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Two examples concerning almost continuous functions

Krzysztof Ciesielski [a,*], Andrzej Rosłanowski [b,c,1]

[a] *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA*
[b] *Department of Mathematics and Computer Science, 1910 University Dr, Boise State University, Boise, ID 83725, USA*
[c] *Mathematical Institute of Wroclaw University, 50384 Wroclaw, Poland*

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Abstract

In this note we will construct, under the assumption that union of less than continuum many meager subsets of $\mathbb{R}$ is meager in $\mathbb{R}$, an additive connectivity function $f:\mathbb{R}\to\mathbb{R}$ with Cantor intermediate value property which is not almost continuous. This gives a partial answer to a question of Banaszewski (1997). (See also Question 5.5 of Gibson and Natkaniec (1996–97).) We will also show that every extendable function $g:\mathbb{R}\to\mathbb{R}$ with a dense graph satisfies the following stronger version of the SCIVP property: for every $a<b$ and every perfect set K between $g(a)$ and $g(b)$ there is a perfect set $C\subset(a,b)$ such that $g[C]\subset K$ and $g\restriction C$ is continuous *strictly increasing*. This property is used to construct a ZFC example of an additive almost continuous function $f:\mathbb{R}\to\mathbb{R}$ which has the strong Cantor intermediate value property but is not extendable. This answers a question of Rosen (1997–98). This also generalizes Rosen's result (1997–98) that a similar (but not additive) function exists under the assumption of the Continuum Hypothesis, and gives a full answer to Question 3.11 of Gibson and Natkaniec (1996–1997). © 2000 Elsevier Science B.V. All rights reserved.

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1. Preliminaries

Our terminology is standard and follows [7]. We consider only real-valued functions of one or two real variables. No distinction is made between a function and its graph. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the set of all real and rational numbers, respectively. We will consider

* Corresponding author. Email: KCies@wvnvms.wvnet.edu. The first author was partially supported by NSF Cooperative Research Grant INT-9600548 with its Polish part financed by KBN.
1 Email: roslanow@math.idbsu.edu.

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$\mathbb{R}$ and $\mathbb{R}^2$ as linear spaces over $\mathbb{Q}$. In particular, for a subset X of either $\mathbb{R}$ or $\mathbb{R}^2$ we will use the symbol $\mathrm{LIN}_{\mathbb{Q}}(X)$ to denote the smallest linear subspace (of $\mathbb{R}$ or $\mathbb{R}^2$) over $\mathbb{Q}$ that contains X. Recall also that if $D \subset \mathbb{R}$ is linearly independent over $\mathbb{Q}$ and $f : D \to \mathbb{R}$ then $F = \mathrm{LIN}_{\mathbb{Q}}(f) \subset \mathbb{R}^2$ is an additive function (see definition below) from $\mathrm{LIN}_{\mathbb{Q}}(D)$ into $\mathbb{R}$. Any linear basis of $\mathbb{R}$ over $\mathbb{Q}$ will be referred as a *Hamel basis*. By a Cantor set we mean any nonempty perfect nowhere dense subset of $\mathbb{R}$.

The ordinal numbers will be identified with the sets of all their predecessors and cardinals with the initial ordinals. In particular $2 = \{0, 1\}$, and the first infinite ordinal ω number is equal to the set of all natural numbers $\{0, 1, 2, \ldots\}$. The family of all functions from a set X into Y is denoted by Y^X. The symbol $|X|$ stands for the cardinality of a set X. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$ and referred as *continuum*. A set $S \subset \mathbb{R}$ is said to be $\mathfrak{c}$-*dense* if $|S \cap (a, b)| = \mathfrak{c}$ for every $a < b$. The closure of a set $A \subseteq \mathbb{R}$ is denoted by $\mathrm{cl}(A)$, its boundary by $\mathrm{bd}(A)$, and its diameter by $\mathrm{diam}(A)$. For a set $A \subseteq X \times Y$ and points $x \in X$ and $y \in Y$ we let $(A)_x = \{y \in Y\colon \langle x, y\rangle \in A\}$ and $(A)^y = \{x \in X\colon \langle x, y\rangle \in A\}$. In a similar manner we define $(A)_{\langle x,y\rangle}$ and $(A)^z$ for a set $A \subseteq X \times Y \times Z$.

We will use also the following terminology [12]. A function $f : \mathbb{R} \to \mathbb{R}$

- is *additive* if $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$;
- is *almost continuous* (in sense of Stallings) if each open subset of $\mathbb{R} \times \mathbb{R}$ containing the graph of f contains also a continuous function from $\mathbb{R}$ to $\mathbb{R}$ [26];
- has the *Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$;
- has the *strong Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$ and the restriction $f \restriction C$ of f to C is continuous;
- is an *extendability* function if there is a connectivity function $F : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in \mathbb{R}$, where
- for a topological space X a function $f : X \to \mathbb{R}$ is a *connectivity* function if the graph of the restriction $f \restriction Z$ of f to Z is connected in $Z \times \mathbb{R}$ for any connected subset Z of X.

The above classes of functions (from $\mathbb{R}$ to $\mathbb{R}$) will be denoted by Add, AC, CIVP, SCIVP, Ext, and Conn, respectively.

Recall that if the graph of $f : \mathbb{R} \to \mathbb{R}$ intersects every closed subset B of $\mathbb{R}^2$ which projection $\mathrm{proj}(B)$ onto the x-axis has nonempty interior then f is almost continuous. (See, e.g., [21].) Similarly, if the graph of $f : \mathbb{R} \to \mathbb{R}$ intersects every compact connected subset K of $\mathbb{R}^2$ with $|\mathrm{proj}(K)| > 1$ then f is connectivity.

We will finish this section with the following well-known fact. (See [5, Theorem 4.A.12], [19, & 47III], or [20, Ch. V, Section 2].)

Proposition 1.1 (Boundary bumping theorem). *If U is a nonempty open proper subset of a compact connected Hausdorff space K and C is a connected component of U then $\mathrm{cl}_K(C) \cap \mathrm{bd}_K(U) \neq \emptyset$. In particular every connected component of U has more than one point.*

2. Additive connectivity function on $\mathbb{R}$ which is not almost continuous

We start this section with recalling the following construction of Roberts [22] of zero-dimensional closed subset Z_0 of $[0,1]^2$ which is intersected by a graph of every continuous function $f:[0,1]\to[0,1]$. Let $C\subset[0,1]$ be a Cantor set of Lebesgue measure 1/2. (Roberts defines it as $C=\bigcap_{n<\omega}C_n$, where $C_0=[0,1]$, each C_n is the union of 2^n disjoint intervals, and C_{n+1} is obtained from C_n by taking out of each of these 2^n intervals a concentric open interval of length $1/2^{2n+2}$.) Define $x,y:[0,1]\to[0,1]$ by $x(t)=2m(C\cap[0,t])$, where m is a Lebesgue measure, and $y(t)=4m(C\cap[0,t])-t=2x(t)-t$. Then $F_0:[0,1]\to[0,1]^2$, $F_0(t)=\langle x(t),y(t)\rangle$, is a continuous embedding, so $M_0=F_0[[0,1]]$ is an arc joining $\langle 0,0\rangle$ with $\langle 1,1\rangle$. Note that each component interval I of $[0,1]\setminus C$ is mapped by F_0 onto an open vertical segment $F_0[I]$. The set Z_0 defined as $F_0[C]$. It is equal to the arc M_0 from which all vertical segments $F_0(I)$ are removed. Note also that an arc $F_0[I]$ has been removed from the section $(M_0)_x$ if and only if $x\in D_0$, where D_0 is the set of all dyadic numbers ($x=k/2^n$) from $(0,1)$. Moreover, $|(Z_0)_x|=2$ for $x\in D_0$ and $(Z_0)_x=(M_0)_x$ is a singleton for all other x from $[0,1]$.

For what follows we will need the following version of this construction, where $\overline{C}=\mathbb{Z}+C$.

