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Darboux-like functions within the classes of Baire one, Baire two, and additive functions ☆

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Abstract

In the paper we present an exhaustive discussion of the relations between Darboux-like functions within the classes of Baire one, Baire two, Borel, and additive functions from $\mathbb{R}^n$ into $\mathbb{R}$. In particular we construct an additive extendable discontinuous function $f:\mathbb{R}\to\mathbb{R}$, answering a question of Gibson and Natkaniec (1996–97, p. 499), and show that there is no similar function from $\mathbb{R}^2$ into $\mathbb{R}$. We also describe a Baire class two almost continuous function $f:\mathbb{R}\to\mathbb{R}$ which is not extendable. This gives a negative answer to a problem of Brown, Humke, and Laczkovich (1988, Problem 1). (See also Problem 3.21 of Gibson and Natkaniec (1996–97).) © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The study of different generalizations of continuity of functions from $\mathbb{R}^n$ into $\mathbb{R}$ has a long history. In this paper we will be interested in the functions with some of these generalized continuities that are known under the common name of Darboux-like functions. The readers unfamiliar with their definitions can find them in the next section.

The basic relations between these classes, for the functions from $\mathbb{R}$ to $\mathbb{R}$, are given in the following chart, in which arrows $\to$ denote strict inclusions. Moreover, all other possible

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"natural intersection" inclusions (in a form of $\mathrm{AC} \cap \mathrm{CIVP} \subseteq \mathrm{Conn} \cap \mathrm{CIVP}$) obtained from different classes of this chart remain strict.

$$\begin{tikzcd}
 & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] \arrow[dr] & \mathrm{PR} \arrow[ur] & \\
 & & & & \mathrm{WCIVP} &
\end{tikzcd}$$

Chart 1.

The inclusions $\mathrm{C} \subset \mathrm{Ext}$, $\mathrm{Conn} \subset \mathrm{D} \subset \mathrm{PC}$, $\mathrm{SCIVP} \subset \mathrm{CIVP} \subset \mathrm{WCIVP}$, and $\mathrm{PR} \subset \mathrm{PC}$ are obvious from the definitions. The inclusions $\mathrm{Ext} \subset \mathrm{AC} \subset \mathrm{Conn}$ were proved by Stallings [27]. The inclusion $\mathrm{CIVP} \subset \mathrm{PR}$ was stated without the proof in [11]. The proof can be found in [10, Theorem 3.8]. The inclusion $\mathrm{Ext} \subset \mathrm{SCIVP}$ was proved by Rosen, Gibson, and Roush in [26]. An excellent discussion of this chart can be found in a recent survey by Gibson and Natkaniec [10, Section 3]. The examples concerning the properness of all intersection inclusions can be also found, in stronger versions, in Theorems 1.1, 1.2, and 1.5 stated below. Also, function F from Corollary 3.3 is the first simple ZFC example of almost continuous SCIVP function which is not extendable.

For the functions from $\mathbb{R}^n$ into $\mathbb{R}$ with $n > 1$ the classes from the lower part of Chart 1 are not defined. The relations between the classes in the upper part of the chart change to the following.

$$\begin{tikzcd}
 & & & \mathrm{AC}(\mathbb{R}^n) \\
\mathrm{C}(\mathbb{R}^n) \arrow[r] & \mathrm{Ext}(\mathbb{R}^n) = \mathrm{Conn}(\mathbb{R}^n) = \mathrm{PC}(\mathbb{R}^n) \arrow[r] & \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \arrow[ur] \arrow[dr] & \\
 & & & \mathrm{D}(\mathbb{R}^n)
\end{tikzcd}$$

Chart 2.

The inclusions $\mathrm{C}(\mathbb{R}^n) \subset \mathrm{Ext}(\mathbb{R}^n) \subset \mathrm{Conn}(\mathbb{R}^n)$ are obvious from the definitions. The inclusion $\mathrm{Conn}(\mathbb{R}^n) \subset \mathrm{Ext}(\mathbb{R}^n)$ was recently proved by Ciesielski, Natkaniec, and Wojciechowski [6]. The containment $\mathrm{Conn}(\mathbb{R}^n) \subset \mathrm{PC}(\mathbb{R}^n)$ was proved by Hamilton [15] and Stallings [27], and the inclusion $\mathrm{PC}(\mathbb{R}^n) \subset \mathrm{Conn}(\mathbb{R}^n)$ by Hagan [14]. (See also Whyburn [28] and [10, Theorem 8.1].) The relation $\mathrm{Conn}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n)$ was proved by Stallings [27]. The examples concerning the properness of the inclusions can be found, in the Baire class one, in Theorem 1.3.

The main goal of this paper is to discuss these two charts when we restrict the function in all these classes to the following four classes of functions: Baire one B_1, Baire two B_2, Borel $\mathcal{B}\mathrm{or}$, and additive functions Add. Notice that an intersection of any two of these classes is trivial, since $B_1 \subset B_2 \subset \mathcal{B}\mathrm{or}$ and $\mathrm{Add} \cap \mathcal{B}\mathrm{or} \subset \mathrm{C}$.

Theorem 1.1. *For the Baire one functions B_1 from $\mathbb{R}$ to $\mathbb{R}$ the following holds.*

$$\mathrm{C} \subsetneq \mathrm{Ext} = \mathrm{AC} = \mathrm{Conn} = \mathrm{D} = \mathrm{PC} = \mathrm{SCIVP} = \mathrm{CIVP} = \mathrm{PR} \subsetneq \mathrm{WCIVP}.$$

Proof. The proof of the equation $\mathrm{Ext} \cap B_1 = \mathrm{PC} \cap B_1$ can be found in [3]. This equation and Chart 1 imply all the other equations. The properness of the inclusions is justified as follows.

- $B_1 \cap \mathrm{D} \setminus \mathrm{C} \neq \emptyset$. It is witnessed by the function $f_0 : \mathbb{R} \to \mathbb{R}$ defined by $f_0(x) = \sin(1/x)$ for $x \neq 0$ and $f_0(0) = 0$. (See [22, Example 1.1].)
- $B_1 \cap \mathrm{WCIVP} \setminus \mathrm{PC} \neq \emptyset$. It is witnessed by the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ for $x \neq 0$ and $g(0) = -1$. □

Theorem 1.2. *The classes from Chart* 1 *restricted to either Baire two functions B_2 or Borel functions from $\mathbb{R}$ to $\mathbb{R}$ leads to the following chart. Moreover, all possible "natural intersection" inclusions obtained from different classes of this chart remain strict.*

$$
\begin{array}{ccc}
 & & \mathrm{PR} \longrightarrow \mathrm{PC} \\
\mathrm{C} \longrightarrow \mathrm{Ext} \longrightarrow \mathrm{AC} \longrightarrow \mathrm{Conn} \longrightarrow \mathrm{D} \longrightarrow \mathrm{SCIVP} = \mathrm{CIVP} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & & \mathrm{WCIVP}
\end{array}
$$

Proof. To see that the inclusion $\mathrm{D} \subseteq \mathrm{SCIVP}$ holds in the class of Borel functions let $x < y$ and K be a perfect set between $f(x)$ and $f(y)$. Since f is Darboux, $f^{-1}(K) \cap (x, y)$ is an uncountable Borel set. Thus, it contains a perfect subset C_0. Moreover, we can find a perfect set $C \subset C_0$ for which $f \restriction C$ is continuous. Similarly we can argue that the inclusion $\mathrm{CIVP} \subseteq \mathrm{SCIVP}$ holds in the class of Borel functions.

The properness of the inclusions is justified as follows.

