SUMS OF CONNECTIVITY FUNCTIONS ON $\mathbb{R}^n$

KRZYSZTOF CIESIELSKI *and* JERZY WOJCIECHOWSKI

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ABSTRACT

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a *connectivity function* if the graph of its restriction $f|C$ to any connected $C \subset \mathbb{R}^n$ is connected in $\mathbb{R}^n \times \mathbb{R}$. The main goal of this paper is to prove that every function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a sum of $n+1$ connectivity functions (Corollary 2.2). We will also show that if $n > 1$, then every function $g\colon \mathbb{R}^n \to \mathbb{R}$ which is a sum of n connectivity functions is continuous on some perfect set (see Theorem 2.5) which implies that the number $n+1$ in our theorem is the best possible (Corollary 2.6).

To prove the above results, we establish and then apply the following theorems which are of interest on their own.

For every dense G_δ-subset G of $\mathbb{R}^n$ there are homeomorphisms $h_1, \ldots, h_n$ of $\mathbb{R}^n$ such that $\mathbb{R}^n = G \cup h_1(G) \cup \ldots \cup h_n(G)$ (Proposition 2.4).

For every $n > 1$ and any connectivity function $f\colon \mathbb{R}^n \to \mathbb{R}$, if $x \in \mathbb{R}^n$ and $\varepsilon > 0$ then there exists an open set $U \subset \mathbb{R}^n$ such that $x \in U \subset B^n(x, \varepsilon)$, $f|\mathrm{bd}(U)$ is continuous, and $|f(x) - f(y)| < \varepsilon$ for every $y \in \mathrm{bd}(U)$ (Proposition 2.7).

1. *Preliminaries*

Our basic terminology and notation is standard. (See, for example, [**4**].) The terminology and preliminaries from dimension theory and the theory of simplicial triangulations will be used only in some parts of the paper and will be introduced on the 'as needed' basis.

For a topological space X and $U \subset X$ we will use the symbols $\mathrm{cl}(U)$ and $\mathrm{bd}(U)$ to denote the *closure* and the *boundary* of U, respectively. Also, we will consider the following classes of functions $f\colon X \to \mathbb{R}$ (we will use them only when $X \subset \mathbb{R}^n$):

$\mathrm{Conn}(X)$: the set of *connectivity functions* $f\colon X \to \mathbb{R}$, that is, functions such that the graph of $f|C$ is connected in $X \times \mathbb{R}$ for every connected subset C of X;

$\mathrm{PC}(X)$: the set of *peripherally continuous functions* $f\colon X \to \mathbb{R}$, that is, functions such that for every $x \in X$ and any pair $U \subset X$ and $V \subset \mathbb{R}$ of open neighbourhoods of x and $f(x)$, respectively, there exists an open neighbourhood W of x with $\mathrm{cl}(W) \subset U$ and $f[\mathrm{bd}(W)] \subset V$;

$\mathrm{Ext}(X)$: the set of *extendable functions* $f\colon X \to \mathbb{R}$, that is, functions such that there exists a connectivity function $g\colon X \times [0,1] \to \mathbb{R}$ with $f(x) = g(x, 0)$ for every $x \in X$.

We will write Conn, PC and Ext in place of $\mathrm{Conn}(X)$, $\mathrm{PC}(X)$ and $\mathrm{Ext}(X)$ when the space X is clear from the context.

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It is immediate from the definition that $\mathrm{Ext}(X) \subset \mathrm{Conn}(X)$ for every connected space X. In what follows we will use the following theorem. (The inclusion '$\subset$' was proved by Hamilton [**8**] and Stallings [**14**], and the inclusion '$\supset$' by Hagan [**7**].)

THEOREM 1.1. *If* $n \geqslant 2$, *then* $\mathrm{Conn}(\mathbb{R}^n) = \mathrm{PC}(\mathbb{R}^n)$.

To place our results within a wider context we need to define two other classes of real functions. However, the rest of this section will not be used in an essential way in the proofs of our main results.

$\mathrm{D}(X)$: the set of *Darboux functions* $f\colon X \to \mathbb{R}$, that is, functions such that $f[C]$ is connected in $\mathbb{R}$ for every connected subset C of X;

$\mathrm{AC}(X)$: the set of *almost continuous functions* $f\colon X \to \mathbb{R}$, that is, functions such that for every open subset U of $X \times \mathbb{R}$ containing the graph of f, there is a continuous function $g\colon X \to \mathbb{R}$ with $g \subset U$.

We will write D and AC in place of $\mathrm{D}(X)$ and $\mathrm{AC}(X)$ when X is clear from the context.

For $X = \mathbb{R}$ we have the following proper inclusions [**1**]:

$$\mathrm{Ext} \subset \mathrm{AC} \subset \mathrm{Conn} \subset \mathrm{D} \subset \mathrm{PC}. \tag{1}$$

In the case when $X = \mathbb{R}^n$ with $n \geqslant 2$ the following relations are known to hold:

$$\mathrm{Ext} \subset \mathrm{PC} = \mathrm{Conn} \subset \mathrm{D} \cap \mathrm{AC}, \quad \mathrm{D} \cap \mathrm{AC} \not\subset \mathrm{Conn}, \quad \mathrm{D} \not\subset \mathrm{AC}, \quad \mathrm{AC} \not\subset \mathrm{D}.$$

The equality $\mathrm{PC} = \mathrm{Conn}$ is a restatement of Theorem 1.1 and the inclusion $\mathrm{Ext} \subset \mathrm{Conn}$ is obvious from the definition. We do not know whether it is proper. The proof of the inclusion $\mathrm{Conn} \subset \mathrm{AC}$ can be found in [**14**]. The inclusion $\mathrm{Conn} \subset \mathrm{D}$ is clear from the definition. This gives $\mathrm{Conn} \subset \mathrm{D} \cap \mathrm{AC}$. A simple Baire class 1 function in $\mathrm{D} \cap \mathrm{AC} \setminus \mathrm{Conn}$ was described in [**13**, Example 1]. The examples showing that $\mathrm{D} \not\subset \mathrm{AC}$ and $\mathrm{AC} \not\subset \mathrm{D}$ can be found in [**11**, Examples 1.1.9 and 1.1.10].

Our investigations in this paper are motivated by the following result of Natkaniec [**11**, Proposition 1.7.1].