Lemma 2.1. *Let X be a countable dense subset of $(-1,1)$. Then there exists an embedding $F=\langle F_0,F_1\rangle:\mathbb{R}\to(-1,1)\times\mathbb{R}$ such that F_0 is non-decreasing,*

(a) *an open arc $M=F[\mathbb{R}]$ is closed in $\mathbb{R}^2$,*
(b) *if $Z=F[\overline{C}]\subset M$ then $g\cap Z\neq\emptyset$ for every continuous $g:[-1,1]\to\mathbb{R}$,*
(c) *$Z_x=M_x$ is a singleton for all $x\in(-1,1)\setminus X$, and*
(d) *for each $x\in X$ the section M_x is a non-trivial closed interval and Z_x consists of the two endpoints of that interval.*

Proof. Let F_0 be Roberts' function defined above. Define $F_1:\mathbb{R}\to\mathbb{R}^2$ by putting $F_1(n+x)=\langle n,n\rangle+F_0(x)$ for every $n\in\mathbb{Z}$ and $x\in[0,1)$. Then F_1 is a continuous embedding extending F_0. Also choose an order isomorphism $h:\mathbb{R}\to(-1,1)$ such that $h[\mathbb{Z}+D_0]=X$ and define a homeomorphism $H:\mathbb{R}^2\to(-1,1)\times\mathbb{R}$ by $H(x,y)=\langle h(x),y\rangle$. It easily follows from the properties of F_0 that $F=H\circ F_1$ satisfies (a)–(d). $\square$

Note that by (b) of Lemma 2.1 if the graph of $f:\mathbb{R}\to\mathbb{R}$ is disjoint with Z then f is not almost continuous, since then the set $U=\mathbb{R}^2\setminus Z$ is an open set containing f which does not contain any continuous function $g:\mathbb{R}\to\mathbb{R}$. Thus the main idea of the next theorem is to construct an additive connectivity function with the graph disjoint with Z.

In our argument it will be also convenient to use the following easy lemma.

Lemma 2.2. *Let $\{I_\alpha:\ \alpha<\mathfrak{c}\}$ be an enumeration, with possible repetitions, of all nonempty open intervals in $\mathbb{R}$. Then there exists a family of pairwise disjoint perfect sets $\{P_\alpha\subset I_\alpha:\ \alpha<\mathfrak{c}\}$ such that $P=\bigcup_{\alpha<\mathfrak{c}}P_\alpha$ is meager in $\mathbb{R}$ and linearly independent over $\mathbb{Q}$. Moreover, we can assume that there is a meager F_σ-set S containing P such that $S=\mathrm{LIN}_{\mathbb{Q}}(S)$ and S is of co-dimension continuum.*

Proof. Take a linearly independent perfect subset K of $\mathbb{R}$. (Such a set has been first constructed by von Neumann [27]. See also [17, Thm. 2, Ch. XI, Section 7].) Partition K into perfect sets $\{F, H, L\}$ and further partition F into pairwise disjoint perfect sets $\{F'_\alpha\colon \alpha < \mathfrak{c}\}$. Choose a countable subset $H_0 = \{x_n\colon n < \omega\}$ of H and for every $\alpha < \mathfrak{c}$ choose a sequence of non-zero rational numbers $\langle q_n^\alpha\colon n < \omega\rangle$ such that $F_\alpha = \bigcup_{n<\omega} q_n^\alpha \cdot x_n + F'_\alpha$ is dense in $\mathbb{R}$. Then the sets F_α are pairwise disjoint and $\bigcup_{\alpha<\mathfrak{c}} F_\alpha$ is linearly independent over $\mathbb{Q}$. For every $\alpha < \mathfrak{c}$ choose perfect $P_\alpha \subset F_\alpha \cap I_\alpha$. Then P_α's are pairwise disjoint and $P = \bigcup_{\alpha<\mathfrak{c}} P_\alpha \subset \mathbb{Q}\cdot H_0 + F$ is meager. Also if $S = \mathrm{LIN}_{\mathbb{Q}}(H \cup F)$ then $S = \mathrm{LIN}_{\mathbb{Q}}(S)$, and it is an F_σ-set. It is of co-dimension continuum (so meager) since S is disjoint with L. □

Theorem 2.3. *If union of less than $\mathfrak{c}$ many meager subsets of $\mathbb{R}$ is meager in $\mathbb{R}$ then there exists an $f \in \mathrm{Add} \cap \mathrm{CIVP} \cap \mathrm{Conn} \setminus \mathrm{AC}$.*

Proof. Let $\langle\langle I_\alpha, C_\alpha\rangle\colon \alpha < \mathfrak{c}\rangle$ be a list of all pairs $\langle I, C\rangle$ such that I is a nonempty open interval in $\mathbb{R}$ and C is a perfect subset of $\mathbb{R}$ and take $\{P_\alpha \subset I_\alpha\colon \alpha < \mathfrak{c}\}$ as in Lemma 2.2.

Let $\{C, D\}$ be a partition of $\mathfrak{c} \setminus \omega$ onto sets of cardinality continuum. Take an enumeration $\{K_\xi\colon \xi \in D\}$ of the family of all compact connected subsets K of $\mathbb{R}^2$ with $|\mathrm{proj}[K]| = \mathfrak{c}$. Also, let H be a Hamel basis containing $P = \bigcup_{\alpha<\mathfrak{c}} P_\alpha$ such that there is a countable set $X \subset (H \setminus P) \cap (-1, 1)$ dense in $(-1, 1)$. Let Z be as in Lemma 2.1 for this X and $\{h_\xi\colon \xi \in C\}$ be an enumeration of H. By induction on $\xi < \mathfrak{c}$ we will choose functions f_ξ from finite subsets H_ξ of H into $\mathbb{R}$ such that for every $\xi < \mathfrak{c}$ the following conditions hold.

(i) $H_\xi \cap \bigcup_{\zeta<\xi} H_\zeta = \emptyset$.
(ii) If $\xi \in C$ then $h_\xi \in \bigcup_{\zeta\leqslant\xi} H_\zeta$.
(iii) If $\xi \in D$ then $K_\xi \cap \mathrm{LIN}_{\mathbb{Q}}(\bigcup_{\zeta\leqslant\xi} f_\zeta) \neq \emptyset$.
(iv) $Z \cap \mathrm{LIN}_{\mathbb{Q}}(\bigcup_{\zeta\leqslant\xi} f_\zeta) = \emptyset$.
(v) If $x \in H_\xi \cap P_\alpha$ for some $\alpha < \mathfrak{c}$ then $f_\xi(x) \in C_\alpha$.

Before we describe the inductive construction note first how it can be used to construct a function as desired. First notice that, by (i) and (ii), $\bigcup_{\xi<\mathfrak{c}} f_\xi$ is a function from H into $\mathbb{R}$. Thus

$$f = \mathrm{LIN}_{\mathbb{Q}}\left(\bigcup_{\xi<\mathfrak{c}} f_\xi\right)$$

is an additive function from $\mathbb{R}$ to $\mathbb{R}$. It is connectivity by (iii). It is not almost continuous by (iv) and remark after Lemma 2.1. It has Cantor intermediate value property by (v) and the choice of $\langle I_\alpha, C_\alpha\rangle$.

The main difficulty in our inductive construction will be the preservation of condition (iv). To handle this easier note that if g is an additive function from $E \subset \mathbb{R} \setminus \{x\}$ into $\mathbb{R}$ such that $Z \cap g = \emptyset$ then $Z \cap \mathrm{LIN}_{\mathbb{Q}}(g \cup \{\langle x, y\rangle\}) = \emptyset$ if and only if

$$\langle x, y\rangle \notin \bigcup \bigl\{qZ + \langle p, g(p)\rangle\colon\ p \in E \text{ and } q \in \mathbb{Q}\bigr\}. \tag{1}$$

In particular, if x is fixed, than $Z \cap \mathrm{LIN}_{\mathbb{Q}}(g \cup \{\langle x, y\rangle\}) = \emptyset$ if and only if

$$y \notin \bigcup \bigl\{\bigl(qZ + \langle p, g(p)\rangle\bigr)_x\colon\ p \in E \text{ and } q \in \mathbb{Q}\bigr\}. \tag{2}$$

We will make the construction in two main steps. First we will construct the functions f_n for $n<\omega$. For this choose an enumeration $\{x_n\colon n<\omega\}$ of X. We put $H_n=\{x_n\}$ and define $f_n(x_n)$ inductively such that

$$\langle x_n, f_n(x_n)\rangle \in M\setminus Z, \tag{3}$$

where M is the set from Lemma 2.1.

