypothesis CH, for every separable metric space X there exists a set $M \subset X$, called a *magic set*, such that any $\langle \emptyset, M\rangle$-approximation g of a nowhere constant function f must be a *truncation* of f, that is, g must be constant on each connected component of $\{x \in X\colon f(x) \neq g(x)\}$. This motivates the introduction of the class *TUA* of *truncation-UA* functions, that is, functions $f \in C(X)$ such that for every compact set $K \subseteq X$ there is a u.c. truncation g of f which coincides with f on K. Clearly $TUA \to C$ for every locally compact space X. The result quoted above shows that, under CH, $UA \to TUA$ for nowhere constant functions on every separable metric space X. (Take a $\langle K, M\rangle$-approximation of the constant function f with respect to a magic set M.) Since the *TUA* functions have a simpler geometrical description, this stimulated the further study of the magic sets and their properties and lead to a deep investigation of the question whether the existence of magic sets can be proved without the assumption of CH ([1, Question 14.1]). After some preliminary negative results (see [5,6]), Shelah and the first named author showed that this cannot be done even for the reals $\mathbb{R}$ [9].

In the comparison of *TUA* and *UA* in separable metric spaces (and in particular, in $\mathbb{R}^n$), Berarducci, Pelant and the second named author [2] noticed recently that uniform approachability provides also a good connection to properties of the functions related to fibers. A function $f : \mathbb{R}^n \to \mathbb{R}$ has *distant connected components* of fibers (briefly, *DCF*) if any two connected components of distinct fibers $f^{-1}(x)$ and $f^{-1}(y)$ are at positive distance. They proved [2, Corollary 6.20] that for the functions on $\mathbb{R}^n$ one has

$$UA \to WUA \to TUA \leftrightarrow DCF.$$

They also proved that $UA \leftrightarrow WUA \leftrightarrow TUA$ for all polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ and, more generally, for all functions with fibers having finitely many connected components. The following question was left open in [2, Question 8.2(1)]:

Question 1.2. Do the properties *UA*, *WUA*, and *TUA* coincide for *all* continuous functions $\mathbb{R}^n \to \mathbb{R}$?

Also, the strength of the condition $g[M] \subseteq f[M]$ suggested that the difference between *UA* and *WUA* is very small. In fact, the following open problem was raised in [1]:

Question 1.3. Let X be a connected metric space and let $f : X \to \mathbb{R}$ be a *WUA* function. Is then f also *UA*?

In this paper we will answer negatively these questions. More precisely, we give contributions mainly in three directions:

(1) We answer negatively Question 1.2 by constructing a function $f \in C(\mathbb{R}^2)$ which shows that, in $\mathbb{R}^n$ with $n \geqslant 2$, *TUA* does not imply even *WUA*.

This shows that *UA* and *WUA* are too strong conditions to participate in a set of equivalent conditions containing *TUA* and *DCF*. This motivated us to introduce here the following weaker version of *UA*: a function $f : X \to \mathbb{R}$ is UA_{d} (*densely uniformly approachable*) if it admits uniform $\langle K, M\rangle$-approximations for every *dense* set M and for every compact set K. One can define analogously WUA_{d}. Let us mention here, that all known examples of non-*UA* (respectively, non-*WUA*) spaces (constructed in [1–3]) are actually non-UA_{d} (respectively, non-WUA_{d}). As a corollary to Theorems 2.1 and 4.3 we see that *UA* does not coincide with UA_{d} for $f \in C(\mathbb{R}^n)$. In the last part (Section 4) we show that the example from Theorem 2.1 may serve also to distinguishing WUA_{d} from *WUA*. (This requires a much more careful choice of the set M witnessing non-*WUA*.)

(2) In a certain sense we improve the main result of [2] by showing that $TUA = DCF = WUA_{\mathrm{d}} = UA_{\mathrm{d}}$ for functions on $\mathbb{R}^n$. (See Theorem 4.3.) This is also the first general theorem on coincidence of (a form of) *UA* with (a form of) *WUA*. (See Question 1.3.)

(3) In Theorem 2.3 we answer negatively Question 1.3 by proving that the restriction to a connected subspace of the function $f : \mathbb{R}^2 \to \mathbb{R}$ constructed in Theorem 2.1 is both *WUA* and UA_{d}. The proof of Theorem 2.1 shows that this restriction is not *UA*, hence our example shows that even the implication $(WUA \;\&\; UA_{\mathrm{d}}) \Rightarrow UA$ may fail for continuous functions on a connected subspace of $\mathbb{R}^2$.

We leave open the last part of Question 1.2. (See also [2, Question 8.3].)

Problem 1.4. Does *WUA* imply *UA* in $C(\mathbb{R}^n)$? What about $C(\mathbb{R}^2)$?

In the diagram below we summarize, for reader's convenience, our results and the open question (invertibility of (1)):

$$\begin{tikzcd} WUA \arrow[r, "(2)"'] & WUA_{\mathrm{d}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & DCF \arrow[d, leftrightarrow] \\ UA \arrow[u, "(1)"'] \arrow[r, "(3)"'] & UA_{\mathrm{d}} \arrow[r, leftrightarrow] & TUA \end{tikzcd}$$

The equivalences in the right hand square are proved in Theorem 4.3. The implication (1) is trivial. The properness of the implication (2) is proved by the example given in Theorem 2.1. (For the proof see Section 5.) This proves also properness of the implication (3) established directly in Theorem 2.1.

1.1. Preliminaries on truncations and approximations

The interior, closure, boundary, and diameter of a set A in a metric space X are denoted by $\operatorname{int}(A)$, $\operatorname{cl}(A)$, $\operatorname{bd}(A)$, and $\operatorname{diam}(A)$, respectively. In what follows for $x, y \in \mathbb{R}^n$, $n = 1, 2, 3, \ldots,$ we will write $||x - y||$ for the Euclidean distance between x and y.

For $f \in C(X)$ and $a, b \in \mathbb{R}$ with $a < b$ define the (a, b)-*truncation* g of f by putting $g(x) = f(x)$ when $f(x) \in [a, b]$, $g(x) = b$ when $f(x) \geqslant b$, and $g(x) = a$ when $f(x) \leqslant a$. For $f, g \in C(X)$ we will write $[f = g]$ and $[f \neq g]$ for the sets $\{x \in X\colon f(x) = g(x)\}$ and $\{x \in X\colon f(x) \neq g(x)\}$, respectively.