THEOREM 1.2. *For every* $n > 0$, *any function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is the sum of two almost continuous functions.*

In general, given a class $\mathcal{F}$ of functions $f\colon X \to \mathbb{R}$ where $X \subseteq \mathbb{R}^n$, let the *repeatability* $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$ be defined as the minimum integer k such that any function $f\colon X \to \mathbb{R}$ can be expressed as the sum of k functions from $\mathcal{F}$. Since the class $\mathrm{AC}(\mathbb{R}^n)$ is a proper subset of $\mathbb{R}^{\mathbb{R}^n}$, Theorem 1.2 says that

$$\mathcal{R}(\mathrm{AC}(\mathbb{R}^n)) = 2, \tag{2}$$

for every $n \geqslant 1$. Since the classes $\mathrm{Conn}(\mathbb{R})$, $\mathrm{D}(\mathbb{R})$ and $\mathrm{PC}(\mathbb{R})$ are proper subsets of $\mathbb{R}^{\mathbb{R}}$, it follows from (2) and (1) that

$$\mathcal{R}(\mathrm{Conn}(\mathbb{R})) = \mathcal{R}(\mathrm{D}(\mathbb{R})) = \mathcal{R}(\mathrm{PC}(\mathbb{R})) = 2.$$

Moreover, Ciesielski and Recław [**3**] and Rosen [**12**] independently proved that

$$\mathcal{R}(\mathrm{Ext}(\mathbb{R})) = 2.$$

In this paper we show that the following general result holds.

THEOREM 1.3. *For every* $n \geqslant 1$,

$$\mathcal{R}(\mathrm{Ext}(\mathbb{R}^n)) = \mathcal{R}(\mathrm{Conn}(\mathbb{R}^n)) = \mathcal{R}(\mathrm{PC}(\mathbb{R}^n)) = n+1.$$

Theorem 1.3 follows from Theorem 2.1 and Corollary 2.6 which are stated and proved in the next section. We do not know† whether a similar result is true for either of the classes $\mathrm{D} \cap \mathrm{AC}$ or D when $n > 1$.

2. *The main results*

In this section we will prove the main theorems of the paper modulo three groups of technical results, each of which will be proved in one of the sections that follow.

The following theorem and Theorem 2.5 are the main results of the paper.

THEOREM 2.1. *Every function* $g\colon \mathbb{R}^n \to \mathbb{R}$ *can be represented as a sum* $g = g_0 + g_1 + \ldots + g_n$ *of* $n+1$ *extendable functions* $g_0, \ldots, g_n\colon \mathbb{R}^n \to \mathbb{R}$.

Since $\mathrm{Ext}(\mathbb{R}^n) \subset \mathrm{Conn}(\mathbb{R}^n)$, Theorem 2.1 immediately implies the following corollary.

COROLLARY 2.2. *Every function* $g\colon \mathbb{R}^n \to \mathbb{R}$ *can be represented as a sum* $g = g_0 + g_1 + \ldots + g_n$ *of* $n+1$ *connectivity functions* $g_0, \ldots, g_n\colon \mathbb{R}^n \to \mathbb{R}$.

For $n = 1$, Theorem 2.1 has been proved in [**3**]. For $n > 1$, it follows from the next two propositions, which will be proved in §§ 3 and 4, respectively.

PROPOSITION 2.3. *For every* $n > 1$, *there exist a function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and a dense* G_δ*-subset* G *of* $\mathbb{R}^n$ *such that any function* $g\colon \mathbb{R}^n \to \mathbb{R}$ *with* $g(x) = f(x)$ *for* $x \notin G$ *is a connectivity function.*

PROPOSITION 2.4. *If* $G \subseteq \mathbb{R}^n$ *is a dense* G_δ*-set, then there are homeomorphisms* $h_j\colon \mathbb{R}^n \to \mathbb{R}^n$ *for* $j \in \{1, \ldots, n\}$ *such that*

$$G \cup \bigcup_{j=1}^{n} h_j(G) = \mathbb{R}^n.$$

Proof of Theorem 2.1. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be an arbitrary function and let $\hat{f}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and a dense G_δ-subset $\hat{G}$ of $\mathbb{R}^n \times \mathbb{R}$ be as in Proposition 2.3. By the Kuratowski–Ulam theorem (a category analogue of the Fubini theorem) there exists $y \in \mathbb{R}$ such that a G_δ-set $G = \{x \in \mathbb{R}^n\colon \langle x, y\rangle \in \hat{G}\}$ is dense in $\mathbb{R}^n$.

Notice that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x) = \hat{f}(x, y)$ for every $x \in \mathbb{R}^n$, then

$$g\colon \mathbb{R}^n \to \mathbb{R} \quad \text{is extendable provided } g(x) = f(x) \text{ for every } x \notin G. \tag{3}$$

†It has been settled recently by Francis Jordan (private communication) who proved that for every $n > 1$ there exists a Baire 1 class function $f\colon \mathbb{R}^n \to \mathbb{R}$ which is not a sum of n Darboux functions. This clearly implies that $\mathcal{R}(\mathrm{D}(\mathbb{R}^n)) \geqslant n+1$, while the other inequality follows from Theorem 1.3.

Let $h_j\colon \mathbb{R}^n \to \mathbb{R}^n$ for $j \in \{1, \ldots, n\}$ be the homeomorphisms from Proposition 2.4, and let $h_0 : \mathbb{R}^n \to \mathbb{R}^n$ be the identity homeomorphism. Notice that for every $j \in \{1, \ldots, n\}$,

$$g_j\colon \mathbb{R}^n \to \mathbb{R} \text{ is extendable} \tag{4}$$

provided $g_j(x) = (f \circ h_j^{-1})(x)$ for every $x \notin h_j(G)$.

Indeed, if g_j satisfies the hypothesis of (4), then $g_j = g \circ h_j^{-1}$ where g is defined by

$$g(x) = \begin{cases} (g_j \circ h_j)(x) & \text{if } x \in G, \\ f(x) & \text{if } x \notin G. \end{cases}$$

But, by (3), g is extendable and so is g_j as a composition of a homeomorphism and an extendable function.