To see that such a choice can be made, note first that (i) is satisfied, and (ii), (iii), and (v) are satisfied in void. Thus, we have to take care only of the condition (iv). However, for each $n<\omega$ we have an entire interval of possible choices for $f_n(x_n)$ (see Lemma 2.1(d)) while, by (2), there is only a countable many exceptional points we have to avoid. (Since $|Z_{x_n}|=2$ and $E=\mathrm{LIN}_{\mathbb{Q}}(\{x_i\colon i<n\})$ in this case.)

Now, assume that for some infinite $\xi<\mathfrak{c}$ the sequence $\langle f_\zeta\colon \zeta<\xi\rangle$ has been already constructed. Put $g=\mathrm{LIN}_{\mathbb{Q}}(\bigcup_{\zeta<\xi} f_\zeta)$ and let E be its domain.

First consider case when $\xi\in C$. If $h_\xi\in\bigcup_{\zeta<\xi} H_\zeta$ we put $f_\xi=H_\xi=\emptyset$. So, assume that $h_\xi\notin\bigcup_{\zeta<\xi} H_\zeta$ and put $H_\xi=\{h_\xi\}$. If $h_\xi\in P_\alpha$ for some $\alpha<\mathfrak{c}$ put $P=C_\alpha$. Otherwise put $P=\mathbb{R}$. Then (i) and (ii) are satisfied and (v) will hold if we choose $f_\xi(h_\xi)\in P$. To have (iv) by (2) it is enough to choose $f_\xi(h_\xi)$ from outside of a set $\bigcup\{(qZ+\langle p,g(p)\rangle)_{x_\xi}\colon p\in E \text{ and } q\in\mathbb{Q}\}$, which has cardinality less than continuum.

So, assume that $\xi\in D$. Let S be as in Lemma 2.2 and put $T_0=\mathrm{LIN}_{\mathbb{Q}}(S\cup\bigcup_{\zeta<\xi} H_\zeta)$. Then $T_0\neq\mathbb{R}$ since $\mathrm{LIN}_{\mathbb{Q}}(S)$ is of co-dimension continuum. Moreover T_0 is a union of less than continuum many meager sets $\mathrm{LIN}_{\mathbb{Q}}(S\cup A)$, where A is a finite subset of $\bigcup_{\zeta<\xi} H_\zeta$. Thus, by our assumption, T_0 is meager. Let T be a meager F_σ-set containing T_0. Our next main objective will be to show that either we already have $K_\xi\cap g\neq\emptyset$ or we can find

$$\langle x,y\rangle\in K_\xi\setminus\Bigl((T\times\mathbb{R})\cup\bigcup\bigl\{qZ+\langle p,g(p)\rangle\colon p\in E \text{ and } q\in\mathbb{Q}\bigr\}\Bigr). \tag{4}$$

Before we argue for it, first note how this will finish the construction. If $K_\xi\cap g\neq\emptyset$ we can put $f_\xi=H_\xi=\emptyset$. So, assume that we can find $\langle x,y\rangle$ as in (4). Take a minimal subset $\{k_0,\dots,k_m\}$ of $H\setminus\bigcup_{\zeta<\xi} H_\zeta$ such that $x\in\mathrm{LIN}_{\mathbb{Q}}(\{k_0,\dots,k_m\}\cup\bigcup_{\zeta<\xi} H_\zeta)$. We will define f_ξ on $H_\xi=\{k_0,\dots,k_m\}$ such that $\langle x,y\rangle\in\mathrm{LIN}_{\mathbb{Q}}(g\cup f_\xi)$, implying (iii), while preserving (iv) and (v). First, for $i\leqslant m$ let P^i be equal to C_α if $k_i\in P_\alpha$ for some $\alpha<\mathfrak{c}$ and equal to $\mathbb{R}$ otherwise. To preserve (v) we have to choose $f_\xi(k_i)\in P^i$. Next note that $H_\xi\not\subset P$ since $x\notin T\supset\mathrm{LIN}_{\mathbb{Q}}(P\cup\bigcup_{\zeta<\xi} H_\zeta)$. Assume that $k_m\notin P$. Thus $P^m=\mathbb{R}$. Note that, by (1), $\bar g=\mathrm{LIN}_{\mathbb{Q}}(g\cup\{\langle x,y\rangle\})$ is disjoint with Z. Proceeding as in case when $\xi\in C$ and using (2) we can inductively choose for every $i<m$ a value $f_\xi(k_i)\in P^i$ such that $h=\mathrm{LIN}_{\mathbb{Q}}(\bar g\cup\{\langle k_i,f_\xi(k_i)\rangle\colon i<m\})$ is disjoint with Z. Then function h is already defined on k_m and we can put $f_\xi(k_m)=h(k_m)\in\mathbb{R}=P^m$. Clearly such f_ξ satisfies (iv) and (v).

To argue for (4) we will consider three cases.

Case 1: $\emptyset\neq(I\times\mathbb{R})\cap(qM+v)\subset K_\xi$ *for some* $v=\langle v_0,v_1\rangle\in g$, $q\in\mathbb{Q}\setminus\{0\}$, *and an open interval* I. *Then* $K_\xi\cap g\neq\emptyset$.

Indeed $\frac{1}{q}(I-v_0)$ is an open interval intersecting $(-1,1)$ and we find $n<\omega$ such that $x_n\in H_n\cap\frac{1}{q}(I-v_0)$. By (3) we have $\langle x_n,g(x_n)\rangle\in M\setminus Z$. Therefore

$$\bigl\langle qx_n+v_0, g(qx_n+v_0)\bigr\rangle=q\bigl\langle x_n,g(x_n)\bigr\rangle+v\in(qM+v)\cap(I\times\mathbb{R})\subset K_\xi.$$

Case 2: *There exists an* $x \in \{z \in \mathbb{R}\colon |(K_\xi)_z| = \mathfrak{c}\} \setminus T$. Choose

$$y \in (K_\xi)_x \setminus \bigcup \bigl\{(qZ + \langle p, g(p)\rangle)_x\colon\ p \in E \text{ and } q \in \mathbb{Q}\bigr\}.$$

Then $\langle x, y\rangle$ satisfies (4).

Case 3: *Neither Case* 1 *nor Case* 2 *hold*.

Define Y as $K_\xi \setminus (T \times \mathbb{R})$. Then Y is a G_δ subset of K_ξ so it is a Polish space. Notice also that, since we are not in Case 2, every vertical section of Y is at most countable. We will prove that

$$qZ + v \text{ is meager in } Y \quad \text{for every } v \in g \text{ and } q \in \mathbb{Q}. \tag{5}$$

This clearly implies the possibility of a choice as in (4) since $\mathbb{R}$ (and so, a Polish space Y) is not a union of less than continuum many meager sets.

To prove (5) fix $v = \langle v_0, v_1\rangle \in g$, $q \in \mathbb{Q} \setminus \{0\}$, and an open set $U \subset \mathbb{R}^2$ such that $U \cap Y \neq \emptyset$. We have to show that $U \cap Y \setminus (qZ + v) \neq \emptyset$. So, fix $p = \langle x, y\rangle \in U \cap Y$ and an open set V containing p such that $\mathrm{cl}(V) \subset U$. Let C_0 be a connected component of $K_\xi \cap V$ containing x. Then, by Proposition 1.1, C_0 has more than one point. Consider a compact connected set $K = \mathrm{cl}(C_0) \subset \mathrm{cl}(V) \subset U$. Then $p \in K$ and $\mathrm{proj}(K)$ is a nontrivial interval, say $[c, d]$, since $K_x \subset (K_\xi)_x$ is at most countable. Thus, it is enough to prove that $K \setminus ((T \times \mathbb{R}) \cup (qZ + v)) \neq \emptyset$ which follows easily from the following property:

$$\bigl|\mathrm{proj}(C)\bigr| = \mathfrak{c} \quad \text{for some connected component } C \text{ of } K \setminus (qM + v), \tag{6}$$

where M is an arc from Lemma 2.1 containing Z.