- $B_2 \cap \mathrm{Ext} \setminus \mathrm{C} \neq \emptyset$. See Theorem 1.1.
- $B_2 \cap \mathrm{AC} \setminus \mathrm{Ext} \neq \emptyset$. See Corollary 3.3.
- $B_2 \cap \mathrm{Conn} \setminus \mathrm{AC} \neq \emptyset$. See Brown [2] or Jastrzębski [17].
- $B_2 \cap \mathrm{D} \setminus \mathrm{Conn} \neq \emptyset$. See Brown [2].
- $B_2 \cap \mathrm{SCIVP} \setminus \mathrm{D} \neq \emptyset$. See Example 3.5.
- $B_2 \cap \mathrm{WCIVP} \cap \mathrm{PR} \setminus \mathrm{CIVP} \neq \emptyset$. Let $\{F_q : q \in \mathbb{Q}\}$ be a family of pairwise disjoint $\mathfrak{c}$-dense F_σ sets. Then $f = \sum_{q \in \mathbb{Q} \cap (0,1)} q\, \chi_{F_q}$ is as desired.
- $B_2 \cap \mathrm{WCIVP} \cap \mathrm{PC} \setminus \mathrm{PR} \neq \emptyset$. It is witnessed by $g = f + 2\chi_D$, where f is as above and D is a countable dense subset of F_2.
- $B_2 \cap \mathrm{WCIVP} \setminus \mathrm{PC} \neq \emptyset$. See Theorem 1.1. □

Restricting functions from Chart 2 to Baire one functions has a lot simpler solution.

Theorem 1.3. *For the Baire one functions $B_1(\mathbb{R}^n)$ from $\mathbb{R}^n$ to $\mathbb{R}$, $n > 1$, Chart* 2 *remains unchanged.*

Proof. The properness of all the inclusions, as well as of their other possible combinations, is justified by the following facts.

- $B_1(\mathbb{R}^n) \cap \mathrm{Conn}(\mathbb{R}^n) \setminus \mathrm{C}(\mathbb{R}^n) \neq \emptyset$. It is witnessed by a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \sin(1/(x^2 + y^2))$ for $\langle x, y\rangle \neq \langle 0, 0\rangle$ and $f(0, 0) = 0$. It is in $\mathrm{PC}(\mathbb{R}^2)$ straight from the definition.

- $B_1(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \setminus \mathrm{Conn}(\mathbb{R}^n) \neq \emptyset$. Rosen, Gibson and Roush [26, Example 1] proved that a function $f:[-1,1]\times[0,1]\to[-1,1]$ given by $f(x,y)=\sin(1/y)$ for $y>0$ and $f(x,0)=x$ is Baire one, almost continuous, Darboux, but not connectivity. It is easy to extend it to a finite support function on $\mathbb{R}^2$ with the same properties.
- $B_1(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n) \neq \emptyset$. It is justified by the function $F(x,y)=f_0(x)$, where f_0 is a function from Theorem 1.1. (See Natkaniec [22, Example 1.7] or [23, Example 1.1.9].)
- $B_1(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n) \neq \emptyset$. Let $f(x)=\sin(1/x)$ for $x\neq 0$ and $f(0)=1$, and let $F:[-1,1]^2\to[-1,1]$ be given by the formula $F(x,y)=yf(x)$. It was proved by Natkaniec [23, Example 1.1.10] that F is Baire one, almost continuous, and not Darboux. It is easy to extend F to a function $\overline{F}:\mathbb{R}^2\to[0,1]$ with compact support while preserving these properties. □

The study of classes of additive functions from Charts 1 and 2 were initiated by Banaszewski [1]. (See also [10, Section 5].) In this direction we have the following results.

Theorem 1.4. *For the additive functions* $\mathrm{Add}(\mathbb{R}^n)$ *from* $\mathbb{R}^n$ *to* $\mathbb{R}$, $n>1$, *Chart* 2 *changes as follows*:

$$\begin{tikzcd}[column sep=small] & & \mathrm{AC}(\mathbb{R}^n) \\ \mathrm{C}(\mathbb{R}^n)=\mathrm{Ext}(\mathbb{R}^n)=\mathrm{Conn}(\mathbb{R}^n)=\mathrm{PC}(\mathbb{R}^n)=\mathrm{AC}(\mathbb{R}^n)\cap\mathrm{D}(\mathbb{R}^n) \arrow[urr] \arrow[drr] & & \\ & & \mathrm{D}(\mathbb{R}^n)\end{tikzcd}$$

Proof. The inclusion $\mathrm{Add}(\mathbb{R}^n)\cap\mathrm{AC}(\mathbb{R}^n)\cap\mathrm{D}(\mathbb{R}^n)\subset\mathrm{C}(\mathbb{R}^n)$ is proved in Theorem 4.8. The properness of the inclusions is justified by the following facts.

- $\mathrm{Add}(\mathbb{R}^n)\cap\mathrm{AC}(\mathbb{R}^n)\setminus\mathrm{D}(\mathbb{R}^n)\neq\emptyset$. See Example 4.9.
- $\mathrm{Add}(\mathbb{R}^n)\cap\mathrm{D}(\mathbb{R}^n)\setminus\mathrm{AC}(\mathbb{R}^n)\neq\emptyset$. See Example 4.10. □

Theorem 1.5. *For the additive functions* Add *from* $\mathbb{R}$ *to* $\mathbb{R}$ *we have the equation* PR $=$ WCIVP. *The other inclusions of Chart* 1 *remain unchanged, except possibly for the inclusion* AC $\subset$ Conn. *Thus, we have*

$$\begin{tikzcd} & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\ \mathrm{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\ & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] & \end{tikzcd}$$

Moreover all possible "natural intersection" inclusions obtained from the different classes of this chart and not involving AC $\subset$ Conn *remain strict. The inclusion* Add$\cap$CIVP$\cap$AC $\subset$ Conn *is strict if union of less than continuum many meager subsets of* $\mathbb{R}$ *is meager in* $\mathbb{R}$.

Proof. The properness of all the inclusions is justified by the following facts.

- Add $\cap$ Ext $\setminus$ C $\neq\emptyset$. See Corollary 4.4.

- $\mathrm{Add} \cap \mathrm{SCIVP} \cap \mathrm{AC} \setminus \mathrm{Ext} \neq \emptyset$. See Ciesielski and Rosłanowski [8]. Compare also Ciesielski [5, Theorem 3.1].
- $\mathrm{Add} \cap \mathrm{SCIVP} \cap \mathrm{D} \setminus \mathrm{Conn} \neq \emptyset$. See Example 5.3. (An example of a function from $\mathrm{Add} \cap \mathrm{D} \setminus \mathrm{Conn}$ was earlier given in [1].)
- $\mathrm{Add} \cap \mathrm{PC} \setminus \mathrm{D} \neq \emptyset$. See Example 5.2. (An example of a function from $\mathrm{Add} \cap \mathrm{CIVP} \setminus \mathrm{D}$ was earlier given in [1].)
- $\mathrm{Add} \cap \mathrm{AC} \cap \mathrm{CIVP} \setminus \mathrm{SCIVP} \neq \emptyset$. See Ciesielski [5, Theorem 4.1].
- $\mathrm{Add} \cap \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP} \neq \emptyset$. See Example 5.1.
- $\mathrm{Add} \cap \mathrm{AC} \setminus \mathrm{PR} \neq \emptyset$. See Banaszewski [1].
- $\mathrm{Add} \cap \mathrm{CIVP} \cap \mathrm{Conn} \setminus \mathrm{AC}$. Such a function, under the assumption that union of less than continuum many meager subsets of $\mathbb{R}$ is meager, has been constructed by Ciesielski and Rosłanowski [8]. (Example for $\mathrm{Add} \cap \mathrm{Conn} \setminus \mathrm{AC}$ requires only that union of less than continuum many meager subsets of $\mathbb{R}$ does not cover $\mathbb{R}$.) □

The following questions, which are variants of Banaszewski's question [10, Question 5.5], remain open.