Let $G_0 = G$, and for every $j = 1, 2, \ldots, n$ put

$$G_j = h_j(G) \setminus \bigcup_{i=0}^{j-1} h_i(G).$$

Then the sets G_0, G_1, $\ldots$, G_n form a partition of $\mathbb{R}^n$. For each $i = 0, 1, \ldots, n$, let $g_i\colon \mathbb{R}^n \to \mathbb{R}$ be defined by

$$g_i(x) = \begin{cases} g(x) - \sum_{j\in\{0,\ldots,n\}\setminus\{i\}} (f \circ h_j^{-1})(x) & \text{if } x \in G_i, \\ (f \circ h_i^{-1})(x) & \text{if } x \notin G_i. \end{cases}$$

Then

$$(g_0 + \ldots + g_n)(x) = g(x)$$

for every $x \in \mathbb{R}^n$. Since $g_i(x) = (f \circ h_i^{-1})(x)$ for every $i = 0, 1, \ldots, n$, and every $x \notin h_i(G)$, it follows from (4) that the functions $g_0, g_1, \ldots, g_n$ are extendable.

Next, we will turn to the proof of our second main result.

THEOREM 2.5. *If $n > 1$ and $g_1, g_2, \ldots, g_n\colon \mathbb{R}^n \to \mathbb{R}$ are connectivity functions then there exists a perfect set $P \subseteq \mathbb{R}^n$ such that the restriction of g_j to P is continuous for every $j \in \{1, 2, \ldots, n\}$.*

Notice that Theorem 2.5 immediately implies the following corollary. In particular, the number $n + 1$ in Theorem 2.1 is the best possible.

COROLLARY 2.6. *For every $n > 0$ there exists a function $f\colon \mathbb{R}^n \to \mathbb{R}$ which is not a sum of n peripherally continuous functions.*

Proof. For $n = 1$, the statement follows from the fact that there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ which is not peripherally continuous. (For example, the characteristic function of a singleton.)

For $n > 1$, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be the characteristic function of a Bernstein set, that is, a set $B \subseteq \mathbb{R}^n$ such that $B \cap P \neq \emptyset$ and $B \setminus P \neq \emptyset$ for every perfect set $P \subseteq \mathbb{R}^n$. Then the restriction of f to any perfect subset of $\mathbb{R}^n$ is discontinuous. It follows from Theorem 2.5 that f is not a sum of n connectivity functions.

The proof of Theorem 2.5 is based on the next two propositions, whose proofs are postponed till § 5.

PROPOSITION 2.7. *Let $n > 0$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Then for any $x_0 \in \mathbb{R}^n$ and any open set W in $\mathbb{R}^n$ containing x_0, there exists an open set $U \subseteq W$ such that $x_0 \in U$ and the restriction of f to $\operatorname{bd} U$ is continuous. Moreover, given any $\varepsilon > 0$, the set U can be chosen so that $|f(x_0) - f(y)| < \varepsilon$ for every $y \in \operatorname{bd} U$.*

PROPOSITION 2.8. *Let $n > 1$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be peripherally continuous. If X is a connected perfect subset of $\mathbb{R}^n$, then there exists a perfect subset P of X such that the restriction of g to P is continuous.*

For $X = [0,1]^n$, Proposition 2.8 has been proved earlier by Gibson, Rosen and Roush [**5**].

Given $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$, we will write $\operatorname{bd}_X U$ to denote the *boundary* of $U \cap X$ in X. For the proof of Theorem 2.5 we need to recall the definition of the inductive dimension of subsets X of $\mathbb{R}^n$ (see, for example, [**4**]):

(i) $\operatorname{ind} X = -1$ if and only if $X = \emptyset$;

(ii) $\operatorname{ind} X \leqslant m$ if for any $p \in X$ and any open neighbourhood W of p there exists an open neighbourhood $U \subseteq W$ of p such that $\operatorname{ind} \operatorname{bd}_X U \leqslant m - 1$;

(iii) $\operatorname{ind} X = m$ if $\operatorname{ind} X \leqslant m$ and it is not true that $\operatorname{ind} X \leqslant m - 1$.

Recall that $\operatorname{ind} \mathbb{R}^n = n$.

Proof of Theorem 2.5. We will define a sequence $D_0, D_1, \ldots, D_{n-1}$ of compact subsets of $\mathbb{R}^n$ such that $\operatorname{ind} D_i \geqslant n - i$ and the restriction of g_j to D_i is continuous for every $j \leqslant i < n$.

First note that this will complete the proof, since then we can choose a component X of D_{n-1} (which is perfect and connected) and apply Proposition 2.8 to X and the function g_n.

To construct such a sequence let $D_0 = \mathbb{R}^n$ and assume that D_{i-1} has been defined for some $i \in \{1, 2, \ldots, n-1\}$. Since $\operatorname{ind} D_{i-1} \geqslant n - i + 1$, there exist $p \in D_{i-1}$ and an open neighbourhood $W \subset \mathbb{R}^n$ of p such that $\operatorname{ind} \operatorname{bd}_{D_{i-1}} U \geqslant n - i$ for every open neighbourhood $U \subset W$ of p. Since g_i is peripherally continuous, it follows from Proposition 2.7 that there is an open neighbourhood $U \subseteq W$ of p such that the restriction of g_i to $\operatorname{bd} U$ is continuous. Let

$$D_i = \operatorname{bd}_{D_{i-1}} U \subseteq \operatorname{bd} U \cap D_{i-1}.$$

Then $\operatorname{ind} D_i \geqslant n - i$ and the restriction of g_j to D_i is continuous for every j with $1 \leqslant j \leqslant i$. Therefore the proof is complete.

3. *Proof of Proposition* 2.3

The proof presented here is analogous to the technique used in [**3**]. However, instead of equilateral triangulations of $\mathbb{R}^2$ we will use a more general concept of a simplicial triangulation of $\mathbb{R}^n$. An introduction to simplicial triangulations of $\mathbb{R}^n$ can be found, for example, in [**9**]. For completeness, we will give basic definitions and results.