We give here several easy properties of truncations that will be frequently used in the sequel.

Lemma 1.5. *Let X be a locally connected space and $f, g, h \in C(X)$.*

(a) *If g is a truncation of f and U is a connected component of $[f \neq g]$, then g is constant on $\operatorname{cl}(U)$ and $g = f$ on $\operatorname{bd}(U)$.*
(b) *If g is a truncation of f and $Y \subseteq X$, then also $g|Y$ is a truncation of $f|Y$.*
(c) *If f is constant and g is a truncation of f, then g is locally constant.*
(d) *If g is locally constant on $[f \neq g]$, then g is a truncation of f.*
(e) *If h is a truncation of g and g is a truncation of f, then h is a truncation of f.*

Proof. (a) is proved in [2, Lemma 5.3], while (b) and (d) are obvious.

(c) Let $x \in X$ and W be the connected component of x in X. Then W is open since X is locally connected. Thus it suffices to show that g is constant on W. If f and g agree on W then there is nothing to prove. So assume that $W \cap [f \neq g] \neq \emptyset$. Let $U \subset W$ be a connected component of $W \cap [f \neq g]$. By (b) $g|W$ is a truncation of $f|W$, hence g is already constant on W when $U = W$. Let us see now that the case $U \neq W$ cannot occur. Indeed, by (a), g is constant on $\operatorname{cl}(U)$ and $g = f$ on $\operatorname{bd}(U)$. Since W is connected and $U \neq W$, the set $\operatorname{bd}(U)$ is non-empty, so that these two constants coincide. Hence $g|U = f|U$, a contradiction.

(e) Let $x \in [h \neq f] \subseteq [h \neq g] \cup [g \neq f]$. If $x \in [h \neq g]$ then h is constant on some neighbourhood of x since h is a truncation of g. Suppose $x \in [g \neq f]$. Then there exists a connected neighbourhood U of x such that g is constant on U. By (b) $h|U$ is a truncation of $g|U$ so (c) yields that $h|U$ is constant. This proves that h is locally constant on $[h \neq f]$. Therefore, by (d), h is a truncation of f. $\square$

Lemma 1.6. *Let $g \in C(\mathbb{R}^n)$ be a truncation of $f \in C(\mathbb{R}^n)$. If $\delta > 0$ and $\varepsilon > 0$ are such that*

$$\textit{for every } x, y \in \mathbb{R}^n \textit{ if } ||x-y|| < \delta, \textit{ then } \big|f(x) - f(y)\big| < \varepsilon \tag{1}$$

then

$$\textit{for every } x, y \in \mathbb{R}^n \textit{ condition } ||x-y|| < \delta, \textit{ implies } \big|g(x) - g(y)\big| < \varepsilon. \tag{2}$$

In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ is u.c. then so is every its truncation.

Proof. Let $\delta > 0$ and $\varepsilon > 0$ be such that (1) holds and by way of contradiction assume that (2) fails. Then there are $x, y \in \mathbb{R}^n$ such that $||x-y|| < \delta$ while $|g(x) - g(y)| \geqslant \varepsilon$. Thus x and y cannot belong to the same component of $[f \neq g]$. Let I be a straight interval connecting x and y. Then there are $x', y' \in I$ such that $f(x') = g(x') = g(x)$ and $f(y') = g(y') = g(y)$. But this implies that $|f(x') - f(y')| = |g(x) - g(y)| \geqslant \varepsilon$ while $||x' - y'|| \leqslant ||x - y|| < \delta$, contradicting (1). $\square$

Lemma 1.7. *If $f : \mathbb{R}^n \to \mathbb{R}$ is TUA and $g : \mathbb{R}^n \to \mathbb{R}$ is a truncation of f then g is also TUA.*

Proof. Recall [2, Corollary 6.20] that $h \in C(\mathbb{R}^n)$ is *TUA* if and only if h has *DCF*. So, assume that f is *TUA*. Then f has *DCF*. It is enough to show that g has *DCF*. So, take different $y, z \in g[\mathbb{R}^n]$ and let U and V be connected components of $g^{-1}(y)$ and $g^{-1}(z)$, respectively. Since the boundary $\mathrm{bd}(U)$ separates V from the interior $\mathrm{int}(U)$ of U there is a connected component S of $\mathrm{bd}(U)$ which separates V from $\mathrm{int}(U)$. (This follows from the unicoherence of $\mathbb{R}^n$, cf. [2, Lemma 4.11].) Similarly, there is a component T of $\mathrm{bd}(V)$ which separates U from the interior $\mathrm{int}(V)$. Now, for every $x \in U$ and $y \in V$ there are $x' \in S$ and $y' \in T$ such that $||x - y|| \geqslant ||x' - y'||$. So, $\mathrm{dist}(U, V) = \mathrm{dist}(S, T)$. But $f|S \cup T = g|S \cup T$, and S and T are subsets of different connected components of fibers $f^{-1}(y)$ and $f^{-1}(z)$ of f. Thus $\mathrm{dist}(U, V) = \mathrm{dist}(S, T) > 0$, since f has *DCF*. $\square$

Remark 1.8. Note that $\mathbb{R}^n$ cannot be replaced by $\mathbb{R} \setminus \{0\}$ in either Lemma 1.6 or Lemma 1.7. Indeed, here the identity function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$ has truncations that are not *TUA*.