By way of contradiction assume that (6) is false. Then every connected component of $K \setminus (qM + v)$ is vertical. Note that there exists a number $r \in (qX + v_0) \cap (c, d)$ such that the vertical section $\{r\} \times (qM + v)_r$ of $qM + v$ is not contained in K, since otherwise we would have

$$\bigl((c, d) \times \mathbb{R}\bigr) \cap (qM + v) \subset \mathrm{cl}\left(\bigcup_{r \in (qX + v_0) \cap (c, d)} \{r\} \times (qM + v)_r\right) \subset K \subset K_\xi$$

contradicting the fact that Case 1 does not hold. Let $a < b$ be such that $\langle r, a\rangle$ and $\langle r, b\rangle$ are the endpoints of the vertical segment $\{r\} \times [a, b]$ of $qM + v$ above r, i.e., such that $(qM + v)_r = [a, b]$. Since $\{r\} \times [a, b]$ is not a subset of a compact set K, we can find $s \in (a, b)$ such that $\langle r, s\rangle \notin K$. Take an $\varepsilon_0 > 0$ such that

(α) $\varepsilon_0 < \frac{1}{4}\min\{s - a, b - s, r - c, d - r\}$ and

(β) the closed rectangle $[r - \varepsilon_0, r + \varepsilon_0] \times [s - \varepsilon_0, s + \varepsilon_0]$ is disjoint from K.

It follows from Lemma 2.1 (in particular, the fact that F_0 is non-decreasing) that we may find a positive $\varepsilon_1 < \varepsilon_0$ such that either

(γ) $(\forall x \in (r, r + \varepsilon_1])(\forall y \in (qM + v)_x)(a - \varepsilon_0 < y < a + \varepsilon_0)$, and

(δ) $(\forall x \in [r - \varepsilon_1, r))(\forall y \in (qM + v)_x)(b - \varepsilon_0 < y < b + \varepsilon_0)$,

or symmetrical conditions interchanging $(r, r + \varepsilon_1]$, $[r - \varepsilon_1, r)$ hold. Without loss of generality we may assume that we have the clauses (γ), (δ) as formulated above. (For Z and M as constructed in Lemma 2.1 this happens when $q > 0$.) Consider the set

$$D_{\varepsilon_1} \stackrel{\text{def}}{=} \bigl([r - \varepsilon_1, r + \varepsilon_1] \times \{s\}\bigr) \cup \bigl\{\langle(r + \varepsilon_1, y)\rangle\colon\ y \geqslant s\bigr\} \cup \bigl\{\langle r - \varepsilon_1, y\rangle\colon\ y \leqslant s\bigr\}.$$

We claim that $D_{\varepsilon_1} \cap K = \emptyset$. Why? Suppose that $\langle x, y\rangle \in D_{\varepsilon_1} \cap K$. By the choice of ε_0 (clause (β)) we know that either $x = r + \varepsilon_1$ and $y > s$, or $x = r - \varepsilon_1$ and $y < s$. The two cases are handled similarly, so suppose that the first one takes place. By the choice of ε_1 (clause (γ)) we know that $\langle x, y\rangle \notin qM + v$ (as $y > s > a + \varepsilon_0$). We have assumed that each connected component of $K \setminus (qM + v)$ is contained in a vertical line, so look at the connected component $C_{\langle x,y\rangle}$ of $K \setminus (qM + v)$ to which $\langle x, y\rangle$ belongs. By Proposition 1.1 we know that $\mathrm{cl}(C_{\langle x,y\rangle}) \cap (qM + v) \neq \emptyset$. Hence, by clause (γ), we conclude that $\langle x, s\rangle \in C_{\langle x,y\rangle}$ (remember $y > s > a + \varepsilon_0$), a contradiction with clause (β).

To obtain a final contradiction note that D_{ε_1} separates non-empty subsets $K \cap (\{c\} \times \mathbb{R})$ and $K \cap (\{d\} \times \mathbb{R})$, which contradicts connectedness of K. The proof is complete. □

It is also worth to mention that essentially the same proof as above gives the following theorem with a slightly weaker set theoretical assumption.

Theorem 2.4. *If $\mathbb{R}$ is not a union of less than continuum many of its meager subsets then there exists an $f \in \mathrm{Add} \cap \mathrm{Conn} \setminus \mathrm{AC}$.*

Sketch of proof. The argument can be obtained by the following modification of the proof of Theorem 2.3. Repeat the proof with replacing sets S, P, and P_α's with the empty set. Then (v) is always satisfied in void and T will become $\mathrm{LIN}_{\mathbb{Q}}(\bigcup_{\zeta<\xi} H_\zeta)$, which has cardinality less than $\mathfrak{c}$, but certainly does not have to be F_σ. Then we note that the set $A = \{z \in \mathbb{R}\colon |(K_\xi)_z| = \mathfrak{c}\}$ is analytic, so it is either countable, or has cardinality continuum. Thus, if case 2 does not hold then A is countable. The proof is finished when we replace the set X from the proof of Theorem 2.3 with $X = K_\xi \setminus (A \times \mathbb{R})$ and notice that the sets $\{z\} \times \mathbb{R}$ with $z \in \mathrm{LIN}_{\mathbb{Q}}(\bigcup_{\zeta<\xi} H_\zeta)$ are meager in X. □

We will finish this section with the following open problems.

Problem 2.1. Does there exist a ZFC example of an additive connectivity function $f : \mathbb{R} \to \mathbb{R}$ (with the CIVP property or not) which is not almost continuous?

Problem 2.2. Does there exist an $f \in \mathrm{Add} \cap \mathrm{SCIVP} \cap \mathrm{Conn} \setminus \mathrm{AC}$?

3. An additive almost continuous SCIVP function $f : \mathbb{R} \to \mathbb{R}$ which is not extendable

The difficult aspect of constructing a function as in the title will be in making sure that it will not be extendable. Since such a function must have a dense graph (as every discontinuous additive function does) we may restrict our attention to such functions. For these we have the following nice generalization of the SCIVP property.

Theorem 3.1. *If $f : \mathbb{R} \to \mathbb{R}$ is an extendable function with a dense graph then for every $a, b \in \mathbb{R}$, $a < b$, and for each Cantor set K between $f(a)$ and $f(b)$ there is a Cantor set C between a and b such that $f[C] \subset K$ and the restriction $f \restriction C$ is continuous strictly increasing.*

Proof. The basic idea of the proof of this theorem is the same as in the proof from [25] that every extendable function is SCIVP. However, our schema of the proof will be more similar to the one used to show that every normal topological space is completely regular.

Let a, b, and K be as in the theorem and let $\{q_n\colon n<\omega\}$ be an enumeration of some countable subset of K such that the linear ordering $(\{q_n\colon n<\omega\},\leqslant)$ is dense and $q_0=\min K$, $q_1=\max K$. Since the graph of f is dense (and f is Darboux) we can find $a<b_0<b_1<b$ with $f(b_0)=q_0$ and $f(b_1)=q_1$. Let $F\colon\mathbb{R}\times[0,1]\to\mathbb{R}$ be a connectivity function extending f in a sense that $F(x,0)=f(x)$ for every $x\in\mathbb{R}$. By [13] (see also [10]) we can choose F to be continuous outside the line $L_0=\mathbb{R}\times\{0\}$. We can also assume that $F(b_0,y)=F(b_0,0)=q_0$ and $F(b_1,y)=F(b_1,0)=q_1$ for every $y\in[0,1]$. (Indeed, let H be a closed subset of $\mathbb{R}\times[0,1]$ from which we remove two V-shape regions with vertices at $\langle b_0,0\rangle$ and $\langle b_1,0\rangle$. Extend $F\restriction H$ to $\{0,1\}\times[0,1]$ as above. Then, by Tietze extension theorem, we can extend such a function to the reminder of V-shape regions continuously. Such modified F will still be connectivity.)

We will construct a sequence $\langle B_n\colon n<\omega\rangle$ of compact connected subsets of $\mathbb{R}\times[0,1]$ such that the following conditions are satisfied for every $m,n<\omega$, where $L_\ell=\mathbb{R}\times\{\ell\}$ for $\ell=0,1$.

(i) $B_0=\{b_0\}\times[0,1]$ and $B_1=\{b_1\}\times[0,1]$.

(ii) $B_n\cap L_0\neq\emptyset$ and $B_n\cap L_1\neq\emptyset$.

(iii) If $q_m<q_n$ then, for $\ell=0,1$, we have

$$\max\bigl(\bigl\{x\in\mathbb{R}\colon \langle x,\ell\rangle\in B_m\bigr\}\bigr)<\min\bigl(\bigl\{x\in\mathbb{R}\colon \langle x,\ell\rangle\in B_n\bigr\}\bigr).$$

(iv) $F[B_n]=\{q_n\}$.