Let $X = \{x_0, x_1, \ldots, x_m\}$ be a set of $m+1$ points in $\mathbb{R}^n$. The points of X are in *general position* if the vectors $x_1 - x_0$, $x_2 - x_0$, $\ldots$, $x_m - x_0$ are linearly independent. An *m-dimensional simplex* $\Delta = \Delta(X)$ in $\mathbb{R}^n$ is the subset of $\mathbb{R}^n$ of the form

$$\Delta = \left\{ \sum_{x \in X} \beta_x x \colon\ (\forall x \in X)\,(\beta_x > 0)\ \&\ \sum_{x \in X} \beta_x = 1 \right\},$$

where X is a set of points in general position. The elements of X are called the *vertices* of Δ. Any simplex $\Delta(Y)$ with $\emptyset \neq Y \subseteq X$ is a *face* of $\Delta(X)$. A face $\Delta(Y)$ of $\Delta(X)$ is *proper* if $Y \neq X$. The *closure* $\operatorname{cl}\Delta$ of the simplex $\Delta = \Delta(X)$ is the union of all faces of Δ, that is,

$$\operatorname{cl}\Delta = \left\{ \sum_{x \in X} \beta_x x \colon\ (\forall x \in X)\,(\beta_x \geqslant 0)\ \&\ \sum_{x \in X} \beta_x = 1 \right\}.$$

The *boundary* $\operatorname{bd}\Delta$ of the simplex Δ is the union of all proper faces of Δ. Note that if Δ is an n-dimensional simplex in $\mathbb{R}^n$, then $\operatorname{cl}\Delta$ is the topological closure and $\operatorname{bd}\Delta$ is the topological boundary of Δ.

A *simplicial complex* $\mathcal{K}$ is a set of disjoint simplices in $\mathbb{R}^n$ such that:

(i) if $\Delta \in \mathcal{K}$ and Δ' is a face of Δ, then Δ' is also in $\mathcal{K}$; and

(ii) any bounded subset of $\mathbb{R}^n$ intersects only finitely many simplices of $\mathcal{K}$.

A *vertex* of a simplicial complex $\mathcal{K}$ is a vertex of one of its simplices and the *boundary* $\operatorname{bd}\mathcal{K}$ of $\mathcal{K}$ is the union of the boundaries of the simplices of $\mathcal{K}$. If Δ is a simplex, then the symbol $\mathcal{K}_\Delta$ will denote the simplicial complex consisting of all faces of Δ. If $\mathcal{K}$ is a simplicial complex and X is the union of the simplices of $\mathcal{K}$, then we say that $\mathcal{K}$ is a *triangulation* of X.

Given an m-dimensional simplex $\Delta = \Delta(Y)$, the *barycentre* c_Δ of Δ is defined by

$$c_\Delta = \sum_{y \in Y} \frac{1}{m+1}\, y.$$

Let $\mathcal{K}$ be a simplicial complex. The *barycentric subdivision* $\mathcal{B}(\mathcal{K})$ of $\mathcal{K}$ is the simplicial complex consisting of all simplices $\Delta(\{c_{\Delta_1}, c_{\Delta_2}, \ldots, c_{\Delta_s}\})$ where $\Delta_i \in \mathcal{K}$ for every $i = 1, 2, \ldots, s$, and Δ_j is a proper face of Δ_{j+1} for every $j = 1, 2, \ldots, s-1$. For a non-negative integer k, the *kth barycentric subdivision* $\mathcal{B}^k(\mathcal{K})$ of $\mathcal{K}$ is defined inductively by $\mathcal{B}^0(\mathcal{K}) = \mathcal{K}$ and $\mathcal{B}^{k+1}(\mathcal{K}) = \mathcal{B}(\mathcal{B}^k(\mathcal{K}))$.

Let $X \subseteq \mathbb{R}^n$ and $f\colon X \to \mathbb{R}$. Then f is *linear* on X if there are $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n} a_i x_i,$$

for every $\langle x_1, \ldots, x_n \rangle \in X$. If $\mathcal{K}$ is a triangulation of X and $f\colon X \to \mathbb{R}$ is a function that is linear on $\operatorname{cl}\Delta$ for every $\Delta \in \mathcal{K}$, then we say that f is *$\mathcal{K}$-linear*. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then the *variation* of f on X is the difference between the maximal and minimal values of f on X. The following lemmas are well known and easy to prove.

LEMMA 3.1. *If Δ is an n-dimensional simplex, then there exists an n-dimensional simplex $\Delta' \in \mathcal{B}^2(\mathcal{K}_\Delta)$ such that* $\operatorname{cl}\Delta' \subseteq \Delta$.

LEMMA 3.2. *For all positive integers n and m there is an integer k such that if Δ is an n-dimensional simplex, then there is a set $\mathcal{A} \subseteq \mathcal{B}^k(\mathcal{K}_\Delta)$ of cardinality m consisting of n-dimensional simplices such that*

$$\mathrm{cl}\,\Delta' \subseteq \Delta,$$

for any $\Delta' \in \mathcal{A}$ and

$$\mathrm{cl}\,\Delta' \cap \mathrm{cl}\,\Delta'' = \emptyset,$$

for any distinct $\Delta', \Delta'' \in \mathcal{A}$.

Proof. Choose an l such that for some n-dimensional simplex Δ the subdivision $\mathcal{B}^l(\mathcal{K}_\Delta)$ contains m distinct n-dimensional simplices. Note that this is also true for any other n-dimensional simplex. Then, by Lemma 3.1, $\mathcal{B}^{l+2}(\mathcal{K}_\Delta)$ contains the simplices as desired. So, $k = l + 2$ satisfies the lemma.

LEMMA 3.3. *Let $\mathcal{K}$ be a triangulation of $\mathbb{R}^n$ and $\Delta, \Delta' \in \bigcup_{k\in\omega} \mathcal{B}^k(\mathcal{K})$. If the simplex Δ is n-dimensional and a vertex of Δ' belongs to Δ then $\Delta' \subseteq \Delta$.*

Proof. For $k \in \omega$ let $\mathcal{A}_k$ denote the family of all n-dimensional simplices from $\mathcal{B}^k(\mathcal{K})$ and let $k, l \in \omega$ be such that $\Delta \in \mathcal{A}_k$ and $\Delta' \in \mathcal{B}^l(\mathcal{K})$. Notice that $k < l$, since otherwise the vertex from Δ' could not belong to $\bigcup \mathcal{A}_k \supset \Delta$. So, either $\Delta' \subseteq \Delta$ or $\Delta' \cap \Delta = \emptyset$, since simplices from $\mathcal{B}^l(\mathcal{K})$ form a partition of $\mathbb{R}^n$ which is finer than that formed by elements of $\mathcal{B}^k(\mathcal{K})$. But $\Delta' \cap \Delta = \emptyset$ contradicts the assumption that Δ contains a vertex of Δ'. So, $\Delta' \subseteq \Delta$.

LEMMA 3.4. [**9**] *If Δ is an n-dimensional simplex and d is the diameter of Δ, then the diameter of any n-dimensional simplex in $\mathcal{B}(\mathcal{K}_\Delta)$ is at most $(n/(n+1))d$.*

From Lemma 3.4 we immediately obtain the following corollary.