Lemma 1.9. *Let $h : X \to \mathbb{R}$ be a truncation of $f : X \to \mathbb{R}$ and let $\mathcal{V}$ be a family of some components of $[f \neq h]$. For every $V \in \mathcal{V}$ let $g_V : \mathrm{cl}(V) \to \mathbb{R}$ be some truncation of $f|\mathrm{cl}(V)$, and define $g : X \to \mathbb{R}$ by putting $g(x) = g_V(x)$ if $x \in \mathrm{cl}(V)$ for some $V \in \mathcal{V}$, and $g(x) = h(x)$ for all other $x \in X$. Then g is a truncation of f.*

Proof. First note that $[h = f] \subseteq [g = f]$, so that $[g \neq f] \subseteq [h \neq f]$. Let C be a connected component of $[g \neq f]$. Then there exists a connected component W of $[h \neq f]$ such that

$C \subset W$. If $W \notin \mathcal{V}$ then $g|W = h|W$ and $h|W$ is constant, so $g|C$ is constant. If, on the other hand, $W \in \mathcal{V}$ then $g|C = g_V|C$ is again constant. $\square$

$\langle K, M\rangle$-approximations are easy to build via Katětov's extension theorem when K is far from M:

Lemma 1.10 [1]. *Let X be a metric space, $M \subseteq X$, and K a compact subset of X such that* $\mathrm{cl}(M) \cap K = \emptyset$. *Then every $f \in C(X)$ admits a $\langle K, M\rangle$-approximation.*

This gives the following easy criterion for building $\langle x, M\rangle$-approximations.

Corollary 1.11. *Let X be a metric space, $f \in C(X)$, and $M \subseteq X$ such that $f[M]$ is closed (in particular, finite) in $\mathbb{R}$. Then there exists an $\langle x, M\rangle$-approximation of f for every point $x \in X$.*

Proof. Indeed, if $f(x) \in f[M]$ then it suffices to take the constant function with value $f(x)$ as an $\langle x, M\rangle$-approximation. If $f(x) \notin f[M] = \mathrm{cl}(f[M])$, then $x \notin \mathrm{cl}(M)$. Now Lemma 1.10 applies to give an $\langle x, M\rangle$-approximation of f. $\square$

2. A function that is *TUA* but not *UA*

Theorem 2.1. *There exists a TUA function $f : \mathbb{R}^2 \to \mathbb{R}$ that is not UA.*

Proof. Let $h : [0,1] \to [0,1]$ be the classical Cantor increasing function locally constant on an open and dense subset U of $(0,1)$. We assume also that $h[U] \subset (0,1)$.

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x,y) = h(x)$ for $x \in [0,1]$ and $g(x,y) = x$, otherwise. Function f is a modification of g obtained in the following way.

For every $n < \omega$ choose a finite set $S_n \subset U \times \{n\}$ such that $[0,1] \times \{n\}$ contains no interval of length 2^{-n} disjoint with S_n. For each $s = \langle t, n\rangle \in S_n$ choose a closed disk $D(s)$ centered at s on which the function g is constant. We will also assume that the disks are pairwise disjoint. It will be helpful to note also that all these disks are far from the boarder of the rectangle $K_n := [-n, n] \times [-\frac{1}{2}(2n+1), \frac{1}{2}(2n+1)]$.

The function f is obtained by modifying g on each disk $D(s)$, with s from $S = \bigcup_{n<\omega} S_n$, by putting $f(s) = 1$, $f(x) = g(x)$ for every boundary point x of $D(s)$, and extending it to the rest of $D(s)$ to get a cone. (In fact, any continuous extension would do.) The function f is as desired.

Indeed, first note that f is *TUA*. Every compact $K \subset \mathbb{R}^2$ is contained in some K_n, so it suffices to argue with $K = K_n$. Note that the set $[f \neq g]$ (union of disks $D(s)$) is far from the boarder of K_n. Now leaving f unchanged on K_n, and giving value $k(x) = g(x)$ for points x outside of K_n gives a u.c. truncation k of f that agrees with f on K. Indeed, k differs from g, which is *UC*, only on a compact set: the finite union of disks $D(s)$ contained in K_n.

To see that f is not *UA* let M be a union of lines $L_0 = \{0\} \times \mathbb{R}$, $L_1 = \{1\} \times \mathbb{R}$, and the set S of all centers s of disks $D(s)$. Thus $f[M] = \{0, 1\}$. Let $K = \{0, 1\} \times \{0\}$ and by way of contradiction assume that there is a *UC* function k agreeing with f on K and such that $k[M] \subset f[M] = \{0, 1\}$. Note that $k[\{\langle 0, 0\rangle\}] = k[L_0] = \{0\}$, since $L_0 \subset M$ and $k(0, 0) = f(0, 0) = 0$. Similarly $k[\{\langle 1, 0\rangle\}] = k[L_1] = \{1\}$. Now, since k is *UC* there exists an $n < \omega$ such that for every $x, y \in \mathbb{R}^2$ if $||x - y|| < 2^{-n}$ then $|k(x) - k(y)| < 1$. Let $\{s_0, \ldots, s_p\}$ be an increasing enumeration of $S_n \cup \{\langle 0, n\rangle, \langle 1, n\rangle\}$. Then $||s_i - s_{i+1}|| < 2^{-n}$ for every $i < p$. Thus $|k(s_i) - k(s_{i+1})| < 1$ for every $i < p$. But $k(s_i) \in f[M] = \{0, 1\}$ for every $i \leqslant p$. Thus, $k(s_i) = k(s_{i+1})$ for every $i < p$. However this is impossible, since $k(s_0) = k(0, n) = 0$ and $k(s_p) = k(1, n) = 1$. This finishes the proof. □

Remark 2.2. We will show in Section 5 that the above example is actually even non-*WUA*. But we prefer to give Theorem 2.1 in this form since the verification that f is not *UA* is much easier due to the relatively simple form of the set M, or more precisely, the fact that $f[M]$ is just a doubleton. According to Corollary 1.11 such a set cannot witness *WUA* for any singleton $K = \{x\}$. So, in Section 5 we will have the change the set M.

2.1. *WUA* & *TUA* & *UA*$_{\mathrm{d}}$ *does not imply UA on connected subspaces of the plane*

The proof of Theorem 2.1 shows that actually the restriction $f|A$ of the function f to the "ladder space" $A = L_0 \cup L_1 \cup ([0, 1] \times \mathbb{Z})$ is not *UA*, since both M and K are contained in A. Now we show that this restriction is also both *WUA* and *UA*$_{\mathrm{d}}$. Obviously it is also *TUA* since, f is *TUA*.