Clearly B_0 and B_1 satisfy (ii)–(iv). So, assume that for some $n<\omega$, $n>1$, the sets $B_0,\ldots,B_{n-1}$ are already constructed. To find B_n choose $i,j<n$ such that (q_i,q_j) is the smallest interval containing q_n with the endpoints from $\{q_0,\ldots,q_{n-1}\}$. Let

$$b_i=\max\bigl(\bigl\{x\in\mathbb{R}\colon \langle x,1\rangle\in B_i\bigr\}\bigr),\qquad b_j=\min\bigl(\bigl\{x\in\mathbb{R}\colon \langle x,1\rangle\in B_j\bigr\}\bigr).$$

(So $b_i<b_j$.) Let $A^*=\mathrm{cl}(F^{-1}(q_n)\setminus L_0)$. Note that the set $F^{-1}(q_n)\setminus L_0$ is closed in $\mathbb{R}\times(0,1]$ and thus $A^*\setminus F^{-1}(q_n)\subseteq L_0$. Now one easily shows that the sets B_i, B_j are contained in different components of the open set $(\mathbb{R}\times[0,1])\setminus A^*$, so A^* separates B_i, B_j. Applying [28, Thm. 4.12, p. 51] (Property I) we may conclude that there is a connected component B^* of A^* which separates points $\langle b_i,1\rangle$ and $\langle b_j,1\rangle$, and thus separates B_i and B_j. Note that $B^*\cap L_0\neq\emptyset\neq B^*\cap L_1$. Take an $x\in(b_i,b_j)$ such that $\langle x,1\rangle\in B^*$ and let B be the connected component of the set $B^*\setminus L_0$ to which $\langle x,1\rangle$ belongs. Put $B_n=\mathrm{cl}(B^*)$. We claim that the compact connected set B_n satisfies our demands. To check clause (iv) note that, by the definition of the set A^*, $B_n\setminus L_0\subseteq F^{-1}(q_n)$. Now suppose that $y\in L_0\cap B_n$. Assume that $\varepsilon=|F(y)-q_n|>0$. Since every connectivity function on $\mathbb{R}^2$ is peripherally continuous (see, e.g., [12]), there exists an open neighborhood W of the point y with the diameter $<\frac{1}{2}$ and such that $|F(z)-F(y)|<\varepsilon)$ for all $z\in\mathrm{bd}(W)$. But B^* is connected, intersects $W\setminus L_0$ and has the diameter $\geqslant 1$ ($\mathrm{cl}(B^*)$ intersects L_0 and L_1), so there exists a $z \in \mathrm{bd}(W) \cap B^*$, a contradiction. Finally, it should be clear that $B_n \cap L_1 \neq \emptyset$ and $B_n \cap L_0 \neq \emptyset$ (e.g., use Proposition 1.1), and

$$\max\bigl(\bigl\{x \in \mathbb{R}\colon \langle x, \ell\rangle \in B_i\bigr\}\bigr) < \min\bigl(\bigl\{x \in \mathbb{R}\colon \langle x, \ell\rangle \in B_n\bigr\}\bigr),$$
$$\max\bigl(\bigl\{x \in \mathbb{R}\colon \langle x, \ell\rangle \in B_n\bigr\}\bigr) < \min\bigl(\bigl\{x \in \mathbb{R}\colon \langle x, \ell\rangle \in B_j\bigr\}\bigr).$$

The construction is completed.

Let $B = \bigcup_{n<\omega} B_n$ and notice that

$$F \restriction \mathrm{cl}(B) \text{ is continuous.} \tag{7}$$

(Compare [25, Thm. 2].) Indeed, by way of contradiction assume that for some $x \in \mathrm{cl}(B)$ there is a sequence $\langle x_i \in B\colon i < \omega\rangle$ such that $\lim_{i\to\infty} F(x_i) = L \neq F(x)$. Let $\varepsilon \in (0, |L - F(x)|/2)$ and $\delta \in (0,1)$ be such that if $|x - x_i| < \delta$ then $|F(x) - F(x_i)| > \varepsilon$. Using peripheral continuity of the function F (see, e.g., [12]) we find an open neighborhood W of x with the diameter $< \delta$ and such that $|f(x) - f(y)| < \varepsilon$ for every $y \in \mathrm{bd}(W)$. Take $i, n < \omega$ such that $x_i \in W \cap B_n$. Note that B_n is connected and has the diameter $\geqslant 1$, so there exists $y \in \mathrm{bd}(W) \cap B_n$. But then,

$$\varepsilon < \bigl|F(x) - F(x_i)\bigr| = \bigl|F(x) - y_n\bigr| = \bigl|F(x) - F(y)\bigr| < \varepsilon,$$

a contradiction.

Consider L_0 as ordered in natural order and for $n < \omega$ define $x_n = \min(B_n \cap L_0)$. Notice that, by (i) and (iii), $x_n < x_m$ if and only if $f(x_n) = q_n < q_m = f(x_m)$. Since $\{q_n\colon n < \omega\}$ (with the natural order) is a dense linear order, so is $S = \{x_n\colon n < \omega\}$. In particular, $\mathrm{cl}(S)$ contains a perfect set $C_0 = C \times \{0\}$. But F is continuous on $\mathrm{cl}(B) \supset \mathrm{cl}(S)$ and is strictly increasing on S. Consequently we may choose a perfect set $C^* \subset \mathrm{cl}(S)$ such that between every two points of C^* there is some x_n. So, $f \restriction C^*$ is strictly increasing, continuous, and $f[C^*] \subseteq f[\mathrm{cl}(S)] \subset \mathrm{cl}(\{q_n\colon n < \omega\}) = K$. $\square$

Theorem 3.2. *There exists an additive almost continuous* SCIVP *function* $f : \mathbb{R} \to \mathbb{R}$ *which is not extendable.*

Proof. Let $\langle\langle I_\xi, y_\xi\rangle\colon \xi < \mathfrak{c}\rangle$ be a list of all pairs $\langle I, y\rangle$ such that I is a nonempty open interval and $y \in \mathbb{R}$. Choose the enumerations $\langle C_\xi\colon \xi < \mathfrak{c}\rangle$ of all perfect subsets of $\mathbb{R}$ and $\langle B_\xi\colon \xi < \mathfrak{c}\rangle$ of all closed subsets of $\mathbb{R}^2$ whose projections have nonempty interior.

For our construction we will also use a Hamel basis H which can be partitioned onto the sets $\{P_\alpha\colon \alpha \leqslant \mathfrak{c}\}$ such that

- all sets in $\mathcal{T} = \{P_\alpha\colon \alpha < \mathfrak{c}\}$ are perfect, and
- every nonempty open interval contains continuum many $T \in \mathcal{T}$.

The existence of such a basis follows easily from the existence of a linearly independent perfect set [17, Thm. 2, Ch. XI, Section 7] and has been described in detail in [8].

By induction choose a sequence $\langle\langle D_\xi, T_\xi\rangle \in [H]^{<\omega} \times \mathcal{T}\colon \xi < \mathfrak{c}\rangle$ such that the sets $\{D_\xi\colon \xi < \mathfrak{c}\}$ and $\{T_\xi\colon \xi < \mathfrak{c}\}$ are pairwise disjoint and that for every $\xi < \mathfrak{c}$

(i) $T_\xi \subset I_\xi$,

(ii) there exists an $a_\xi \in D_\xi \cap \mathrm{proj}(B_\xi)$,

(iii) there exist $z \in \mathbb{R}$, $0 < n < \omega$, non-zero rational numbers $q_0, \ldots, q_{n-1}$, and $\{b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}\} \in [(D_\xi \setminus \{a_\xi\}) \cup \bigcup_{\eta \leqslant \xi} T_\eta]^{2n}$ with the property that $b_\xi = z + \sum_{j<n} q_j b_j$ and $c_\xi = z + \sum_{j<n} q_j c_j$ belong to C_ξ and that $b_j \in T_\eta$ if and only if $c_j \in T_\eta$ for every $j < n$ and $\eta \leqslant \xi$,

(iv) if $y_\xi \in H$ then $y_\xi \in \bigcup_{\eta \leqslant \xi}(D_\eta \cup T_\eta)$.