COROLLARY 3.5. *Let Δ be an n-dimensional simplex, f be a linear function on $\mathrm{cl}\,\Delta$, and a be the variation of f on $\mathrm{cl}\,\Delta$. If $\Delta' \in \mathcal{B}(\mathcal{K}_\Delta)$, then the variation of f on $\mathrm{cl}\,\Delta'$ is at most $(n/(n+1))a$.*

LEMMA 3.6. *If $\mathcal{K}$ is a triangulation of X, and V is the set of all vertices of $\mathcal{K}$, then any function $f\colon V \to \mathbb{R}$ can be uniquely extended to a $\mathcal{K}$-linear function on X.*

Proof of Proposition 2.3. Fix $n > 1$, let

$$\mathbb{D} = \left\{ \frac{s}{2^m} \colon s \in \mathbb{Z}, m \in \mathbb{N} \right\}$$

be the set of all *dyadic rationals* and let

$$\mathbb{D}_i = \left\{ \frac{-4^i}{2^i}, \frac{-4^i+1}{2^i}, \ldots, \frac{4^i}{2^i} \right\} \subseteq \mathbb{D} \quad \text{for every } i \in \omega.$$

Let $\mathcal{K}$ be any triangulation of $\mathbb{R}^n$. For each $i \in \omega$, we define integers k_i, r_i and ℓ_i, triangulations $\mathcal{K}_i$ and $\mathcal{K}_i'$ of $\mathbb{R}^n$, a function ψ_i on the set $\mathcal{A}_i$ of

n-dimensional simplices of $\mathcal{K}_i$, and a function ξ_i on $\mathcal{A}_i \times \mathbb{D}_i$ such that ψ_i and ξ_i take n-dimensional simplices in $\mathbb{R}^n$ as values. Let $k_0 = 0$.

Assume that $i \in \omega$ and that k_i has been defined. Let $\mathcal{K}_i = \mathcal{B}^{k_i}(\mathcal{K})$. By Lemma 3.1, for each $\Delta \in \mathcal{A}_i$ there exists an n-dimensional simplex $\psi_i(\Delta) \in \mathcal{B}^2(\mathcal{K}_\Delta)$ such that $\mathrm{cl}\,\psi_i(\Delta) \subseteq \Delta$. By Lemma 3.2, there is an integer r_i such that for every $\Delta \in \mathcal{A}_i$ and every $j \in \mathbb{D}_i$ there is an n-dimensional simplex $\xi_i(\Delta, j) \in \mathcal{B}^{r_i}(\mathcal{K}_{\psi(\Delta)})$ with

$$\mathrm{cl}\,\xi_i(\Delta, j) \subseteq \psi_i(\Delta)$$

such that

$$\mathrm{cl}\,\xi_i(\Delta, j) \cap \mathrm{cl}\,\xi_i(\Delta, j') = \emptyset$$

for any distinct $j, j' \in \mathbb{D}_i$. Let

$$\mathcal{K}_i' = \mathcal{B}^{2+r_i}(\mathcal{K}_i),$$

let ℓ_i be an integer such that

$$\left(\frac{n}{n+1}\right)^{\ell_i} \cdot 4^i \leqslant 2^{-i}, \tag{5}$$

and put $k_{i+1} = k_i + 2 + r_i + \ell_i$. This finishes the inductive construction.

Note that

$$\mathcal{K}_{i+1} = \mathcal{B}^{\ell_i}(\mathcal{K}_i') \quad \text{and} \quad \xi_i(\Delta, j) \in \mathcal{K}_i'$$

for every $i \in \omega$, $\Delta \in \mathcal{A}_i$ and $j \in \mathbb{D}_i$.

For the next step of our construction we will need the following additional notation. For each $i \in \omega$, let V_i be the set of vertices of $\mathcal{K}_i$, let V_i' be the set of vertices of $\mathcal{K}_i'$, and put

$$\bar{V}_i = V_i' \cap \bigcup_{\Delta \in \mathcal{A}_i} \mathrm{bd}\ \psi_i(\Delta).$$

Moreover, for every $i \in \omega$ and every $j \in \mathbb{D}_i$, we define

$$V_i^j = \bigcup_{\Delta \in \mathcal{A}_i} V_i^{\Delta, j},$$

where $V_i^{\Delta, j} \subseteq V_i'$ is the set of vertices of $\xi_i(\Delta, j)$. Also, for every $i \in \omega$ and $x \in \mathbb{R}^n$ let $\Delta'_{x,i} \in \mathcal{K}_i'$ be such that $x \in \Delta'_{x,i}$, and for $q \in \omega$ put

$$Y_q = \mathbb{R}^n \setminus \bigcup_{t > q} \bigcup_{\Delta \in \mathcal{A}_t} \psi_t(\Delta).$$

Note that for every $i, q \in \omega$ with $i > q$, the following condition holds:

$$\text{if } x \in Y_q, \text{ then every vertex of } \Delta'_{x,i} \text{ is in } Y_q. \tag{6}$$

Indeed, suppose that some vertex v of $\Delta'_{x,i}$ does not belong to Y_q. Then there are $t > q$ and $\Delta \in \mathcal{A}_t$ such that $v \in \psi_t(\Delta)$. Then, by Lemma 3.3, $\Delta'_{x,i} \subseteq \psi_t(\Delta)$, which contradicts the fact that $x \in Y_q$.

Now, we define recursively a sequence of functions $g_0, g_1, \ldots$ such that the following conditions hold for every $i \in \omega$:

(a) $g_i\colon \mathbb{R}^n \to [-2^{i-1}, 2^{i-1}]$ is $\mathcal{K}_i$-linear,

(b) if $x \in \operatorname{bd} \mathcal{K}_i$, then $g_{i+1}(x) = g_i(x)$,

(c) if $x \in \operatorname{bd} \psi_i(\Delta)$ for some $\Delta \in \mathcal{A}_i$, then $g_{i+1}(x) = 0$,

(d) if $x \in \operatorname{bd} \xi_i(\Delta, j)$ for some $\Delta \in \mathcal{A}_i$ and $j \in \mathbb{D}_i$, then $g_{i+1}(x) = j$,

(e) if there is $q \in \omega$ such that $x \in Y_q$, then $g_i(x) \in [-2^q, 2^q]$,

(f) for every $\Delta \in \mathcal{K}_i$ the variation of g_i on $\operatorname{cl} \Delta$ is at most 2^{-i}.