Problem 1.1.
(1) Does there exist a ZFC example of an additive connectivity function $f:\mathbb{R} \to \mathbb{R}$ which is not almost continuous?
(2) Does there exist an additive connectivity function $f:\mathbb{R} \to \mathbb{R}$ with SCIVP property which is not almost continuous?

2. Definitions and notation

Our terminology is standard and follows [4]. We consider only real-valued functions of one or more real variables. No distinction is made between a function and its graph. A restriction of a function $f: X \to Y$ to a set $A \subset X$ is denoted by $f \restriction A$. Symbol χ_A will be used for a characteristic function of a subset A of a fixed space X. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the set of all real and rational numbers, respectively. We will consider $\mathbb{R}^n$ as linear spaces over $\mathbb{Q}$. In particular, for $X \subset \mathbb{R}^n$ we will use the symbol $\mathrm{LIN}_{\mathbb{Q}}(X)$ to denote the smallest linear subspace of $\mathbb{R}^n$ over $\mathbb{Q}$ that contains X. Recall also that if $D \subset \mathbb{R}^n$ is linearly independent over $\mathbb{Q}$ and $f: D \to \mathbb{R}$ then

$$F = \mathrm{LIN}_{\mathbb{Q}}(f) \subset \mathbb{R}^{n+1}$$

is an additive function (see definition below) from $\mathrm{LIN}_{\mathbb{Q}}(D)$ into $\mathbb{R}$ which extends f. Any linear basis of $\mathbb{R}$ over $\mathbb{Q}$ will be referred as a *Hamel basis*.

The ordinal numbers will be identified with the sets of all their predecessors and cardinals with the initial ordinals. In particular $2 = \{0, 1\}$ and the first infinite ordinal ω number is equal to the set of all natural numbers $\{0, 1, 2, \ldots\}$. The family of all functions from a set X into Y is denoted by Y^X. In particular 2^n will stand for the set of all sequences $s: \{0, 1, 2, \ldots, n-1\} \to \{0, 1\}$, while $2^{<\omega} = \bigcup_{n<\omega} 2^n$ is the set of all finite sequences into 2. The symbol $|X|$ stands for the cardinality of a set X. The cardinality of $\mathbb{R}$ is denoted by

$\mathfrak{c}$ and referred as *continuum*. A set $S \subset \mathbb{R}$ is said to be $\mathfrak{c}$-*dense* if $|S \cap (a, b)| = \mathfrak{c}$ for every $a < b$. The closure of a set $A \subset \mathbb{R}^n$ is denoted by $\mathrm{cl}(A)$, its boundary by $\mathrm{bd}(A)$, and its diameter by $\mathrm{diam}(A)$.

We will use also the following terminology [10]. For $X \subseteq \mathbb{R}^n$ a function $f : X \to \mathbb{R}$ is:

- *additive* if X is closed under the addition and $f(x + y) = f(x) + f(y)$ for every $x, y \in X$;
- *Darboux* if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset K of X;
- *almost continuous* (in sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of f contains also a continuous function from X to $\mathbb{R}$ [27];
- *connectivity* function if the graph of $f \restriction Z$ is connected in $Z \times \mathbb{R}$ for any connected subset Z of X;
- *extendability* function provided there exists a connectivity function $F : X \times [0, 1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$.

The classes of these functions are denoted by $\mathrm{Add}(X)$, $\mathrm{D}(X)$, $\mathrm{AC}(X)$, $\mathrm{Conn}(X)$, $\mathrm{Ext}(X)$, and $\mathrm{PC}(X)$, respectively. The class of continuous functions from X into $\mathbb{R}$ is denoted by $\mathrm{C}(X)$. We will drop the index X if $X = \mathbb{R}$.

Recall also that if the graph of $f : \mathbb{R} \to \mathbb{R}$ intersects every closed subset B of $\mathbb{R}^2$ which projection $\mathrm{pr}(B)$ onto the x-axis has nonempty interior then f is almost continuous. (See, e.g., [22].) Similarly, if the graph of $f : \mathbb{R} \to \mathbb{R}$ intersects every compact connected subset B of $\mathbb{R}^2$ with $|\mathrm{pr}(B)| > 1$ then f is connectivity. This follows from the following well-known fact.

Fact 2.1. *If $S \subset \mathbb{R}^2$ disconnects $\mathbb{R}^2$ then it contains a nontrivial compact connected subset.*

A function $f : \mathbb{R} \to \mathbb{R}$ has:

- *Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$;
- *strong Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous;
- *weak Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set C between x and y such that $f[C] \subset (f(x), f(y))$;
- *perfect road* if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral (i.e., two-sided) limit point for which $f \restriction P$ is continuous at x.

The above classes of these functions are denoted by CIVP, SCIVP, WCIVP, and PR, respectively.

3. Almost continuous Baire two class function which is not extendable

The main example described in this section answers Problem 2 and the main part of Problem 1 from [3], as well as Problem 3.21 from [10].

Let $C \subset [0,1]$ be the ternary Cantor set and let $\mathcal{J}$ be the family of all component intervals of $[0,1] \setminus C$. We put

$$\mathcal{J}_0 = \{J \in \mathcal{J}\colon \text{ the length of } J \text{ is } 3^{-n} \text{ with } n < \omega \text{ even}\}$$

and

$$\mathcal{J}_1 = \{J \in \mathcal{J}\colon \text{ the length of } J \text{ is } 3^{-n} \text{ with } n < \omega \text{ odd}\}.$$

Let $\{(a_n, b_n)\colon n < \omega\}$ and $\{(c_n, d_n)\colon n < \omega\}$ be the enumerations of $\mathcal{J}_0$ and $\mathcal{J}_1$, respectively. Define function $f : [0,1] \to [0,1]$ in the following way:

- for every $n < \omega$ we put $f(a_n) = f(d_n) = 0$, $f(b_n) = f(c_n) = 1$, and extend it linearly on $[a_n, b_n]$ and $[c_n, d_n]$;
- for all other x's we put $f(x) = 0$.

Theorem 3.1. *The function f is almost continuous, Baire class two, but not extendable.*

Proof. We will start with showing that f is not extendable. By way of contradiction, assume that f can be extendable, that is, that there is a connectivity function $F : [0,1]^2 \to [0,1]$ with $F(x,0) = f(x)$ for all $x \in [0,1]$. Thus F is peripherally continuous. We will deduce from this that there exists a perfect set $P \subset C \times \{0\}$ on which F is constantly equal to 1, which evidently contradicts our definition of f.

We will define the following families: $\langle p_s \in C\colon s \in 2^{<\omega}\rangle$, $\langle B_s \subset [0,1]^2\colon s \in 2^{<\omega}\rangle$, and $\langle U_s \subset [0,1]^2\colon s \in 2^{<\omega}\rangle$ such that the following conditions hold for every $s \in 2^n$ and different $t, t' \in 2^{n+1}$ extending s.

(i) $\langle p_s, 0\rangle \in U_s$, $f(p_s) = 1$, and U_s is open with a diameter at most 2^{-n}.
(ii) B_s is closed, connected, and $F[B_s] \subset [1 - 2^{-n}, 1]$.
(iii) $\mathrm{cl}(U_t) \cup B_t \subset U_s$, $\langle p_s, 0\rangle \notin U_t$, and $B_t \cap B_s \neq \emptyset$.
(iv) $\mathrm{cl}(U_t) \cap \mathrm{cl}(U_{t'}) = \emptyset$.