To make an inductive step assume that for some $\xi < \mathfrak{c}$ the sequence $\langle\langle D_\eta, T_\eta\rangle\colon \eta < \xi\rangle$ has been already constructed and let $M_\xi = \bigcup_{\eta<\xi}(D_\eta \cup T_\eta)$. It is easy to find $T_\xi \in \mathcal{T}$ with $T_\xi \subset I_\xi \setminus M_\xi$ and an $a_\xi \in \operatorname{proj}(B_\xi) \setminus (T_\xi \cup M_\xi)$. Next put $\kappa = |\bigcup_{\eta<\xi} D_\eta| + \omega < \mathfrak{c}$ and for $x \in C_\xi$ let $x = \sum_{i<m_x} q_i^x h_i^x$ be a unique representation of x in base H (i.e., q_i^x's are non-zero rationals and h_i^x's are different elements of H). By a combination of the pigeon-hall principle and Δ-system lemma (see, e.g., [18, Thm. 1.6, p. 49]) we can find $m < \omega$, $\Delta \subset H$, and an $E \subset C_\xi$ of cardinality κ^+ such that for every different $x, y \in E$ we have:

$$m_x = m, \quad \Delta = \{h_i^x\colon i < m\} \cap \{h_i^y\colon i < m\}, \quad \text{and} \quad q_i^x = q_i^y \quad \text{for every } i < m.$$

Let $n = m - |\Delta|$. Refining E and reenumerating the sets $\{h_i^x\colon i < m\}$, if necessary, we can also assume that $h_j^x = h_j^y$ and $\Delta = \{h_i^x\colon n \leqslant i < m\}$ for all $x, y \in E$ and $n \leqslant j < m$. Moreover, since $|(\xi+2)^n| \leqslant \kappa < |E|$ we can additionally assume that for every $i < n$ and $\eta \leqslant \xi$ we have $h_i^x \in T_\eta$ if and only if $h_i^y \in T_\eta$. Finally, by the definition of κ, we can also require that $\{h_i^x\colon i < n\} \cap (\{a_\xi\} \cup \bigcup_{\eta<\xi} D_\eta) = \emptyset$ for all $x \in E$. Fix different $x, y \in E$ and notice that $z = \sum_{n \leqslant i < m} q_i^x h_i^x$, $b_i = h_i^x$, $c_i = h_i^y$, and $q_i = q_i^x = q_i^y$ for $i < n$ satisfy (iii). Now we can define D_ξ as $(\{a_\xi\} \cup \{b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}\}) \setminus \bigcup_{\eta \leqslant \xi} T_\eta$ adding to it y_ξ, if necessary, to satisfy (iv). This finishes the inductive construction.

Notice that by (iv) we have

$$H = \bigcup_{\xi<\mathfrak{c}} (D_\xi \cup T_\xi).$$

We define f on H in such a way that for each $\xi < \mathfrak{c}$ we have: $\langle a_\xi, f(a_\xi)\rangle \in B_\xi$, $f \restriction T_\xi \equiv y_\xi$, and $f(b_i) = f(c_i)$ for every $i < n$, where b_i and c_i are the points from (iii). We claim that the unique additive extension of such defined $f \restriction H$ has the desired properties.

Clearly f is additive and almost continuous, since f intersects every set B_ξ. It is SCIVP since for every $a < b$ and perfect K between $f(a)$ and $f(b)$ there is $\xi < \mathfrak{c}$ with $I_\xi = (a, b)$ and $y_\xi \in K$. So, $f \restriction T_\xi$ witness SCIVP. To see that it is not extendable first note that f is clearly discontinuous, so it has a dense graph. Thus, by Theorem 3.1 it is enough to show that $f \restriction C$ is not strictly increasing for every perfect set C. So, let C be perfect. We claim that there are different $b, c \in C$ such that $f(b) = f(c)$, which clearly implies that $f \restriction C$ is not strictly increasing.

Indeed let $\xi < \mathfrak{c}$ be such that $C = C_\xi$. Then points $b_\xi, c_\xi \in C_\xi$ from (iii) are different and the additivity of f implies that

$$f(b_\xi) = f\left(z + \sum_{j<n} q_j b_j\right) = f\left(z + \sum_{j<n} q_j c_j\right) = f(c_\xi).$$

This finishes the proof. $\square$

4. Another ZFC example of almost continuous SCIVP function which is not extendable

In [24] Rosen showed that the Continuum Hypothesis implies the existence of SCIVP almost continuous function $f:\mathbb{R}\to\mathbb{R}$ with a dense graph such that $f[M]\neq\mathbb{R}$ for every meager set $M\subset\mathbb{R}$. He also noticed that such an f is not extendable.[2] In this section we will show that a function with such properties can be constructed in ZFC. (See Theorem 4.2 and Corollary 4.3.) We also show (see Proposition 4.4) that there are serious obstacles to make such a function additive.

Lemma 4.1. *Suppose that $F\subseteq\omega^\omega\times\omega^\omega\times\mathbb{R}$ is a Borel set such that for some basic open sets $U,V\subseteq\omega^\omega$ we have*:

(a) *the set $Z\stackrel{\text{def}}{=}\{\langle x,y\rangle\in U\times V\colon (F)_{\langle x,y\rangle}=\emptyset\}$ is meager,*

(b) *the set $A\stackrel{\text{def}}{=}\{\langle x,y\rangle\in U\times V\colon (F)_{\langle x,y\rangle}$ is uncountable$\}$ is meager,*

(c) *for each $z\in\mathbb{R}$ the section $(F)^z$ is meager.*

Then there is a perfect set $P\subseteq U\times V$ such that

$$\big(\forall\langle x,y\rangle\in P\big)\big((F)_{\langle x,y\rangle}\neq\emptyset\big)$$

and for distinct $\langle x',y'\rangle,\langle x'',y''\rangle\in P$ we have $(F)_{\langle x',y'\rangle}\cap(F)_{\langle x'',y''\rangle}=\emptyset$ and $x'\neq x''$.

Proof. Without loss of generality we may assume that $U=V=\omega^\omega$. (Remember that basic open subsets of ω^ω are homeomorphic with ω^ω.) Let $Z^*\subseteq\omega^\omega\times\omega^\omega$ be a Borel meager set such that $Z\subseteq Z^*$ and let $A^*\subseteq\omega^\omega\times\omega^\omega$ be a Borel meager set such that $A\subseteq A^*$. For a sufficiently large regular cardinal χ take a countable elementary submodel N of $\langle\mathcal{H}(\chi),\in,<^*\rangle$ (where $\mathcal{H}(\chi)$ is the family of sets that are hereditarily of size $<\chi$, and $<^*$ is a fixed well-ordering of $\mathcal{H}(\chi)$) such that the sets F, Z^*, and A^* are in N. (Strictly speaking we require that the Borel codes of these sets are in N.)

For $n<\omega$ a set $T\subseteq\omega^{\leqslant n}\times\omega^{\leqslant n}$ is an n-tree if

$$\langle\sigma_0,\sigma_1\rangle\in T \text{ and } \sigma_0'\subseteq\sigma_0 \text{ and } \sigma_1'\subseteq\sigma_1\Rightarrow\langle\sigma_0',\sigma_1'\rangle\in T$$

and for each $\langle\sigma_0,\sigma_1\rangle\in T$ there is $\langle\sigma_0^*,\sigma_1^*\rangle\in T$ such that $\sigma_0\subseteq\sigma_0^*\in\omega^n$ and $\sigma_1\subseteq\sigma_1^*\in\omega^n$. Let $\mathbb{P}$ be the collection of all $T\subseteq\omega^{<\omega}\times\omega^{<\omega}$ which are n-trees for some n. We equip $\mathbb{P}$ with the end-extension order, that is, if T_0,T_1 are n_0- and n_1-trees, respectively, then T_0 is stronger than T_1, $T_0\leqslant T_1$, if and only if $n_1\leqslant n_0$ and $T_0\cap(\omega^{n_1}\times\omega^{n_1})=T_1$. Note that $\mathbb{P}$ is a countable atomless partial order (and it belongs to N), so it is equivalent to the Cohen forcing notion. (See, e.g., [4, Thm. 3.3.1].) Let $G\subseteq\mathbb{P}$ be a generic filter over N. (It exists since N is countable; of course it is produced by a Cohen real over N.) It is a routine to check that $\bigcup G\subseteq\omega^{<\omega}\times\omega^{<\omega}$ is a perfect tree. Let

$$P=\big\{\langle x,y\rangle\in\omega^\omega\times\omega^\omega\colon (\forall n\in\omega)(\exists T\in G)\big(\langle x\restriction n,y\restriction n\rangle\in T\big)\big\}.$$

One easily shows that P is a perfect subset of $\omega^\omega\times\omega^\omega$ and that each $\langle x,y\rangle\in P$ is a Cohen real over N (i.e., this pair does not belong to any meager subset of $\omega^\omega\times\omega^\omega$ coded in the

[2] In fact, Rosen's function is from [0, 1] to [0, 1], but a minor modification gives one from $\mathbb{R}$ to $\mathbb{R}$.

model N). But even more, all elements of the perfect set P are *mutually* Cohen over N: if $\langle x,y\rangle$, $\langle x',y'\rangle$ are distinct elements of P then $\langle x,y\rangle$ is Cohen over $N[\langle x',y'\rangle]$. (Compare with [4, Lemma 3.3.2].) For our purposes it is enough to note that if $A\subseteq(\omega^\omega\times\omega^\omega)^2$ is a Borel meager set coded in N and $\langle x,y\rangle$, $\langle x',y'\rangle$ are distinct elements of P then $\langle\langle x,y\rangle,\langle x',y'\rangle\rangle\notin A$.)