Let $g_0(x) = 0$ for every $x \in \mathbb{R}^n$. Suppose that $i \in \omega$ and that the function $g_i\colon \mathbb{R}^n \to [-2^{i-1}, 2^{i-1}]$ satisfies conditions (a)–(f). Let g_{i+1} be the unique $\mathcal{K}_i'$-linear extension of the function $h\colon V_i' \to [-2^i, 2^i]$ defined by

$$h(v) = \begin{cases} 0 & \text{if } v \in \bar{V}_i, \\ j & \text{if } v \in V_i^j \text{ for some } j \in \mathbb{D}_i, \\ g_i(v) & \text{otherwise.} \end{cases}$$

It is obvious that the function g_{i+1} satisfies conditions (a)–(d). To see that condition (e) holds, note that if $q < i$ and $x \in Y_q$, then every vertex v of $\Delta'_{x,i}$ is outside $\bigcup_{j \in \mathbb{D}_i} V_i^j$ implying that either $g_{i+1}(v) = g_i(v)$ or $g_{i+1}(v) = 0$. Now it follows from (6) and the inductive hypothesis that $g_{i+1}(v) \in [-2^q, 2^q]$ for any vertex v of $\Delta'_{x,i}$, implying that $g_{i+1}(x) \in [-2^q, 2^q]$. Finally, it follows from Corollary 3.5 and inequality (5) that the function g_{i+1} satisfies condition (f).

For each $i \in \omega$, let f_i be the restriction of g_i to $\operatorname{bd} \mathcal{K}_i$. If follows from condition (b) that f_{i+1} is an extension of f_i for every $i \in \omega$. Let

$$X = \bigcup_{i \in \omega} \operatorname{bd} \mathcal{K}_i,$$

and let

$$f = \bigcup_{i \in \omega} f_i\colon X \to \mathbb{R}.$$

We are going to extend the function f to a function on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n \setminus X$. If there is an integer $q \geqslant 0$ such that $x \in Y_q$, then it follows from condition (e) that $g_i(x) \in [-2^q, 2^q]$ for every $i \in \omega$. Then let $f(x)$ be the limit of some convergent subsequence of the sequence $\langle g_i(x) \rangle_{i=0}^{\infty}$. If such q does not exist, then let $f(x) = 0$. This completes the definition of the function f.

We will show first that f is peripherally continuous.

Denote by X' the set of points $x \in \mathbb{R}^n \setminus X$ for which the integer q as above exists; that is, let

$$X' = (\mathbb{R}^n \setminus X) \cap \bigcup_{q \in \omega} Y_q,$$

and put $X'' = (\mathbb{R}^n \setminus X) \setminus X'$. Note that $f(x) = 0$ for $x \in X''$.

To see that f is peripherally continuous choose $x \in \mathbb{R}^n \setminus X$ and, for each $i \in \omega$, let $\Delta_{x,i}$ be the simplex of $\mathcal{A}_i$ containing x. Since the sequence $k_0, k_1, \ldots$ is strictly increasing, it follows from Lemma 3.4 that the diameters of $\Delta_{x,i}$ converge to 0 as $i \to \infty$. If $x \in X'$, then the peripheral continuity of f at x follows from condition (f). If $x \in X''$, then there are infinitely many integers i such that x belongs to $\psi_i(\Delta)$ for some $\Delta \in \mathcal{A}_i$. Since $f(x) = 0$, the peripheral continuity of

f at x follows from condition (c). If $x \in X$, then for each $i \in \omega$, let $\mathscr{E}_{x,i}$ be the set of simplices $\Delta \in \mathscr{A}_i$ such that $x \in \operatorname{cl}\Delta$ and

$$Z_{x,i} = \bigcup_{\Delta \in \mathscr{E}_{x,i}} \operatorname{cl}\Delta.$$

Since the diameter of $Z_{x,i}$ is at most twice as large as the maximal diameter of a simplex in $\mathscr{E}_{x,i}$, it follows from Lemma 3.4 that the diameters of $Z_{x,i}$ converge to 0 as $i \to \infty$. Thus it follows from condition (f) that f is peripherally continuous at x.

By Theorem 1.1, it remains to define the subset G of $\mathbb{R}^n$ which is a dense G_δ-set and is such that any function $h\colon \mathbb{R}^n \to \mathbb{R}$ with $h(x) = f(x)$ for $x \notin G$ is peripherally continuous.

So, for each $j \in \mathbb{D}$ define

$$G_j = \bigcup_{i \in \{k : j \in \mathbb{D}_k\}} \bigcup_{\Delta \in \mathscr{A}_i} \xi_i(\Delta, j),$$

and notice that G_j is an open and dense subset of $\mathbb{R}^n$. This implies that

$$G = \bigcap_{j \in \mathbb{D}} G_j$$

is a dense G_δ-subset of $\mathbb{R}^n$. We will show that G has the desired property.

So, let $h\colon \mathbb{R}^n \to \mathbb{R}$ be any function with $h(x) = f(x)$ for $x \notin G$. The function h is peripherally continuous at any $x \notin G$, for the same reason that f is. If $x \in G$, then for any $j \in \mathbb{D}$ there is an arbitrarily large $i \in \omega$ such that $x \in \xi_i(\Delta, j)$ for some $\Delta \in \mathscr{A}_i$. Thus it follows from condition (d) that h is peripherally continuous at x. The proof is complete.

4. *Proof of Proposition* 2.4

In what follows we will identify a natural number n with the set of its predecessors, that is, $n = \{0, \ldots, n-1\}$. Let $A \subseteq \mathbb{R}$. We say that A is a *thick meagre* set if A is a countable union of nowhere-dense perfect sets and A is dense in $\mathbb{R}$. If $\langle A_i \colon i \in n\rangle$ is a family of sets then

$$\prod_{i \in n} A_i = A_0 \times \ldots \times A_{n-1}.$$

LEMMA 4.1. *If G is a dense G_δ-set in $\mathbb{R}^n$, then for each $i \in n$ there are a countable dense set $B_i \subseteq \mathbb{R}$ and a thick meagre set $Y_i \subseteq \mathbb{R}$ such that $B_i \cap Y_i = \emptyset$ and*

$$\prod_{i \in n} (B_i \cup Y_i) \subset G.$$

Proof. Let G be a dense G_δ-set in $\mathbb{R}^n$. First note that it is enough to prove that for each $i \in n$ there is a thick meagre set $Y_i \subseteq \mathbb{R}$ such that

$$\prod_{i \in n} Y_i \subset G, \tag{7}$$

since then for every $i \in n$ there exists a countable dense $B_i \subset Y_i$ and a thick meagre set $Y_i' \subset Y_i$ such that $B_i \cap Y_i' = \emptyset$.