For $s = \emptyset \in 2^0$ we put $U_s = B_s = [0,1]^2$ and choose an arbitrary p_s with $f(p_s) = 1$. Farther, the construction goes by the induction on the length n of $s \in 2^{<\omega}$. Thus, assume that for some $s \in 2^n$ the sets B_s, U_s, and the point p_s are already chosen. Let t and t' be different sequences from 2^{n+1} extending s. To choose B_t, p_t, and U_t we proceed as follows.

Note that p_s is an endpoint of some $J \in \mathcal{J}$ since $f(p_s) = 1$. If p_s is a left endpoint of J we put $I = [0, p_s)$. Otherwise we put $I = (p_s, 1]$. Choose an $\varepsilon > 0$ less than the diameters of I and B_s and such that

$$\text{for every } J \in \mathcal{J} \text{ with } J \subset I \text{ if the distance from } J \text{ to } p_s \text{ is less than } \varepsilon \text{ then } J \times \{0\} \subset U_s. \tag{$*$}$$

Let $W_s \subset [0,1]^2$ be an open neighborhood of $\langle p_s, 0\rangle$ with diameter less than ε and such that $F[\mathrm{bd}(W_s)] \subset [1 - 2^{-n}, 1]$. It exists since F is peripherally continuous at $\langle p_s, 0\rangle$ and

$F(p_s,0)=f(p_s)=1$. Without loss of generality we can assume that $\mathrm{bd}(W_s)$ is connected. (Replacing W_s by its component, if necessary, we can assume that W_s is connected. Then we can increase W_s to $[0,1]^2\setminus V_s$, where V_s is an "unbounded" component of $[0,1]^2\setminus W_s$, that is the one which contains the boundary of $[0,1]^2$. This decreases $\mathrm{bd}(W_s)$ and makes it connected.) Note that the choice of ε guarantees that $\mathrm{bd}(W_s)\cap B_s\neq\emptyset\neq\mathrm{bd}(W_s)\cap(I\times\{0\})$ since B_s and $I\times\{0\}$ are connected and $\mathrm{bd}(W_s)$ disconnects $[0,1]^2$. Let $z\in I$ be such that $\langle z,0\rangle\in\mathrm{bd}(W_s)\cap(I\times\{0\})$. Since $F(z,0)\in F[\mathrm{bd}(W_s)]\subset[1-2^{-n},1]\subset(0,1]$ there exists a $J\in\mathcal{J}$ such that $z\in\mathrm{cl}(J)$. Let p_t be the endpoint of $\mathrm{cl}(J)$ for which $f(p_t)=1$. By $(*)$ we have $\langle p_t,0\rangle\in U_s$. The set B_t is defined as a union of $\mathrm{bd}(W_s)$ and a closed segment joining $\langle z,0\rangle$ and $\langle p_t,0\rangle$. The open neighborhood U_t of p_t is chosen such that its diameter is at most $2^{-(n+1)}$ and that $p_s\notin\mathrm{cl}(U_t)\subset U_s$. It is easy that conditions (i)–(iii) are satisfied.

To choose $B_{t'}$, $p_{t'}$, and $U_{t'}$ we replace U_s with $U_s'=U_s\setminus\mathrm{cl}(U_t)$ and repeat the process described above. This finishes the inductive construction.

Now, to finish the argument take an arbitrary $s\in 2^\omega$ and note that by (i) and (iii) the limit $\lim_{n\to\infty}\langle p_{s\restriction n},0\rangle$ exists and is equal to a point p_s which belongs to $C\times\{0\}$. Also, by (ii), the set

$$B_s=\bigcup_{0<n<\omega}B_{s\restriction n}$$

is connected and p_s is its only accumulation point. Since $P=\{p_s\colon s\in 2^\omega\}$ is evidently equal to a perfect set $\bigcap_{n<\omega}\bigcup\{\mathrm{cl}(U_t)\colon t\in 2^n\}$ it is enough to prove that $F(p_s)=1$ for every $s\in 2^\omega$. But if $F(p_s)\neq 1$ then $F(p_s)=0$, since $p_s\in C\times\{0\}$. Take $\varepsilon>0$ is less then the diameter of B_s and let U be an open neighborhood of p_s of diameter less than ε and such that $F[\mathrm{bd}(U)]\subset[0,1/2)$. Then for a point $w\in B_s\cap\mathrm{bd}(U)$ we have $F(w)\in[0,1/2)\cap[1/2,1]$, a contradiction. This finishes the proof that f is not extendable.

Next we will show that f is almost continuous. Let G be an open set contained in $[0,1]^2$ containing the graph of f. For every $x\in[0,1]$ there exists an interval (a_x,b_x) such that

- $x\in(a_x,b_x)$,
- $f(a_x)=f(b_x)=0$, and
- there is a continuous function $g_x:[0,1]\to\mathbb{R}$ with $g_x\restriction[a_x,b_x]\subset G$ and such that $g_x(t)=0$ for $t\notin(a_x,b_x)$.

Indeed, if $f(x)=0$ then it is easy to find (a_x,b_x) for which $g_x\equiv 0$ works. If $f(x)\neq 0$ then $x\in J$ for some $J\in\mathcal{J}$, say $J=(A,B)$. Assume that f is increasing on J, the other case is similar. Then $f(B)=1$. Find an interval $J'=(C,D)\in\mathcal{J}$ on which function f is decreasing and such that $[B,C]\times\{1\}\subset G$. Put $a_x=A$, $b_x=D$, define $g_x(B)=g_x(C)=1$, $g_x(0)=g_x(a_x)=g_x(b_x)=g_x(1)=0$, and extend g_x in a linear way on each interval with these endpoints. Thus g_x has a hat shape.

Now choose a finite subcover $\{(a_{x_i},b_{x_i})\colon i<n\}$, with $n<\omega$, of the cover $\{(a_x,b_x)\colon x\in[0,1]\}$ of the interval $[0,1]$. Then the function

$$g(x)=\max\bigl\{g_{x_i}(x)\colon i<n\bigr\}$$

is continuous and $g\subset G$. This ends the proof that f is almost continuous.

To see that f is of Baire class two it is enough to notice that the preimage of every open (and closed) set U is a countable union of F_σ (closed) sets $f^{-1}(U) \cap \mathrm{cl}(J)$ with $J \in \mathcal{J}$ and, possible, of a G_δ set $C \setminus \bigcup\{\mathrm{cl}(J)\colon J \in \mathcal{J}\}$. □

In [3, Problem 1] the authors asked also whether the function as above can be in the class J_1 of all functions (from $\mathbb{R}$ or $[0,1]$ into $\mathbb{R}$) that are pointwise limits of functions which have only discontinuities of the first class, that is, these functions for which both one sided limits exist at each point. Clearly $B_1 \subset J_1 \subset B_2$. Our function gives also an answer to this question.

Theorem 3.2. *There exists a J_1 function $f:[0,1] \to [0,1]$ which is almost continuous but not extendable.*

Proof. Let f be a function from Theorem 3.1 and let χ_D stand for a characteristic function of $D \subset [0,1]$. If $\{J_n\colon n < \omega\}$ is an enumeration of $\{\mathrm{cl}(J)\colon J \in \mathcal{J}\}$ and $D_n = \bigcup_{i<n} J_i$ then f is a pointwise limit of functions $f\chi_{D_n}$. Thus, f is in J_1. □

Corollary 3.3. *There exists a J_1 function $F:\mathbb{R} \to \mathbb{R}$ which is almost continuous but not extendable.*

Proof. Extend the function f from Theorem 3.2 to F by putting $F(x) = 0$ for all $x \in \mathbb{R} \setminus [0,1]$. □

The main core of the proof that the function f from Theorem 3.1 is not extendable is that the set $f^{-1}(1)$ is countable. The next proposition shows that it is essentially the only obstacle for f to be extendable, in a sense that we can redefine f on a subset of C to get an extendable function.