Theorem 2.3. *The restriction $f|A$ is both WUA and UA$_{\mathrm{d}}$.*

Proof. In the sequel we work only on the space A and accordingly we write simply f instead of $f|A$. We start by proving a property stronger than just *UA*$_{\mathrm{d}}$. Namely, we prove that for every $M \subseteq A$ such that $f[M]$ is dense in $[0, 1]$ one can build a $\langle K, M\rangle$-approximation for every compact $K \subseteq A$. It will suffice to find a $\langle K_n \cap A, M\rangle$-approximation for K_n as defined in the proof Theorem 2.1. Let $U = \bigcup_{t=1}^{\infty} U_t$, where each U_t is an open subinterval of $[0, 1]$ and U is the open set as in the proof of Theorem 2.1 on which the Cantor function $h : [0, 1] \to [0, 1]$ is locally constant. For every $k \in \mathbb{N}$ with $k > n$ truncate f on every set $V_t^{(k)} = U_t \times \{k\}$, $t \in \mathbb{N}$, at a level $f(m)$, where $m \in M$ is such that $h[U_t] \leqslant f(m) < h[U_t] + \frac{1}{k}$. For each $x = \langle x_1, x_2\rangle \in A$ with either $x_1 \notin U$ or $|x_2| \leqslant \frac{1}{2}(2n + 1)$ we leave $f(x)$ unchanged. The function obtained that way (which, by Lemma 1.9, is a truncation of f) will be denoted by f_1. Note that f_1 coincides with f on $K_n \cap A$. Moreover, for $B_k = [0, 1] \times \{k\}$ with $|k| > n$ the oscillation $\mathrm{osc}_{B_k}(f_1 - g)$ of $f_1 - g$ on the set B_k is at most $\frac{1}{n}$, where g is the function from the proof of Theorem 2.1. Thus, we have also $\mathrm{osc}_{A \setminus K_n}(f_1 - g) \leqslant \frac{1}{n}$.

Let us see that f_1 is u.c. Take an $\varepsilon > 0$ and choose a $k \in \mathbb{N}$ with $\frac{4}{k} < \varepsilon$. By the compactness of K_{k+1} there exists a $\delta \in (0, 1)$ such that $|f_1(x) - f_1(y)| < \frac{1}{2}\varepsilon$ for every

$x, y \in K_{k+1}$ with $||x - y|| < \delta$. Since g is u.c. we can assume, decreasing δ if necessary, that

$$\left|g(x) - g(y)\right| < \tfrac{1}{2}\varepsilon \quad \text{for all } x, y \in A \text{ with } ||x - y|| < \delta.$$

Now we show that this δ works for f_1 as well. Indeed, for $x, y \in A$ with $||x - y|| < \delta$ one has either $x, y \in K_{k+1}$ (and then $|f_1(x) - f_1(y)| < \frac{1}{2}\varepsilon$), or one of the points x and y, say x, does not belong to K_{k+1}. Then $\delta < 1$ implies that $x, y \notin K_k$. This yields $|f_1(x) - g(x)| \leqslant \frac{1}{k}$ and $|f_1(y) - g(y)| \leqslant \frac{1}{k}$. Hence $|f_1(x) - f_1(y)| \leqslant |g(x) - g(y)| + \frac{2}{k} \leqslant \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. This proves that $f|A$ is UA_{d}.

To see that $f|A$ is *WUA* fix $x \in A$ and $M \subseteq A$. We will find a u.c. $\langle x, M\rangle$-approximation of $f|A$. Since for each $k \in \mathbb{Z}$ one has $f[B_k] = [0, 1]$, it follows from the above argument that $f|A$ has an $\langle x, M\rangle$-approximation for every $x \in A$ and every $M \subseteq A$ that is dense in some B_k. Thus we will assume that for every $k \in \mathbb{Z}$ the set M avoids the closure of some open non-empty subinterval $\Delta_k = (a_k, b_k) \times \{k\}$ of B_k.

An $\langle x, M\rangle$-approximation is easy to build via Lemma 1.10 when $x \notin \mathrm{cl}(M)$. When $f(x) \in f[M]$ it is easy again: use the constant function with value $f(x)$. The last case shows that when $x \in L_0$ it makes sense to assume $0 = f(x) \notin f[M]$ and, consequently, that $M \cap L_0 = \emptyset$. Analogously, for $x \in L_1$ we will concentrate on the case when $M \cap L_1 = \emptyset$.

Here is the main trick that will allow us to build $\langle x, M\rangle$-approximation for all essential pairs $\langle x, M\rangle$. Suppose that x has an open bounded neighbourhood V_x such that the closure of $M' = M \setminus V_x$ is disjoint with $\mathrm{cl}(V_x)$. Since $\mathrm{cl}(V_x)$ is compact, the distance between M' and V_x is positive. Therefore, Katětov's extension theorem applies to the u.c. function ρ from $Y = \mathrm{cl}(V_x) \cup \mathrm{cl}(M')$ to $[0, 1]$, where $\rho|\mathrm{cl}(V_x) = f|\mathrm{cl}(V_x)$ and $\rho|\mathrm{cl}(M')$ is any constant with value in $f[M]$. The uniform continuity of ρ is granted by the uniform continuity of both restrictions and the positive distance between M' and V_x. Since $M \subseteq Y$ and $\rho[M] \subseteq f[M]$, any u.c. extension $\overline{\rho}$ of ρ will be an $\langle x, M\rangle$-approximation of f. In the sequel we aim to find such an open neighbourhood V_x of x. The argument splits into the following cases.

(a) If $x \in L_0$ then $x \in \mathrm{cl}(M)$ and $M \cap L_0 = \emptyset$ imply $x = \langle 0, n\rangle$ for some $n \in \mathbb{Z}$. Now disjointness of M with Δ_n permits to take as V_x the open T-shaped set $(\{0\} \times (n - \frac{1}{2}, n + \frac{1}{2})) \cup [0, a_n) \times \{n\}$. Analogous argument works for $x \in L_1$.

(b) So assume that $x = \langle x_1, n\rangle \in B_n \setminus (L_0 \cup L_1)$ for some n. Since $M \cap \Delta_n = \emptyset$ and $x \in \mathrm{cl}(M)$ we have $x \notin \Delta_n$ and $x_1 \in (0, 1) \setminus (a_n, b_n)$. We assume that $b_n \leqslant x_1 < 1$, the case $0 < x_1 \leqslant a_n$ being analogous. We have two cases.