We prove (7) by induction on n. If $n = 1$, then it is clear that (7) holds. Assume that $n \geqslant 2$ and that (7) holds for smaller values of n. We claim that

(†) there are a thick meagre set $Y \subseteq \mathbb{R}$, and a dense G_δ-set G' in $\mathbb{R}^{n-1}$, such that $Y \times G' \subseteq G$.

It is obvious that (†) and the induction hypothesis imply that the lemma holds.

To prove (†) we will first show that

($\star$) for every $p < q$ there exist a nowhere-dense perfect set $Y_{p,q} \subseteq (p, q)$ and a dense G_δ-set $G_{p,q} \subseteq \mathbb{R}^{n-1}$ such that $Y_{p,q} \times G_{p,q} \subseteq G$.

Clearly ($\star$) implies (†), since for $\mathcal{A} = \{\langle p, q\rangle \in \mathbb{Q}^2 : p < q\}$ the sets $Y = \bigcup_{\langle p,q\rangle \in \mathcal{A}} Y_{p,q}$ and $G' = \bigcap_{\langle p,q\rangle \in \mathcal{A}} G_{p,q}$, satisfy (†).

Now we show that ($\star$) holds. Assume that

$$G = \bigcap_{m\in\omega} U_m,$$

where U_m is an open dense set in $\mathbb{R}^n$ for every $m \in \omega$, and let $p < q$. Let J_0, J_1, ... be an enumeration of some countable basis of the topology of $\mathbb{R}^{n-1}$, and let $\langle t_0, u_0\rangle$, $\langle t_1, u_1\rangle$, ... be an enumeration of $\omega \times \omega$. Let T_i be the set of all zero-one sequences $g: i \to 2$ of length i, and for $g \in T_i$ and $j \in 2$ let $g * j \in T_{i+1}$ be the *concatenation* of g and j, that is,

$$g * j = \langle s_0, s_1, \ldots, s_{n-1}, j\rangle,$$

where $g = \langle s_0, s_1, \ldots, s_{n-1}\rangle$. For each $i \in \omega$ we define, by induction on i, an open set $V_i \subseteq \mathbb{R}^{n-1}$ and a family $\{W_g : g \in T_i\}$ of non-empty open subsets of (p, q), such that the following conditions hold for every $i \in \omega$:

(i) $V_i \cap J_{t_i} \neq \emptyset$;

(ii) $(\bigcup_{g\in T_i} \operatorname{cl} W_g) \times V_i \subseteq U_{u_i}$;

(iii) $\operatorname{diam} W_g \leqslant 2^{-i}$ for every $g \in T_i$;

(iv) $\operatorname{cl} W_{g*0} \cap \operatorname{cl} W_{g*1} = \emptyset$ for every $g \in T_{i-1}$ provided $i > 0$;

(v) $\operatorname{cl} W_{g*0} \cup \operatorname{cl} W_{g*1} \subseteq W_g$ for every $g \in T_{i-1}$ provided $i > 0$.

For $i = 0$ choose arbitrary $W_\tau \subset (p, q)$, τ being an empty sequence, and $V_0 \subset J_{t_0}$ such that $\operatorname{cl} W_\tau \times V_0 \subseteq U_{u_0}$. Such a choice can be made, since U_{u_0} is dense in $\mathbb{R}^n$. It is clear that with such a choice conditions (i)–(v) are satisfied.

To make the inductive step choose $i < \omega$, $i > 0$, such that V_{i-1} and W_g for each $g \in T_{i-1}$ satisfying (i)–(v) are already defined. Since U_{u_i} is dense open in $\mathbb{R}^n$, there are non-empty open sets $V_i \subseteq J_{t_i}$ and, for every $g \in T_{i-1}$, a non-empty open set $W'_g \subseteq W_g$ such that

$$\left(\bigcup_{g\in T_{i-1}} \operatorname{cl} W'_g\right) \times V_i \subseteq U_{u_i}.$$

For each $g \in T_{i-1}$ choose non-empty open sets $W_{g*0}, W_{g*1} \subseteq W'_g$ satisfying (iii)–(v). This completes our construction.

To prove that ($\star$) holds, it suffices to take

$$Y_{p,q} = \bigcap_{i\in\omega} \bigcup_{g\in T_i} \operatorname{cl} W_g \quad \text{and} \quad G_{p,q} = \bigcap_{m\in\omega} H_m,$$

where

$$H_m = \bigcup \{V_i\colon\ u_i = m\}.$$

Then it is clear that $Y_{p,q}$ is a nowhere-dense perfect subset of (p,q) and that $G_{p,q}$ is a G_δ-subset of $\mathbb{R}^{n-1}$. To see that $G_{p,q}$ is dense in $\mathbb{R}^{n-1}$, it is enough to note that for every $m \in \omega$ the set H_m intersects every element of the basis $\{J_i\colon\ i \in \omega\}$ of $\mathbb{R}^{n-1}$. It remains to verify that

$$Y_{p,q} \times G_{p,q} \subseteq G.$$

So, choose arbitrary $x \in Y_{p,q}$, $y \in G_{p,q}$ and $m \in \omega$. Then $y \in H_m$ and there exists $i \in \omega$ such that $u_i = m$ and $y \in V_i$, which implies that

$$\langle x, y\rangle \in \left(\bigcup_{g \in T_i} \operatorname{cl} W_g\right) \times V_i \subseteq U_{u_i} = U_m.$$

Therefore $\langle x, y\rangle \in \bigcap_{m\in\omega} U_m = G$ and so the proof is complete.