We claim that P is as required. First note that $P\cap Z^*=\emptyset$ (so $(F)_{\langle x,y\rangle}\neq\emptyset$ for every $\langle x,y\rangle\in P$) and that $P\cap A^*=\emptyset$ (implying that $(F)_{\langle x,y\rangle}$ is countable for every $\langle x,y\rangle\in P$). Now suppose that $\langle x',y'\rangle,\langle x'',y''\rangle\in P$ are distinct. So $\langle x',y'\rangle$ is a Cohen real over $N[\langle x'',y''\rangle]$ and in particular $x'\neq x''$ (as $\{x''\}\times\omega^\omega$ is a Borel meager set coded in $N[\langle x'',y''\rangle]$). We know that $(F)_{\langle x'',y''\rangle}$ is a countable set from $N[\langle x'',y''\rangle]$, and hence $\bigcup\{(F)^z\colon z\in(F)_{\langle x'',y''\rangle}\}$ is a meager Borel set coded in $N[\langle x'',y''\rangle]$. Thus $\langle x',y'\rangle$ does not belong to it. Consequently $(F)_{\langle x',y'\rangle}\cap(F)_{\langle x'',y''\rangle}=\emptyset$ and the proof is finished. □

Theorem 4.2. *There is a function $f:\mathbb{R}\to\mathbb{R}$ such that*

- $(\otimes_1)$ *if $F\subseteq\mathbb{R}^2$ is a Borel set such that the projection* proj$[F]$ *is not meager then $f\cap F\neq\emptyset$,*
- $(\otimes_2)$ *if $P\subseteq\mathbb{R}$ is a perfect set and $B\subseteq\mathbb{R}$ is a non-meager Borel set then there are a perfect set $Q\subseteq B$ and a real $y\in P$ such that $f(x)=y$ for all $x\in Q$,*
- $(\otimes_3)$ *if $M\subseteq\mathbb{R}$ is meager then $f[M]\neq\mathbb{R}$.*

Proof. First note that $\mathbb{R}\setminus\mathbb{Q}$ is homeomorphic to $\omega^\omega\times\omega^\omega$, so it is enough to construct a function $f:\omega^\omega\times\omega^\omega\to\mathbb{R}$ such that

- $(\otimes_1^*)$ if $F\subseteq(\omega^\omega\times\omega^\omega)\times\mathbb{R}$ is a Borel set such that the projection of F onto $\omega^\omega\times\omega^\omega$ is not meager then $f\cap F\neq\emptyset$,
- $(\otimes_2^*)$ if $P\subseteq\mathbb{R}$ is a perfect set and $B\subseteq\omega^\omega\times\omega^\omega$ is a non-meager Borel set then there are a perfect set $Q\subseteq B$ and a real $z\in P$ such that $f\restriction Q\equiv z$,
- $(\otimes_3^*)$ if $M\subseteq\omega^\omega\times\omega^\omega$ is meager then $f[M]\not\supset\mathbb{R}\setminus\{0\}$.

(If a function $f:\mathbb{R}\setminus\mathbb{Q}\to\mathbb{R}$ satisfies the demand $(\otimes_1^*)$–$(\otimes_3^*)$, then the function $\bar f:\mathbb{R}\to\mathbb{R}$ such that $f\subset\bar f$ and $\bar f\restriction\mathbb{Q}\equiv 0$ is as required in the theorem.)

Fix enumerations

- $\{\langle r_\alpha,s_\alpha\rangle\colon\alpha<\mathfrak{c}\}$ of $\omega^\omega\times\omega^\omega$,
- $\{M_\alpha\colon\alpha<\mathfrak{c}\}$ of all Borel meager subsets of $\omega^\omega\times\omega^\omega$,
- $\{\langle P_\alpha,B_\alpha\rangle\colon\alpha<\mathfrak{c}\}$ of pairs $\langle P,B\rangle$ such that $P\subseteq\mathbb{R}$ is a perfect set, and $B\subseteq\omega^\omega\times\omega^\omega$ is a Borel non-meager set,
- $\{F_\alpha\colon\alpha<\mathfrak{c}\}$ of all Borel sets $F\subseteq\omega^\omega\times\omega^\omega\times\mathbb{R}$ such that the projection of F onto $\omega^\omega\times\omega^\omega$ is not meager.

By induction on $\alpha<\mathfrak{c}$ we will choose perfect sets $Q_\alpha\subseteq\omega^\omega$ and reals $x_\alpha^0,x_\alpha^1,w_\alpha\in\omega^\omega$, $y_\alpha,z_\alpha\in\mathbb{R}$, $v_\alpha\in\mathbb{R}\setminus\{0\}$ such that for $\alpha,\beta<\mathfrak{c}$:

- (i) $(\{w_\alpha\}\times Q_\alpha)\subseteq B_\alpha,\ z_\alpha\in P_\alpha$,
- (ii) $\langle x_\alpha^0,x_\alpha^1\rangle\notin\{w_\beta\}\times Q_\beta$ and if $\alpha\neq\beta$ then $\langle x_\alpha^0,x_\alpha^1\rangle\neq\langle x_\beta^0,x_\beta^1\rangle$, $w_\alpha\neq w_\beta$, and $v_\alpha\neq v_\beta$,
- (iii) $\langle x_{2\alpha}^0,x_{2\alpha}^1,y_{2\alpha}\rangle\in F_\alpha$,

(iv) $\langle r_\alpha, s_\alpha\rangle \in \{\langle x^0_\gamma, x^1_\gamma\rangle\colon \gamma \leqslant 2\alpha+1\} \cup \bigcup_{\gamma\leqslant 2\alpha+1}\{w_\gamma\}\times Q_\gamma$,

(v) $z_\alpha \neq v_\beta$, and if $\langle x^0_\alpha, x^1_\alpha\rangle \in M_\beta$ then $y_\alpha \neq v_\beta$.

Assume that we can carry out the construction so that the demands (i)–(v) are satisfied. Define a function $f : \omega^\omega \times \omega^\omega \to \mathbb{R}$ by:

$$f \restriction \big(\{w_\alpha\}\times Q_\alpha\big) \equiv z_\alpha \quad \text{and} \quad f\big(x^0_\alpha, x^1_\alpha\big) = y_\alpha \quad \forall \alpha < \mathfrak{c}.$$

It follows from the clauses (ii) and (iv), that the above condition defines a function on $\omega^\omega\times\omega^\omega$. This function has the required properties: ($\otimes^*_1$) holds by clause (iii), ($\otimes^*_2$) follows from clause (i), and ($\otimes^*_3$) is a consequence of (v) since $v_\alpha \notin f[M_\alpha]$.

So let us show how the construction may be carried out. Assume that we have defined $x^0_\beta, x^1_\beta, w_\beta \in \omega^\omega$, $y_\beta, z_\beta \in \mathbb{R}$, $v_\beta \in \mathbb{R}\setminus\{0\}$, and $Q_\beta \subseteq \omega^\omega$ for $\beta < \alpha$. First choose non-zero numbers $v_\alpha \in \mathbb{R} \setminus \bigcup\{\{v_\beta, y_\beta, z_\beta\}\colon \beta < \alpha\}$ and $z_\alpha \in P_\alpha \setminus \{v_\beta\colon \beta \leqslant \alpha\}$. The set B_α is not meager so we find $w_\alpha \in \omega^\omega \setminus \bigcup\{\{x^0_\beta, w_\beta\}\colon \beta<\alpha\}$ such that the section $(B_\alpha)_{w_\alpha}$ is not meager. Pick a perfect set $Q_\alpha \subseteq (B_\alpha)_{w_\alpha}$. Next we consider two separate cases to choose x^0_α, x^1_α, and y_α.