  (b1) M is not dense in $[x_1, 1] \times \{n\}$. Then there is an open subinterval (c_n, d_n) of $[x_1, 1]$ such that $((c_n, d_n) \times \{n\}) \cap M = \emptyset$. Now take $V_x = (b_n, c_n) \times \{n\}$ and use the trick described above.

  (b2) M is dense in $[x_1, 1] \times \{n\}$. Here we have again two cases.

   (i) $1 \notin f[M]$. This means of course that M does not meet L_1. Now take $V_x = (b_n, 1) \times \{n\}$ and repeat the trick.

   (ii) $1 \in f[M]$. Consider the function $\rho : A \to \mathbb{R}$ that coincides with f on $[b_n, 1] \times \{n\}$, takes value 1 on the complement of $[a_n, 1] \times \{n\}$ in A and ρ

is linear on Δ_n. Clearly ρ is u.c. and is the desired $\langle x, M\rangle$-approximation of f. $\square$

Remark 2.4. The above function $f \in C(A)$ is *TUA*, *WUA*, UA_{d} but not *UA*. This should be compared with the function $g \in C(A)$ constructed in [1] that is non-*WUA*. Actually, that function has countable fibers and has no uniformly continuous non-constant truncations, so that its non-*WUA*-ness was established in [1] by the existence of a magic set M_g of g that forces all $\langle x, M_g\rangle$-approximations of g to be truncations of g. Obviously such a function g is *DCF*.[2] Since M_g must be dense, this proves actually that g is not even WUA_{d}. This should be compared with Theorem 4.3 where we prove that $DCF = WUA_{\mathrm{d}}$ for $C(\mathbb{R}^n)$.

3. *TUA* implies UA_{d}

Theorem 3.1. *TUA implies UA_{d} in $C(\mathbb{R}^k)$.*

Sketch of the proof. Let $f:\mathbb{R}^k \to \mathbb{R}$ be *TUA*, $K \subset \mathbb{R}^k$ be compact, and M be a dense subset of $\mathbb{R}^k$. We will construct a u.c. function $h:\mathbb{R}^k \to \mathbb{R}$ such that $h|K = f|K$ and

$$h[M] \subseteq f[M]. \tag{3}$$

It seems natural to take a u.c. truncation h_0 of f that agrees with f on K. But then (3) need not be satisfied. The main difficulty to overcome is to ensure the inclusion (3). Our plan is to define a sequence $\langle h_n\colon n < \omega\rangle$ of u.c. functions from $\mathbb{R}^k$ into $\mathbb{R}$ that modify h_0 and approximate f by means of a sequence $\langle g_n\colon n < \omega\rangle$ of truncations of f (hence, by Lemma 1.7, of *TUA* functions) starting with $g_0 = f$ and such that each g_n satisfies $g_n[M] \subseteq f[M]$, and agrees with h_{n-1} on K_{n-1}, where K_{n-1} is the closed ball with radius n and center 0. With this assumption the common limit h of the sequences g_n and h_n is u.c., agrees with f on K, and satisfies (3).

Detailed description of the construction. We construct the sequences $\langle g_n\colon n < \omega\rangle$ and $\langle h_n\colon n < \omega\rangle$ by induction on $n < \omega$. It makes no harm to think that our original compact set K is contained in K_0. (Otherwise the induction should start from some $n_0 < \omega$.)

To carry out the (much easier) *h-part of the construction* note that if g_n is a *TUA* function, imitating the first step with h_0, we can more generally define at each step $n < \omega$ a function h_n such that:

$$h_n \text{ is a u.c. truncation of } g_n \text{ and } h_n|K_n = g_n|K_n. \tag{4}$$

The existence of such a truncation h_n is an immediate consequence of the definition of *TUA*. The following fact will be used in the sequel.

Lemma 3.2. *If V is a component of $[g_n \neq f]$ intersecting K_n then it is also a component of $[h_n \neq f]$ and $g_n|V = h_n|V$.*

[2] Every *light* function (i.e., with totally disconnected fibers) is *DCF*.

Proof. Let $x \in V \cap K_n$. Then $f(x) \neq g_n(x) = h_n(x)$, thus $x \in [h_n \neq f]$. Clearly $g_n[V] = \{g_n(x)\}$. Thus $h_n|V$ is a truncation of the constant function $g_n|V$ (Lemma 1.5(b)), so it is constant (Lemma 1.5(c)), since V is connected and locally connected. Therefore $g_n[V] = \{g_n(x)\} = \{h_n(x)\} = h_n[V]$ and so the functions h_n, g_n, and f agree on the boundary of V. Thus, V is a component of $[h_n \neq f]$. □

Next we describe the more complicated *g-part of our construction*. We shall build a function g_{n+1} with properties G_n(i)–G_n(iv) given below under the assumption that for some $n < \omega$ the functions h_n and g_n are already constructed with the properties G_{n-1}(i)–G_{n-1}(iv).

G_n(i): $g_{n+1}|K_n = g_n|K_n$. (Hence $g_{n+1}|K_n = g_n|K_n = h_n|K_n$.)

G_n(ii): g_{n+1} is a truncation of f (so a *TUA* function) such that $|g_{n+1}(x) - h_n(x)| < 2^{-n}$ for every $x \in \mathbb{R}^k$.

G_n(iii): If V is a component of $[g_n \neq f]$ intersecting K_n then it is also a component of $[g_{n+1} \neq f]$ and $g_n|V = g_{n+1}|V$.

G_n(iv): $g_{n+1}[M] \subseteq f[M]$.

Definition of the truncation g_{n+1} of f. Define $g_{n+1}(x) = h_n(x)$ on $[h_n = f]$. To extend g_{n+1} on $[h_n \neq f]$ note that since h_n is a truncation of g_n and g_n is a truncation of f we can conclude by Lemma 1.5(e) that h_n is a truncation of f. For every component U of $[h_n \neq f]$ we define the restriction g_U of g_{n+1} on $\mathrm{cl}(U)$ as follows.