Proposition 3.4. *If f is from Theorem 3.1 then there exists a meager F_σ subset B of $C_0 = C \setminus \bigcup\{\mathrm{cl}(J)\colon J \in \mathcal{J}\}$ such that $f_0 = f + \chi_B$ is extendable.*

Proof. Gibson and Roush [12] proved that a function $g:[0,1] \to [0,1]$ is extendable if and only if there exists a sequence $\langle\langle I_n, J_n\rangle\colon n < \omega\rangle$ of pairs of open intervals, called a *PI family for g*, such that

(a) $\lim_{n\to\infty} \mathrm{diam}(I_n) = 0$,
(b) $g[\mathrm{bd}(I_n)] \subset J_n$ for every $n < \omega$,
(c) for every $x \in [0,1]$ and $\varepsilon > 0$ there exists an $n < \omega$ such that $x \in I_n$, $g(x) \in J_n$, and $\max\{\mathrm{diam}(I_n), \mathrm{diam}(J_n)\} < \varepsilon$,
(d) for every $n < m < \omega$ if the sets $I_n \cap I_m$, $I_n \setminus I_m$, and $I_m \setminus I_n$ are nonempty then $J_n \cap J_m \neq \emptyset$.

It was also noticed in [6] that in (c) it is enough to consider only discontinuity points x of g. In what follows we will construct a PI family for our future f_0.

Let T be the set of all endpoints of all intervals $J \in \mathcal{J}$ and let $\{t_n\colon\ n<\omega\}$ be an enumeration of T. We construct by induction on $n<\omega$ the finite families F_n of triples $\langle t, I, J\rangle$ such that

(1) I and J are open intervals in $[0,1]$, $t \in T \cap I$, and $f(t) \in J$;

(2) $f[\mathrm{bd}(I)] \subset (0,1) \cap J$ and $\mathrm{diam}(J) \leqslant 2^{-n}$;

(3) $I \cap I' = \emptyset$ for different $\langle t, I, J\rangle, \langle t', I', J'\rangle \in F_n$;

(4) $C \subset \bigcup\{I\colon\ \langle t, I, J\rangle \in F_n\}$ and $\{t_i\colon\ i<n\} \subset \{t\colon\ \langle t, I, J\rangle \in F_n\}$;

(5) if $n>0$ then for every $\langle t, I, J\rangle \in F_n$ there exists $\langle t_0, I_0, J_0\rangle \in F_{n-1}$ such that $I \subset I_0$.

The induction can be started with $F_0 = \{\langle t_0, (0,1), (0,1)\rangle\}$, and can be easily carried through since C is compact zero-dimensional and the sets $f^{-1}(1)$ and $T \cap f^{-1}(0)$ are dense in C.

Now, let $\{\langle t_n, I_n, J_n\rangle\colon\ n<\omega\}$ be an enumeration of $\bigcup_{n<\omega} F_n$. We claim that the sequence $\langle\langle I_n, J_n\rangle\colon n<\omega\rangle$ is a PI family for $f_0 = f + \chi_B$ for an appropriately chosen set B.

Clearly (3) and (5) imply that condition (d) is satisfied in void. Condition (a) can be deduced from the density of T in C and (1)–(3). (b) For $g = f_0$ is implied by (2) and the fact that $f_0(x) = f(x)$ for $x \in [0,1] \setminus C$. Similarly (c) for the points $x \in T$ is implied by (4) and $f_0 \restriction T = f \restriction T$. To finish the proof it is enough to show that (c) holds for points $x \in C_0$ for an appropriate choice of B. But every $x \in C_0$ the set $S_x = \{k<\omega\colon\ x \in I_k\}$ is infinite. Let B be the set of all those points $x \in C_0$ for which the set $\{k \in S_x\colon\ f(t_k) = 0\}$ is finite. Then for every $x \in C_0$ the set $\{k \in S_x\colon\ f_0(x) \in J_k\}$ is infinite proving (b).

The fact that B is a meager F_σ subset of C_0 is left as an exercise. □

Example 3.5. There exists an $f \in B_2 \cap \mathrm{SCIVP} \setminus \mathrm{D}$.

Proof. Define g from $[-1,1]$ onto $(0,1]$ by $g(x) = (x^2-1)\sin^2(1/x) + 1$ for $x \neq 0$ and $f(0) = 1$. Note that $g(-1) = g(1) = 1$ and that g is SCIVP. For each component J of $[0,1] \setminus C$ of length $1/3^{-n}$ define $f \restriction \mathrm{cl}(J)$ as $(-1)^n g \circ h_J$, where h_J is an increasing linear function with $h_J[\mathrm{cl}(J)] = [-1,1]$. For all other points x we put $f(x) = 1$. Note that $f[\mathbb{R}] = [-1,0) \cup (0,1]$ and that preimage of every open set is a union of a G_δ and an F_σ set. So f is Baire two and not Darboux. To see that it is SCIVP take $x<y$ and a perfect set K between $f(x)$ and $f(y)$. We have to find a perfect set $P \subset (x,y)$ on which f is continuous and with $f[P] \subset K$. If both x and y belong to the closure of the same component of C then the existence of K follows from SCIVP property of g. Otherwise there exist two components J_0 and J_1 of C between x and y with $f[J_0 \cup J_1] = [-1,0) \cup (0,1]$. This we can choose appropriate $K \subset J_0 \cup J_1$. □

4. Additive extendable discontinuous function

Let $h \in \mathrm{Ext}(\mathbb{R})$. We say that a set $G \subset \mathbb{R}$ is *h-negligible for a function h* provided $f \in \mathrm{Ext}$ for every function $f:\mathbb{R} \to \mathbb{R}$ for which $f = h$ on a set $\mathbb{R} \setminus G$.

Lemma 4.1 (Rosen [25], Ciesielski and Recław [7]). *There exists an extendable function $h:\mathbb{R}\to\mathbb{R}$ such that some dense G_δ-set $G\subset\mathbb{R}$ is h-negligible.*

Lemma 4.2. *If $g:\mathbb{R}\to\mathbb{R}$ is homeomorphism, $h\in\mathrm{Ext}$, and G is h-negligible then $h\circ g^{-1}\in\mathrm{Ext}$ and $g[G]$ is $(h\circ g^{-1})$-negligible.*

Proof. This is a simple corollary from [18, Lemma 2.2]. (See also [24].) □

Proposition 4.3. *For every $\mathfrak{c}$-dense meager F_σ-set $F\subset\mathbb{R}$ there exists an extendable function $f:\mathbb{R}\to\mathbb{R}$ such that $\mathbb{R}\setminus F$ is f-negligible.*

Proof. Let h and G be as in Lemma 4.1. By Lemma 4 of [13] there exists a homeomorphism $g:\mathbb{R}\to\mathbb{R}$ such that $g[\mathbb{R}\setminus G]\subset F$. Then $\mathbb{R}\setminus F\subset g[G]$ so, by Lemma 4.2, function $f=h\circ g^{-1}$ is extendable and the set $\mathbb{R}\setminus F$ is f-negligible. □

Corollary 4.4. *There exists an additive extendable function $g:\mathbb{R}\to\mathbb{R}$ with a dense graph. In particular $g\in\mathrm{Add}\cap\mathrm{Ext}\setminus\mathrm{C}$.*

Proof. Let F be $\mathfrak{c}$-dense meager F_σ-set which is linearly independent over $\mathbb{Q}$. Such a set can be easily constructed from a linearly independent perfect set, which description can be found in [20, Theorem 2, Chapter XI, Section 7]. By Proposition 4.3 there exists an $f\in\mathrm{Ext}$ such that $\mathbb{R}\setminus F$ is f-negligible. In particular $f\restriction F$ must be discontinuous. Let g be a linear extension of $f\restriction F$. Then $g\in\mathrm{Ext}$ since $\mathbb{R}\setminus F$ is f-negligible. Clearly g is additive and discontinuous, so it has a dense graph. □

Next we prove that $\mathrm{Add}(\mathbb{R}^n)\cap\mathrm{AC}(\mathbb{R}^n)\cap\mathrm{D}(\mathbb{R}^n)\subset\mathrm{C}(\mathbb{R}^n)$ for $n>1$. Its proof will be based on the following two propositions, the first of which was proved by Lipiński [19]. (See also Maliszewski and Natkaniec [21].) This fact was noticed independently by the authors of this paper and our proof is enclosed below.