Case 1: *α is odd, say $\alpha = 2\alpha_0+1$.* Let $\langle x^0_\alpha, x^1_\alpha\rangle \in \omega^\omega\times\omega^\omega \setminus (\{\langle x^0_\beta, x^1_\beta\rangle : \beta<\alpha\} \cup \bigcup_{\beta\leqslant\alpha}\{w_\beta\}\times Q_\beta)$ be such that

$$\langle r_{\alpha_0}, s_{\alpha_0}\rangle \in \big\{\langle x^0_\beta, x^1_\beta\rangle\colon \beta\leqslant\alpha\big\} \cup \bigcup_{\beta\leqslant\alpha}\{w_\beta\}\times Q_\beta,$$

and let $y_\alpha \in \mathbb{R}\setminus\{v_\beta, z_\beta\colon \beta\leqslant\alpha\}$.

Case 2: *α is even, say $\alpha = 2\alpha_0$.* Look at the set F_{α_0}. If there is $y\in\mathbb{R}$ such that the section $(F_{\alpha_0})^y$ is not meager then take such an y as y_α. Pick

$$x^0_\alpha \in \omega^\omega \setminus \bigcup\big\{\{w_\beta, x^0_\beta\} : \beta<\alpha\big\} \setminus \{w_\alpha\}$$

such that $((F_{\alpha_0})^{y_\alpha})_{x^0_\alpha}$ is not meager and

$$\text{if } v_\beta = y_\alpha,\ \beta\leqslant\alpha, \quad \text{then } (M_\beta)_{x^0_\alpha} \text{ is meager.}$$

(Note that there is at most one β as above.) Next choose $x^1_\alpha \in \omega^\omega$ such that $\langle x^0_\alpha, x^1_\alpha, y_\alpha\rangle \in F_{\alpha_0}$ and $\langle x^0_\alpha, x^1_\alpha\rangle \notin M_\beta$ provided $v_\beta = y_\alpha$, $\beta\leqslant\alpha$.

So suppose now that for each $y\in\mathbb{R}$ the section $(F_{\alpha_0})^y$ is meager. Let

$$A \stackrel{\text{def}}{=} \big\{\langle x_0, x_1\rangle \in \omega^\omega\times\omega^\omega\colon (F_{\alpha_0})_{\langle x_0,x_1\rangle} \text{ is uncountable}\big\}.$$

It is an analytic set, so it has the Baire property. If A is not meager then we may choose $x^0_\alpha \in \omega^\omega \setminus \bigcup\{\{w_\beta, x^0_\beta\}\colon \beta<\alpha\}\setminus\{w_\alpha\}$ and $x^1_\alpha\in\omega^\omega$ and $y_\alpha \in \mathbb{R}\setminus\{v_\beta\colon \beta\leqslant\alpha\}$ such that $\langle x^0_\alpha, x^1_\alpha, y_\alpha\rangle \in F_{\alpha_0}$.

So assume that the set A is meager. Take basic open sets $U, V \subseteq \omega^\omega$ such that $\{\langle x_0, x_1\rangle \in U\times V\colon (F_{\alpha_0})_{\langle x_0,x_1\rangle} = \emptyset\}$ is meager. Note that the sets U, V and F_{α_0} satisfy the assumptions of Lemma 4.1. So we get a perfect set $P\subseteq U\times V$ such that $(F_{\alpha_0})_{\langle x_0,x_1\rangle} \neq \emptyset$ for every $\langle x_0,x_1\rangle \in P$ and that for distinct $\langle x'_0, x'_1\rangle, \langle x''_0, x''_1\rangle \in P$:

$$(F_{\alpha_0})_{\langle x'_0,x'_1\rangle} \cap (F_{\alpha_0})_{\langle x''_0,x''_1\rangle} = \emptyset \quad \text{and} \quad x'_0 \neq x''_0.$$

Now we may easily find $\langle x^0_\alpha, x^1_\alpha\rangle \in P$ and $y_\alpha \in \mathbb{R} \setminus \{v_\beta\colon \beta \leqslant \alpha\}$ such that

$$x^0_\alpha \notin \{w_\beta, x^0_\beta\colon \beta < \alpha\} \cup \{w_\alpha\} \quad \text{and} \quad \langle x^0_\alpha, x^1_\alpha, y_\alpha\rangle \in F_{\alpha_0}.$$

This finishes the inductive step of the construction. Checking that the demands (i)–(v) are satisfied is straightforward in all cases. (Note that it follows from $(\otimes^*_1) + (\otimes^*_3)$ that for each meager set $M \subset \omega^\omega \times \omega^\omega$, the set $\mathbb{R} \setminus f[M]$ is uncountable. One may easily guarantee that these sets are of size $\mathfrak{c}$, but there is no need for this.)

Thus the proof of the theorem is complete. □

Corollary 4.3. *There exists an almost continuous function $f:\mathbb{R} \to \mathbb{R}$ which has the strong Cantor intermediate value property but is not an extendability function.*

Proof. Let $f:\mathbb{R} \to \mathbb{R}$ be the function constructed in Theorem 4.2. The property $(\otimes_1)$ implies that the function f is almost continuous and $\mathrm{rng}(f)$ is dense in $\mathbb{R}^2$, and the property $(\otimes_2)$ guarantees that $f \in \mathrm{SCIVP}$. To show that f is not an extendability function we use the third property listed in Theorem 4.2. So by way of contradiction assume that $f \in \mathrm{Ext}$. Then, by Rosen [23], there is a meager set $M \subseteq \mathbb{R}$ such that

($\oplus$) if $g:\mathbb{R} \to \mathbb{R}$ and $g \restriction M = f \restriction M$ then g is an extendability function.

We may additionally demand that $\mathrm{cl}(f[M]) = \mathbb{R}$. (Just increase M if necessary.) Pick any $r^* \in f[M]$ and define a function $g:\mathbb{R} \to \mathbb{R}$ by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in M, \\ r^* & \text{otherwise.} \end{cases}$$

By $(\oplus)$, g is an extendability function (and thus Darboux) and by its definition $\mathrm{rng}(g) = f[M]$ is a dense subset of $\mathbb{R}$ (so it has to be $\mathbb{R}$). But $f[M] \neq \mathbb{R}$ (remember $(\otimes_3)$ of Theorem 4.2), a contradiction. □

One would hope for getting an additive function as in Theorem 4.2. Unfortunately this approach cannot work.

Proposition 4.4. *Suppose that $f:\mathbb{R} \to \mathbb{R}$ is an additive function such that*

(1) *for some perfect set $P \subseteq \mathbb{R}$, the restriction $f \restriction P$ is continuous,*

(2) *for each nowhere dense set $M \subseteq \mathbb{R}$, the image $f[M]$ is not $\mathbb{R}$.*

Then there is a closed set $F \subseteq \mathbb{R}^2$ such that $\mathrm{proj}[F] = \mathbb{R}$ *and* $f \cap F = \emptyset$.

Proof. Let $P \subseteq \mathbb{R}$ be a compact perfect set such that $f \restriction P$ is continuous. By Erdős, Kunen, and Mauldin [11], we find a compact perfect set Q of Lebesgue measure 0 (and so nowhere dense) such that $P + Q$ contains the interval $[0, 1]$. By the second assumption, we may pick a real $r \in \mathbb{R} \setminus f[Q]$. Let F be the subset of the plane $\mathbb{R}^2$ described by:

$$\langle x, y\rangle \in F \qquad \text{if and only if}$$

$$(\exists w \in P)(\exists z \in Q)(\exists m \in \mathbb{Z})(x = w + z + m \text{ and } y = f(w) + f(m) + r).$$

Since P, Q are compact and $f \restriction P$ is continuous, the set F is closed. By the choice of the perfect Q we know that $\mathrm{proj}[F] = \mathbb{R}$. Finally, suppose that $\langle x, y\rangle \in F \cap f$. Take $w \in P$, $z \in Q$ and $m \in \mathbb{Z}$ witnessing $\langle x, y\rangle \in F$. Then

$$f(w) + f(m) + r = y = f(x) = f(w + z + m) = f(w) + f(z) + f(m),$$

and hence $f(z) = r$, a contradiction with the choice of r. $\square$

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