We do not change h_n on $\mathrm{cl}(U)$, i.e., we leave $g_U = h_n|\mathrm{cl}(U)$ if U intersects K_n. Otherwise, setting $\{z\} = h_n[U]$, we choose $a \in (z - 2^{-n}, z] \cap f[M]$ and $b \in [z, z + 2^{-n}) \cap f[M]$. Such a and b exist by the density of $f[M]$ in $f[\mathbb{R}^k]$, which follows from the density of M in $\mathbb{R}^k$. In this case let $g_U : \mathrm{cl}(U) \to [a, b]$ be the (a, b)-truncation of $f|\mathrm{cl}(U)$.

Lemma 3.3. *g_{n+1} satisfies* $\mathrm{G}_n(\mathrm{i})-\mathrm{G}_n(\mathrm{iv})$.

Proof. G_n(i): Take an $x \in K_n$ and note that $h_n(x) = g_n(x)$. If $h_n(x) = f(x)$ then $g_{n+1}(x) = h_n(x) = g_n(x)$. On the other hand if $h_n(x) \neq f(x)$ then take the connected component U of $[h_n \neq f]$ containing x and notice that $x \in K_n \cap U$. So $g_{n+1}(x) = g_U(x) = h_n(x) = g_n(x)$. This proves G_n(i).

G_n(ii): g_{n+1} is a truncation of f by Lemma 1.9 and consequently g_{n+1} is *TUA* by Lemma 1.7, since f is *TUA*. The rest of the condition G_n(ii) is clear from the definition.

G_n(iii): Let V be a connected component of $[g_n \neq f]$ intersecting K_n. Then, by Lemma 3.2, it is also a connected component of $[h_n \neq f]$. So $V \cap K_n \neq \emptyset$ implies $g_{n+1}|\mathrm{cl}(V) = g_V = h_n|\mathrm{cl}(V)$. In particular $g_{n+1}|\mathrm{bd}(V) = h_n|\mathrm{bd}(V)$ and we can also conclude that $g_{n+1}|\mathrm{bd}(V) = f|\mathrm{bd}(V)$. This proves that V is also a component of $[g_{n+1} \neq f]$ and $g_n|V = g_{n+1}|V$.

G_n(iv): Let $m \in M$. If $m \in [g_{n+1} = f]$ then obviously $g_{n+1}(m) \in f[M]$. Therefore assume that $m \in [g_{n+1} \neq f]$ and let V be the component of $[g_{n+1} \neq f]$ containing m. Since by the definition of g_{n+1} we have $[g_{n+1} \neq f] \subseteq [h_n \neq f]$, it is clear that such a connected component V must be contained in a connected component U of $[h_n \neq f]$.

If U intersects K_n then g_{n+1} coincides with h_n on U. Therefore $U = V$. Choose an $x \in K_n \cap U$. Then $g_n(x) = h_n(x) \neq f(x)$, so x belongs to a connected component W of $[g_n \neq f]$. By Lemma 3.2 W is also a connected component of $[h_n \neq f]$, hence $W = V$. As $V = W = U$ turned out to be a connected component of $[g_n \neq f]$ that intersects K_n, condition G_n(iii) and the inductive hypothesis G_{n-1}(iv) imply that $g_{n+1}(m) = g_n(m) \in f[M]$. Hence G_n(iv) is satisfied in this case.

Now assume that U does not intersect K_n. Then let a and b be as described above in the definition of g_{n+1}. We have now necessarily $g_{n+1}(m) \in f[M]$ as g_{n+1} is an (a, b)-truncation of f and $g_{n+1}(m)$, being distinct from $f(m)$, must coincide with a or b. □

This finishes the inductive construction.

Now, by conditions G_n(i), for every $x \in \mathbb{R}^k$ the sequence $\langle g_n(x)\colon n < \omega\rangle$ is eventually constant. Let $g(x) = \lim_{n\to\infty} g_n(x)$. Note that

$$g\big|K_n = g_n\big|K_n. \tag{5}$$

Lemma 3.4. *g is u.c. and $g[M] \subseteq f[M]$.*

Proof. Inclusion $g[M] \subseteq f[M]$ follows directly from condition G_n(iv) and the definition of g. We will next show that g is u.c.

So, fix an $\varepsilon > 0$. We will find a $\delta > 0$ such that

$$\text{if } ||x - y|| < \delta \text{ then } \big|g(x) - g(y)\big| < \varepsilon.$$

For this first find an $n < \omega$ such that $\sum_{m=n}^{\infty} 2^{-m} < \frac{1}{3}\varepsilon$. Since h_n is u.c. we can find $\delta > 0$ such that

$$\text{if } ||x - y|| < \delta \text{ then } \big|h_n(x) - h_n(y)\big| < \frac{\varepsilon}{3}.$$

So, by G_n(ii), if $||x - y|| < \delta$ then

$$\big|g_{n+1}(x) - g_{n+1}(y)\big| \leqslant \big|h_n(x) - h_n(y)\big| + 2 \cdot 2^{-n} < \frac{\varepsilon}{3} + 2 \cdot 2^{-n}$$

and, by Lemma 1.6, since h_{n+1} is a truncation of g_{n+1},

$$\text{if } ||x - y|| < \delta \text{ then } \big|h_{n+1}(x) - h_{n+1}(y)\big| < \frac{\varepsilon}{3} + 2 \cdot 2^{-n}.$$

Continuing by induction we show that for every $0 < k < \omega$ if $||x - y|| < \delta$ then

$$\big|h_{n+k}(x) - h_{n+k}(y)\big| < \frac{\varepsilon}{3} + 2 \cdot \sum_{i=0}^{k-1} 2^{-(n+i)} < \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon.$$

Since for every $x, y \in \mathbb{R}^k$ there is an $m > n$ such that $g(x) = h_m(x)$ and $g(y) = h_m(y)$, the above condition implies that for this fixed δ

$$\text{if } ||x - y|| < \delta \text{ then } \big|g(x) - g(y)\big| < \varepsilon.$$

Thus g is u.c. □

This finishes the proof of Theorem 3.1 since g is a $\langle K, M\rangle$-approximation of f (as g coincides with f on K by (5)).

Remark 3.5.