Proposition 4.5. *If $f:\mathbb{R}\to\mathbb{R}$ is discontinuous then $F:\mathbb{R}^2\to\mathbb{R}$ given by $F(x,y)=f(x)$ is not almost continuous.*

Proof. Assume that $f:\mathbb{R}\to\mathbb{R}$ is discontinuous at some point x. Taking a translation of a graph of f, if necessary, we can assume that $x=0$ and $f(0)=0$. So, there exists a sequence $\{x_n\}_{n<\omega}$ converging to 0 such that $\lim_{n\to\infty}f(x_n)=L_0\neq 0$. Multiplying f by $3/L_0$, if necessary, we can assume that $L_0=3$. We will also assume that $f(x_n)>2$ for all $n<\omega$.

Consider the closed set

$$A=\bigl\{\langle x,y,z\rangle\colon\ x=x_n \text{ for some } n<\omega,\ y\in\mathbb{R},\ \text{and } z\leqslant 2\bigr\}\cup\bigl(\{0\}\times\mathbb{R}^2\bigr)$$

and let $G=\mathbb{R}^3\setminus A$. Then G is an open set containing all the graph of F except of a line $L=\{0\}\times\mathbb{R}\times\{0\}$. Let

$$H=\bigl\{\langle x,y,z\rangle\in\mathbb{R}^3\colon\ x^2+z^2<\mathrm{e}^{-y^2}\bigr\}.$$

Then H is an open and contains L, so $U = G \cup H$ contains F. We will show that U does not contain a graph of a continuous function.

By way of contradiction assume that there exists a continuous $g : \mathbb{R}^2 \to \mathbb{R}$ with $g \subset U$. Then $\langle 0, 0, g(0,0)\rangle \in H$. In particular, $g(0,0) \in (-1, 1)$. So, by the continuity of g, there exists an $n < \omega$ such that $g(x_n, 0) \in (-1, 1)$. We claim that $g \restriction (\{x_n\} \times \mathbb{R})$ is discontinuous. Indeed, notice that

$$\left(\{x_n\} \times \mathbb{R}^2\right) \cap U \subset \{x_n\} \times (G_0 \cup H_0),$$

where $G_0 = \mathbb{R} \times (2, \infty)$, $H_0 = (-b, b) \times (-\infty, 1)$, and $b > 0$ is such that $\mathrm{e}^{-(|x_n|+b)^2} = 1$. Moreover, $\langle x_n, 0, g(x_n, 0)\rangle \in \{x_n\} \times H_0$. But clearly there is no continuous function on $\mathbb{R}$ whose graph is contained in $G_0 \cup H_0$ and intersects H_0. This contradiction finishes the proof. □

Notice that Proposition 4.5 stay in contrast with the following fact.

Fact 4.6 (Natkaniec [22, Corollary 4.2(1)]). *If $f : \mathbb{R} \to \mathbb{R}$ is almost continuous and Y is a compact topological space then $F : \mathbb{R} \times Y \to \mathbb{R}$ given by $F(x, y) = f(x)$ is almost continuous.*

Proposition 4.7. *If $n > 1$ and $F \in \mathrm{Add}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n)$ then $F^{-1}(0)$ contains a straight line.*

Proof. If F is constantly equal to 0 then there is nothing to prove. So, assume that this is not the case. Then $f[\mathbb{R}^2] = \mathbb{R}$. In particular $V = F^{-1}(0)$ disconnects $\mathbb{R}^2$. (Otherwise its complement $\mathbb{R}^2 \setminus V$ would be connected, while $F[\mathbb{R}^2 \setminus V] = \mathbb{R} \setminus \{0\}$ would not, contradicting Darboux property.) Thus, by Fact 2.1, V contains a nontrivial compact connected subset K. Pick $a, b \in K$ with $\mathrm{diam}(\{a, b\}) = \mathrm{diam}(K)$. Since every rotation r is a linear homeomorphism, replacing F with $F \circ r$ for an appropriate r if necessary, we can assume that a and b are on the same vertical line. If $\mathrm{pr}(K)$ is a singleton, then K contains a straight line segment connecting a with b. This, and the fact that V is linear over $\mathbb{Q}$, easily imply that V contains a straight line. So, assume that $\mathrm{pr}(K) = [x_0, x_1]$ for some $x_0 < x_1$. We claim that this implies that

$$\text{there exists a bounded open set } U \subset \mathbb{R}^2 \text{ with } \mathrm{bd}(U) \subset V. \tag{$*$}$$

To see it take $c, d \in K$ with $\mathrm{pr}(c) = x_0$ and $\mathrm{pr}(d) = x_1$, and let $[y_0, y_1]$ be a projection of K to the second coordinate. Let $P = [x_0, x_1] \times [y_0, y_1]$. Notice that $K \subset P$ and that a and b lie on the opposite horizontal sides of P, and c and d on opposite vertical sides of it. Note also that vectors $v = b - a$ and $w = d - c$ belong to V. Now consider the parallelogram-like set

$$B = \bigcup_{m=0}^{4} \left[mv + \left(K \cup (4w + K)\right) \cup mw + \left(K \cup (4v + K)\right)\right] \subset V.$$

(The “sides” are formed from translated “roads” from a to b and from c to d.) Note that the interior U_0 of $2v + w + P$ is disjoint with B and that B separates it from the infinity. Then the component U of $\mathbb{R}^2 \setminus B$ containing U_0 satisfies $(*)$.

Now, as in [9, Lemma 5.4] we can find a nonempty bounded connected open set $W \subset \mathbb{R}^2$ with connected boundary $\mathrm{bd}(W) \subset B \subset V$. Take an $\varepsilon > 0$ such that some open disk of radius ε is contained in B. Then, if $B(\varepsilon) = \{v \in \mathbb{R}^2\colon$ length of v is less than $\varepsilon\}$ then for every $v \in B(\varepsilon)$ we have $\mathrm{bd}(W) \cap (v + \mathrm{bd}(W)) \neq \emptyset$. So, $v \in \mathrm{bd}(W) - \mathrm{bd}(W) \subset V$. Therefore, $B(\varepsilon) \subset V$, and so $V = \mathbb{R}^2$ contradicting our assumption that F is non-zero. □

Theorem 4.8. $\mathrm{Add}(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \subset \mathrm{C}(\mathbb{R}^n)$ *for* $n > 1$.

Proof. Let $F \in \mathrm{Add}(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n)$. We will show that it is continuous.

Since it is enough to prove continuity of any restriction of F to a plane containing the origin, we can assume that $n = 2$. Now, by Proposition 4.7, $f^{-1}(0)$ contains a line L_0. Since $f^{-1}(0)$ is close under addition, it contains also a parallel line L which contains the origin. Since all the classes under consideration are closed under inner composition with a rotation, we can assume that L is a vertical line. But this means that there exists additive $f : \mathbb{R} \to \mathbb{R}$ such that $F(x, y) = f(x)$ for every $x, y \in \mathbb{R}$. So, by Proposition 4.5, function f is continuous. Thus $F \in \mathrm{C}(\mathbb{R}^2)$. □

Example 4.9. There exists an $f \in \mathrm{Add}(\mathbb{R}^2) \cap \mathrm{AC}(\mathbb{R}^2) \setminus \mathrm{D}(\mathbb{R}^2)$.