(a) g is a truncation of f. For this let U be a connected open subset of $[f \neq g]$. We will see that g is locally constant of U. Indeed, the connectedness of U yields that g is constant on U. Also, by (5) we have $[f \neq g] \subseteq \bigcup_n (K_n \cap [f \neq g_n])$. So every $x \in U$ has an open neighbourhood V with compact closure such that $V \subseteq K_n$ for some n. Then $V \subseteq K_n \cap [f \neq g_n]$. As g_n is a truncation of f it follows that g_n is constant on V. By (5) again this means that g is constant on V too.

(b) The above proof uses the density of $f[M]$ in $f[\mathbb{R}^k]$ rather than the density of M in $\mathbb{R}^k$.

(c) The implication *TUA* $\Rightarrow$ *UA*$_\mathrm{d}$ is not always true, so that the choice of $\mathbb{R}^n$ plays an important role. An example of a metric space X and a continuous *TUA* function $f: X \to \mathbb{R}$ that is not even *WUA*$_\mathrm{d}$ is given in [3]. (Actually, X is the Hedgehog space with $\mathbf{b}$ many spikes; the function f admits a magic set M_f with $f(0) \notin f[M_f]$ and has no uniformly continuous truncations g with $g(0) = f(0)$; therefore f has no $\langle x, M_f\rangle$-approximation and so f is non-*WUA*. One can easily check that such a magic set must be necessarily dense, hence we get automatically $f \notin$ *WUA*$_\mathrm{d}$.)

(d) The proof of Theorem 2.3 shows that it is possible to replace $\mathbb{R}^n$ by other nice spaces – for example the ladder space A from Section 2.1. For a proof in a more general setting one needs more general forms of Lemmas 1.6–1.9. While Lemma 1.9 works in a general situation, we are not aware if this is possible with Lemmas 1.6 and 1.7. (See Remark 1.8.)

4. *UA*$_\mathrm{d}$ $\leftrightarrow$ *WUA*$_\mathrm{d}$ $\leftrightarrow$ *TUA* $\leftrightarrow$ *DCF* in $\mathbb{R}^n$

We already know that *DCF* $\leftrightarrow$ *TUA* $\to$ *UA*$_\mathrm{d}$ $\to$ *WUA*$_\mathrm{d}$ in $\mathbb{R}^n$: the equivalence *DCF* $\leftrightarrow$ *TUA* was proved in [2, Corollary 6.20], the implication *TUA* $\to$ *UA*$_\mathrm{d}$ is a restatement of Theorem 3.1, and *UA*$_\mathrm{d}$ $\to$ *WUA*$_\mathrm{d}$ follows immediately from the definition. Thus it is enough to prove that *WUA*$_\mathrm{d}$ $\to$ *DCF* in $\mathbb{R}^n$. The argument is essentially the same as for [2, Corollary 6.10] that *WUA* $\to$ *DCF* in $\mathbb{R}^n$. In particular the proof of the next theorem is similar to that of [2, Theorem 6.8] – we only need to show that by taking additional care the set M witnessing non-*WUA* can be chosen to be dense, in order to witness also non-*WUA*$_\mathrm{d}$.

Theorem 4.1. *Let X be a separable metric space and suppose that there is an uncountable set $Y \subseteq \mathbb{R}$ and for each $y \in Y$ a connected component C^y of $f^{-1}(y)$ such that for some $z \in Y$*

the distance between C^y and C^z is equal to 0 for every $y \in Y$.

Then f is not WUA$_\mathrm{d}$.

Proof. Let $N = \bigcup_{y \in Y} C^y$. Since $N \subseteq X$ and X is a separable metric space, N is separable. Let Y_0 be the set of all $y \in Y$ for which either C^y has a non-empty interior in N or $f^{-1}(y)$ has a non-empty interior in X. Since Y_0 is at most countable, we can pick $u \in Y \setminus Y_0$, $u \neq z$. Then the set $N \setminus C^u$ is not closed in N and therefore there is a countable subset $\{y_n\colon n < \omega\}$ of $Y \setminus \{u\}$ and for each $n < \omega$ a point $x_n \in C^{y_n}$ such that the sequence $\langle x_n \rangle$ converges to an $x \in C^u$. Note that $f(x) \notin f[M_0]$, where $M_0 = \bigcup_n C^{y_n} \subseteq N \setminus C^u$. Since, by the choice of u, the complement of the set $f^{-1}(u)$ is dense in X we can choose a dense countable subset M_1 of X that does not meet $f^{-1}(u)$. Hence $f(x) \notin f[M_1]$. Therefore the set $M = M_0 \cup M_1 \cup C^z$ is dense in X and $f(x) \notin f[M]$. We show now that f is not *WUA*$_\mathrm{d}$. Suppose for a contradiction that there is an $\langle x, M \rangle$-approximation $g \in C(X)$ of f. Then $g[M] \subseteq f[M]$ is countable, hence totally disconnected. So g restricted to each of the connected set C^{y_n} must be constant. In particular g is constant on C^z. Since g is u.c. and the distance between each C^{y_n} and C^z is equal 0, g must be constant on the entire M_0, and so also on its closure $\mathrm{cl}(M_0)$. Since $x \in \mathrm{cl}(M_0)$, g has the constant value $g(x) = f(x)$ on $\mathrm{cl}(M_0)$. This however contradicts the inclusion $g[M] \subseteq f[M]$ since $f(x)$ does not belong to the latter set. $\square$

Corollary 4.2. *If a function $f \in C(\mathbb{R}^n)$ is WUA$_\mathrm{d}$, then it is DCF.*

Proof. It was proved in [2, Theorem 6.9] that if $f \in C(\mathbb{R}^n)$ has two connected components A, B of distinct fibers at distance zero, then it has a family, of cardinality of the continuum, of connected components of distinct fibers such that each member of the family has distance zero from both A and B. Combined with Theorem 4.1, this shows that a function with two connected components of distinct fibers at distance zero is not *WUA*$_\mathrm{d}$. $\square$

Corollary 4.2 and the above discussion imply immediately the following theorem.

Theorem 4.3. *UA$_\mathrm{d}$ $\leftrightarrow$ WUA$_\mathrm{d}$ $\leftrightarrow$ TUA $\leftrightarrow$ DCF in $\mathbb{R}^n$.*

The next corollary is valid also for the larger class of semialgebraic functions, but we give it here for polynomial ones. It follows immediately from Theorem 4.3 and [2, Lemma 6.21].