Proof. Let $\mathcal{B}$ be a family of all closed subsets B of $\mathbb{R}^2 \times \mathbb{R}$ for which the projection $\mathrm{pr}(B)$ onto first coordinate $\mathbb{R}^2$ has cardinality $\mathfrak{c}$. (Thus, it contains a perfect set.) It is known that if $f : \mathbb{R}^2 \to \mathbb{R}$ intersects every element of $\mathcal{B}$ then $f \in \mathrm{AC}(\mathbb{R}^2)$. (See [22, Proposition 1.2].) Let $\{F_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{B}$. By induction on $\xi < \mathfrak{c}$ define a sequence $\langle\langle a_\xi, b_\xi, y_\xi\rangle\colon \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$ the following conditions hold:

(i) $a_\xi \in \mathrm{pr}(B_\xi) \setminus \mathrm{LIN}_{\mathbb{Q}}(\{a_\zeta\colon \zeta < \xi\} \cup \{b_\zeta\colon \zeta < \xi\})$,

(ii) $b_\xi \in \mathrm{pr}(B_\xi) \setminus \mathrm{LIN}_{\mathbb{Q}}(\{a_\zeta\colon \zeta \leqslant \xi\} \cup \{b_\zeta\colon \zeta < \xi\})$,

(iii) $\langle a_\xi, y_\xi\rangle \in B_\xi$.

Let $A = \{a_\xi\colon \xi < \mathfrak{c}\}$, $B = \{b_\xi\colon \xi < \mathfrak{c}\}$, and H be a Hamel base containing $A \cup B$. Define $f_0 : H \to \mathbb{R}$ by putting $f_0(a_\xi) = y_\xi$ and $f_0(x) = 1$ for $x \in H \setminus A$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a linear extension of f_0. Then f is additive and almost continuous, since it intersects every $F \in \mathcal{F}$. However, it is not Darboux, since the set $D = B \cup \{0\}$ is connected, as it intersects every perfect subset of $\mathbb{R}^2$, while $f[D] = \{0, 1\}$ is not connected. □

Example 4.10. There exists an $f \in \mathrm{Add}(\mathbb{R}^2) \cap \mathrm{D}(\mathbb{R}^2) \setminus \mathrm{AC}(\mathbb{R}^2)$.

Proof. Take an additive Darboux function $g : \mathbb{R} \to \mathbb{R}$ with the property that

$$g\big[(a, b)\big] = \mathbb{R} \quad \text{for every } a < b. \tag{$*$}$$

A function f from Example 5.3 has the property. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = g(x)$. Clearly f is additive and discontinuous. To see that f is Darboux take a nonempty connected set $D \subset \mathbb{R}^2$. If $J = \mathrm{pr}(D)$ is a singleton then $f[D] = g[\mathrm{pr}(D)]$ is a singleton, so it is connected. Otherwise $\mathrm{pr}(D)$ contains an interval $(a, b) \neq \emptyset$ and, by $(*)$, $f[D] = g[\mathrm{pr}(D)] \supset g[(a, b)] = \mathbb{R}$. Now, by Theorem 4.8, f is not almost continuous. □

As noted above, function f from Example 4.10 cannot be almost continuous. It is interesting however, that if function g used to define f is almost continuous, which can be easily constructed, then $f \restriction \mathbb{R} \times [-k, k]$ is almost continuous for every $k > 0$. This follows from Fact 4.6.

5. Some missing examples of additive Darboux-like functions on $\mathbb{R}$

Example 5.1. There exists an $f \in \mathrm{Add} \cap \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}$. Moreover, $f[K]$ is not nowhere dense for every perfect set $K \subset \mathbb{R}$.

Proof. Let $\mathcal{P}$ be a family of pairwise disjoint perfect sets such that the set $\bigcup \mathcal{P}$ is linearly independent and $|\{P \in \mathcal{P}\colon\ P \subset (a, b)\}| = \mathfrak{c}$ for every $a < b$. Such a family can be easily constructed from a linearly independent perfect set. (See, e.g., [20, Theorem 2, Chapter XI, Section 7].) Let $\mathcal{J}$ be a family of all nonempty open intervals and let $\{\langle I_\xi, J_\xi\rangle\colon\ \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{J} \times \mathcal{J}$. By an easy induction we can find a one-to-one sequence $\{P_\xi \in \mathcal{P}\colon\ \xi < \mathfrak{c}\}$ such that $P_\xi \subset I_\xi$ for every $\xi < \mathfrak{c}$. Let $H \subset \mathbb{R}$ be a Hamel basis containing $\bigcup \mathcal{P}$ and for each $h \in H$ let $J_h = J_\xi$ if $h \in P_\xi$, and $J_h = (0, 1)$ otherwise. Our f will be a linear extension of some function $f_0 : H \to \mathbb{R}$ such that

$$f_0(h) \in J_h \quad \text{for every } h \in H. \tag{$*$}$$

It is easy to see that any such an f will have a perfect road. To make sure that f has the additional property, which make it not CIVP, we will need to make some more work.

Let $\mathcal{F}$ be the family of all perfect nowhere dense subsets of $\mathbb{R}$ and let $\{\langle K_\xi, S_\xi\rangle\colon\ \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{F} \times \mathcal{F}$. We will make sure for every $\xi < \mathfrak{c}$ there exists an $x_\xi \in K_\xi$ such that $f(x_\xi) \notin S_\xi$. Clearly such a function will have all the desired property. For this we will construct a sequences $\langle H_\xi\colon\ \xi < \mathfrak{c}\rangle$ of pairwise disjoint finite subsets of H and $\langle g_\xi\colon\ \xi < \mathfrak{c}\rangle$ of functions from H_ξ into $\mathbb{R}$ such that for every $\xi < \mathfrak{c}$

(i) $g_\xi(h) \in J_h$ for every $h \in H_\xi$,

(ii) there exists an $x_\xi \in K_\xi \cap \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{H_\zeta\colon\ \zeta \leqslant \xi\})$ such that $G_\xi(x_\xi) \notin S_\xi$, where $G_\xi = \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{g_\zeta\colon\ \zeta \leqslant \xi\})$.

Now, the inductive choice of H_ξ and g_ξ is quite simple. We choose an $x_\xi \in K_\xi \setminus \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{H_\zeta\colon\ \zeta < \xi\})$ and represent x_ξ as $z + q_1h_1 + \cdots + q_nh_n$ where $z \in \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{H_\zeta\colon\ \zeta < \xi\})$, $h_1, \ldots, h_n \in H \setminus \bigcup\{H_\zeta\colon\ \zeta < \xi\}$, and $q_1, \ldots, q_n \in \mathbb{Q}$. We put $H_\xi = \{h_1, \ldots, h_n\}$ and define g_ξ such that (i) and (ii) are satisfied. This can be done, since we have an open interval of possible choices for each value of $g_\xi(h_i)$, while we have to omit only a nowhere dense set S_ξ for the value of $G_\xi(x_\xi) = G_\xi(z) + q_1g_\xi(h_1) + \cdots + q_ng_\xi(h_n)$. Now, if $f_0 : H \to \mathbb{R}$ is any extension of $\bigcup_{\xi<\mathfrak{c}} g_\xi$ which satisfies $(*)$ then a linear extension f of f_0 has all the desired properties. □

Example 5.2. There exists an $f \in \mathrm{Add} \cap \mathrm{SCIVP} \cap \mathrm{PC} \setminus \mathrm{D}$.

Proof. Let H_0 be a Hamel basis such that $|H_0 \cap K| = \mathfrak{c}$ for every perfect set $K \subset \mathbb{R}$. (See [4, Corollary 7.3.7].) Let $h_0 \in H_0$ and $V = \mathrm{LIN}_{\mathbb{Q}}(H_0 \setminus \{h_0\})$. Then V is a proper linear subspace of $\mathbb{R}$ with $|V \cap K| = \mathfrak{c}$ for every perfect $K \subset \mathbb{R}$.

As in the example above, let $\mathcal{P}$ be a family of pairwise disjoint perfect sets such that the set $\bigcup \mathcal{P}$ is linearly independent and $|\{P \in \mathcal{P}\colon\ P \subset (a,b)\}| = \mathfrak{c}$ for every $a < b$. Let $\mathcal{J}$ be a family of all nonempty open intervals and let $\{\langle I_\xi, v_\xi\rangle\colon\ \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{J} \times V$. Find a one-to-one sequence $\{P_\xi \in \mathcal{P}\colon\ \xi < \mathfrak{c}\}$ such that $P_\xi \subset I_\xi$ for every $\xi < \mathfrak{c}$ and let $H \subset \mathbb{R}$ be a Hamel basis containing $\bigcup \mathcal{P}$.

Define $f_0 : H \to V$ by putting $f_0(h) = v_\xi$ for $h \in P_\xi$ and choose an arbitrary $f_0(h) \in V$ for all other $h \in H$. Let $f : \mathbb{R} \to \mathbb{R}$ be a linear extension of f_0. Then f is additive and non-zero, so f has a dense graph. Thus $f \in \mathrm{PC}$. Also $f[\mathbb{R}] = V$ implying $f \notin \mathrm{D}$. It is SCIVP since for every perfect set $K \subset \mathbb{R}$ and $a < b$ there exist $v \in V \cap K$ and $\xi < \mathfrak{c}$ such that $\langle I_\xi, v_\xi\rangle = \langle (a,b), v\rangle$. So $f[P_\xi] = \{v_\xi\} = \{v\} \subset K$ and $f \restriction P_\xi$ is continuous. □

Example 5.3. There exists an $f \in \mathrm{Add} \cap \mathrm{SCIVP} \cap \mathrm{D} \setminus \mathrm{Conn}$.

Proof. The construction is very similar to that for function g from Example 4.10. Let $L = \{\langle x, x+1\rangle\colon\ x \in \mathbb{R}\}$. As above choose a family $\mathcal{P}$ of pairwise disjoint perfect sets such that the set $\bigcup \mathcal{P}$ is linearly independent and $|\{P \in \mathcal{P}\colon\ P \subset (a,b)\}| = \mathfrak{c}$ for every $a < b$. Let $\{A, B\}$ be a partition of continuum $\mathfrak{c}$ with $|A| = |B| = \mathfrak{c}$. Put $\mathcal{J} = \{(a,b)\colon\ a < b\}$ and let $\{\langle I_\xi, r_\xi\rangle\colon\ \xi \in A\}$ be an enumeration of $\mathcal{J} \times \mathbb{R}$. Also, find a one-to-one sequence $\{P_\xi \in \mathcal{P}\colon\ \xi < \mathfrak{c}\}$ such that $P_\xi \subset I_\xi$ for every $\xi < \mathfrak{c}$. Finally, let $H \subset \mathbb{R}$ be a Hamel basis containing $\bigcup \mathcal{P}$ and let $H = \{h_\xi\colon\ x \in B\}$.

By induction on $\xi < \mathfrak{c}$ construct a sequence $\langle\langle K_\xi, y_\xi\rangle\colon\ \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$ the following conditions hold.

(i) $K_\xi \cap \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\zeta\colon\ \zeta < \xi\}) = \emptyset$.
(ii) $L \cap \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\zeta \times \{y_\zeta\}\colon\ \zeta \leqslant \xi\}) = \emptyset$.
(iii) If $\xi \in A$ then K_ξ is a perfect subset of P_ξ and $y_\xi = r_\xi$.
(iv) If $\xi \in B$ then $|K_\xi| \leqslant 1$ and $h_\xi \in \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\zeta\colon\ \zeta \leqslant \xi\})$.

To find such a sequence assume that a sequence $\langle\langle a_\zeta, y_\zeta\rangle\colon\ \zeta < \xi\rangle$ is already constructed for some $\xi < \mathfrak{c}$. If $\xi \in A$ put $y_\xi = r_\xi$ and look at the set

$$Z = L \cap \mathrm{LIN}_{\mathbb{Q}}\left(\bigcup\{K_\zeta \times \{y_\zeta\}\colon\ \zeta \leqslant \xi\}\right) \cup (P_\xi \times \{y_\xi\}).$$

Notice that Z has cardinality less than $\mathfrak{c}$ since $Z \subset \mathbb{R} \times \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{y_\zeta\colon \zeta \leqslant \xi\})$ and L intersects every horizontal line at exactly one point. Let $T = \mathrm{pr}(Z)$ and notice that by the inductive assumption of (ii) we have $T \cap \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\zeta\colon\ \zeta < \xi\}) = \emptyset$. For every $t \in T$ choose an $h_t \in H \setminus \bigcup\{K_\zeta\colon\ \zeta < \xi\}$ with $t \notin \mathrm{LIN}_{\mathbb{Q}}(H \setminus \{h_t\})$. Then the set

$$S = \{h_t\colon\ t \in T\} \cup \bigcup\{K_\zeta\colon\ \zeta < \xi \text{ and } \zeta \in B\}$$

has cardinality less than $\mathfrak{c}$. Choose a perfect set $K_\xi \subset P_\xi \setminus S$. Then K_ξ satisfies (i)–(iii).

Next assume that $\xi \in B$ and let $V = \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\zeta\colon \zeta < \xi\})$. If $h_\xi \in V$ we put $K_\xi = \emptyset$ and choose y_ξ arbitrarily. So assume that $h_\xi \notin V$. We put $K_\xi = \{h_\xi\}$. This guarantees (i) and (iv). To get (ii) we have to find y_ξ such that the set

$$\mathrm{LIN}_{\mathbb{Q}}\bigl(V \cup \{\langle h_\xi, y_\xi\rangle\}\bigr) = \bigl\{\langle x, y\rangle + q\langle h_\xi, y_\xi\rangle\colon \langle x, y\rangle \in V \text{ and } q \in \mathbb{Q}\bigr\}$$

is disjoint with L. Thus, we must have $y + qy_\xi \neq x + qh_\xi + 1$, that is, $y_\xi \neq px + h_\xi - py + p$ for every $\langle x, y\rangle \in V$ and $q = p^{-1} \in \mathbb{Q} \setminus \{0\}$. Therefore it is enough to choose

$$y_\xi \notin \mathrm{LIN}_{\mathbb{Q}}\left(\bigcup\{K_\zeta\colon \zeta \leqslant \xi\} \cup \{y_\zeta\colon \zeta < \xi\} \cup \{1\}\right)$$

which can be done, since $\mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\zeta\colon \zeta \leqslant \xi\})$ has co-dimension $\mathfrak{c}$. This finishes the inductive construction.

Now put $f = \mathrm{LIN}_{\mathbb{Q}}(\bigcup\{K_\xi \times \{y_\xi\}\colon \xi < \mathfrak{c}\}$. Then $f\colon \mathbb{R} \to \mathbb{R}$ is additive and misses L, so it is not connectivity. It is Darboux and SCIVP since for every $a, b, y \in \mathbb{R}$, $a < b$, there exists $\xi < \mathfrak{c}$ such that $\langle I_\xi, y_\xi\rangle = \langle (a, b), y\rangle$. So $K_\xi \subset P_\xi \subset I_\xi = (a, b)$, $f[K_\xi] = \{y_\xi\}$, and $f \restriction K_\xi$ is continuous. □

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