Corollary 4.4. *UA $\leftrightarrow$ WUA $\leftrightarrow$ UA$_\mathrm{d}$ $\leftrightarrow$ WUA$_\mathrm{d}$ $\leftrightarrow$ TUA $\leftrightarrow$ DCF $\leftrightarrow$ DF*[3] *for polynomial functions $f:\mathbb{R}^n \to \mathbb{R}$.*

This corollary shows in particular that for polynomial functions $f:\mathbb{R}^n \to \mathbb{R}$ *UA* coincides with *UA*$_\mathrm{d}$ and *WUA* coincides with *WUA*$_\mathrm{d}$. The example from Theorem 2.1 along with Theorem 3.1 shows that *UA* does not coincide with *UA*$_\mathrm{d}$ in $C(\mathbb{R}^n)$. Therefore the next objective will be to clarify whether *WUA* coincides with *WUA*$_\mathrm{d}$. According to the above corollary, it suffices to check the implication *TUA* $\to$ *WUA*.

[3] f is *DF* (*has distant fibers*) if any distinct fibers $f^{-1}(x)$ and $f^{-1}(y)$ are of positive distance.

5. *TUA* does not imply *WUA*

We will show that the function f from Theorem 2.1 is not even *WUA*. This will be shown with K being the singleton point $x = \langle 0,0\rangle$ and a set M constructed below. We will use here the same notation as in the theorem.

Consider the intervals $I_n = (-2^{-n}, -2^{-n-1})$, let $J = \bigcup_{n<\omega} I_{2n+1}$, and put $P = (J \setminus \mathbb{Q}) \times \mathbb{R}$. In what follows we will find an $M^0 \subset P$ such that for every continuous function h with $h(0,0) = 0$

$$\text{if } h\big[M^0\big] \subset f\big[M^0\big] \cup \{1\} \text{ then } h \text{ is constant on } L_0. \tag{$\star$}$$

Let us see first how the proof will proceed once such an M^0 is found. For this we define $M = M^0 \cup S \cup L_1$ and by way of contradiction assume that there exists a u.c. function $k:\mathbb{R}^2 \to \mathbb{R}$ such that $k(0,0) = 0$ and $k[M] \subset f[M]$. Note that $f[M] \subset (J \setminus \mathbb{Q}) \cup \{1\}$, so $f[M]$ is totally disconnected. Thus $k[L_1] = \{c\} \subset f[M]$ for some $c \neq 0$ and, by ($\star$), $k[L_0] = \{0\}$. By the definition of P, between 0 and c there exists a nonempty open interval I (one of of the intervals I_{2n}, if $c < 0$, and $(0,1)$ if $c = 1$) which is disjoint with $f[M]$. Let $\varepsilon > 0$ be the length of I. By the uniform continuity of k there exists a $\delta > 0$ such that for every $x, y \in \mathbb{R}^2$ if $||x-y|| < \delta$ then $|k(x) - k(y)| < \varepsilon$. Choose an $n < \omega$ such that $2^{-n} < \delta$ and let $\{s_0, \ldots, s_p\}$ be an increasing enumeration of $S_n \cup \{\langle 0,n\rangle, \langle 1,n\rangle\}$. Then $||s_i - s_{i+1}|| < 2^{-n} < \delta$ for every $i < p$. Thus $|k(s_i) - k(s_{i+1})| < \varepsilon$ for every $i < p$. In particular $k(s_i)$ and $k(s_{i+1})$ stay on the same side of I for every $i < p$. But this implies that all $k(s_i)$, with $i \leqslant p$, stay of the on the same side of I. However this contradicts the fact that $k(s_0) = 0$ and $k(s_p) = c$ are on the opposite sides of I. This contradiction shows that f is not *WUA*.

In order to construct M^0 satisfying ($\star$) let $\langle h_\xi\colon \xi < \mathbf{c}\rangle$ be an enumeration of all continuous functions $h:\mathbb{R}^2 \to \mathbb{R}$ such that $h(0,0) = 0$ and $h[L_0] \neq \{0\}$. We will construct by induction on $\xi < \mathbf{c}$ a sequence $\langle m_\xi\colon \xi < \mathbf{c}\rangle$ of elements of P aiming for $M^0 = \{m_\xi\colon \xi < \mathbf{c}\}$. At stage ξ we assume that all m_γ with $\gamma < \xi$ are chosen and let $M_\xi = \{m_\gamma\colon \gamma < \xi\}$. We will add to M^0 a point $m_\xi \in P$ aiming for

$$h_\xi(m_\xi) \not\subset f\big[M^0\big] \cup \{1\}. \tag{7}$$

Clearly (7) will imply ($\star$). To have (7) it is enough to choose an $m_\xi \in P$ such that

(a) $h_\xi(m_\xi) \neq f(m_\xi)$,
(b) $h_\xi(m_\xi) \notin f[M_\xi]$, and
(c) $f(m_\xi) \neq h_\gamma(m_\gamma)$ for all $\gamma < \xi$.

For this note that since h_ξ is not identically 0 on L_0, there exists a point $p = \langle 0,y\rangle \in L_0$ such that $h_\xi(p) \neq 0 = h_\xi(0,0)$. So, there is an $a < 0$ such that

$$h_\xi\big[(a,0) \times \{0\}\big] \cap h_\xi\big[(a,0) \times \{y\}\big] = \emptyset. \tag{8}$$

Since f restricted to $(a,0) \times \{0\}$ is one-to-one we can find an $x \in (a,0) \cap (J \setminus \mathbb{Q})$ for which $f(x,0) \neq h_\gamma(m_\gamma)$ for all $\gamma < \xi$. Since we will choose m_ξ as $\langle x,z\rangle$ for some z, this guarantees satisfaction of (c). Now, let I^0 be an interval (in $\mathbb{R}$) with endpoints 0 and y and let $I^1 = \{x\} \times I^0$. Note that, by (8), h_ξ has different values on the endpoints of I^1. Thus

$h_\xi[I^1]$ has cardinality continuum. Therefore it is easy to choose $m_\xi \in I^1$ for which (b) holds and $h_\xi(m_\xi) \neq f(x,0) = f(m_\xi)$. This finishes the construction and the proof of the theorem.

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