LEMMA 4.2. *If $B \subseteq \mathbb{R}$ is a countable dense set, $Y \subseteq \mathbb{R}$ is a thick meagre set, and $Z \subseteq \mathbb{R}$ is a meagre set such that $B \cap Y = B \cap Z = \emptyset$, then there is an increasing homeomorphism $g\colon \mathbb{R} \to \mathbb{R}$ such that $Z \subseteq g(Y)$ and $g(B) = B$.*

Proof. It is clear that we can assume that the set Z is thick meagre. Let

$$Z = \bigcup_{i\in\omega} Z_i,$$

where $\{Z_i\colon\ i \in \omega\}$ is a family of mutually disjoint nowhere-dense perfect sets. Let $\langle b_i\colon\ i \in \omega\rangle$ be an enumeration of B and $\langle I_i\colon\ i \in \omega\rangle$ be an enumeration of all non-empty open intervals (p,q) with rational endpoints $p, q \in \mathbb{R}$. We construct, by induction on $i \in \omega$, two strictly increasing sequences $\langle n_i \in \omega\colon\ i \in \omega\rangle$ and $\langle m_i \in \omega\colon\ i \in \omega\rangle$, and a sequence $\langle f_i\colon\ i \in \omega\rangle$ of functions such that the following conditions hold for every $k \in \omega$:

(i) $f_k\colon\ \bigcup_{i\leqslant k} Z_{n_i} \cup \{b_{m_i}\colon\ i \leqslant k\} \to Y \cup B$ is a strictly increasing continuous function extending $\bigcup_{i<k} f_i$ such that

$$f_k\left[\bigcup_{i\leqslant k} Z_{n_i}\right] \subseteq Y \quad \text{and} \quad f_k[\{b_{m_i}\colon\ i \leqslant k\}] \subseteq B;$$

(ii) if $k = 4j$, then $\bigcup_{i\leqslant j} Z_i \subseteq \operatorname{dom} f_k$;

(iii) if $k = 4j+1$, then $f_k[\bigcup_{i\leqslant k} Z_{n_i}] \cap I_j \neq \emptyset$;

(iv) if $k = 4j+2$, then $\{b_i\colon\ i \leqslant j\} \subseteq \operatorname{dom} f_k$;

(v) if $k = 4j+3$, then $\{b_i\colon\ i \leqslant j\} \subseteq \operatorname{range} f_k$.

Then the function

$$f = \bigcup_{i\in\omega} f_i\colon\ Z \cup B \to Y \cup B$$

is strictly increasing, $f[Z] \subseteq Y$ is dense in $\mathbb{R}$, and $f[B] = B$. Thus f can be extended to a homeomorphism h from $\mathbb{R}$ to $\mathbb{R}$ and $g = h^{-1}$ satisfies the requirements. This completes the proof.

In the remainder of this section we will use the following non-standard notation. If $\langle A_i \colon i \in n\rangle$ is a family of sets, C is a set and $j \in n$, then let

$$A_i \vee_j C = \begin{cases} C & \text{if } i = j, \\ A_i & \text{if } i \neq j. \end{cases}$$

If moreover $\langle B_i \colon i \in n\rangle$ is a family of sets and f is a function from n into $2 = \{0, 1\}$, then define

$$A_i \vee_f B_i = \begin{cases} A_i & \text{if } f(i) = 0, \\ B_i & \text{if } f(i) = 1. \end{cases}$$

We will also use the notation $A_i \vee_f B_i \vee_j C$ to denote the set $D_i \vee_j C$ where $D_i = A_i \vee_f B_i$, that is,

$$A_i \vee_f B_i \vee_j C = \begin{cases} C & \text{if } i = j, \\ B_i & \text{if } i \neq j \text{ and } f(i) = 1, \\ A_i & \text{if } i \neq j \text{ and } f(i) = 0. \end{cases}$$

LEMMA 4.3. *Let $G \subseteq \mathbb{R}^n$ be a G_δ-set. If $f \colon n \to 2$ is a function, $i \in n$ and $\langle b_0, \ldots, b_{n-1}\rangle \in \mathbb{R}^n$, then the set*

$$\left\{x \in \mathbb{R}\colon \prod_{t\in n}(\{b_t\} \vee_f \mathbb{R} \vee_i \{x\}) \subseteq G\right\}$$

is a G_δ-subset of $\mathbb{R}$.

Proof. Assume that

$$G = \bigcap_{k\in\omega} U_k,$$

with $U_k \subseteq \mathbb{R}^n$ being open for every $k \in \omega$. Let

$$D_x^r = \prod_{t\in n}(\{b_t\} \vee_f [-r, r] \vee_i \{x\}) \subseteq \mathbb{R}^n,$$

for every $x \in \mathbb{R}$ and $r \in \omega$, and let

$$V_k^r = \{x \in \mathbb{R}\colon D_x^r \subseteq U_k\},$$

for every $k, r \in \omega$. Then

$$\left\{x \in \mathbb{R}\colon \prod_{t\in n}(\{b_t\} \vee_f \mathbb{R} \vee_i \{x\}) \subseteq G\right\} = \bigcap_{k\in\omega}\bigcap_{r\in\omega} V_k^r.$$

To complete the proof it remains to show that the set V_k^r is open in $\mathbb{R}$ for every $k, r \in \omega$.

Suppose that $x \in V_k^r$. Then $D_x^r \subseteq U_k$ and since U_k is open, it follows that for every $y \in D_x^r$ there is an open neighbourhood W_y of y in $\mathbb{R}^n$ with $W_y \subseteq U_k$. Since D_x^r is compact, there is a finite subfamily of $\{W_y \colon y \in D_x^r\}$ that covers D_x^r, which implies that there is an open neighbourhood $A \subseteq \mathbb{R}$ of x such that

$$\prod_{t\in n}(\{b_t\} \vee_f [-r, r] \vee_i A) \subseteq U_k.$$

So $A \subseteq V_k^r$, which implies that V_k^r is open and hence completes the proof.

Proof of Proposition 2.4. Assume that $G \subseteq \mathbb{R}^n$ is a dense G_δ-set. By Lemma 4.1, for each $i \in n$ there are a countable dense set $B_i \subseteq \mathbb{R}$ and a thick meagre set $Y_i \subseteq \mathbb{R}$ such that $B_i \cap Y_i = \emptyset$ and

$$\prod_{i\in n}(B_i \cup Y_i) \subseteq G.$$

We will define homeomorphisms $g^i_j\colon \mathbb{R} \to \mathbb{R}$ for every $i \in n$ and $j \in \{1, 2, \dots, n\}$ such that if

$$h_j = g^0_j \times \dots \times g^{n-1}_j\colon \mathbb{R}^n \to \mathbb{R}^n,$$

then

$$\prod_{i\in n}(B_i \vee_f \mathbb{R}) \subseteq G \cup \bigcup_{j=1}^{k} h_j(G) \tag{8}$$

for every $k \in n+1$ and every function $f\colon n \to 2$ such that $|f^{-1}(1)| = k$. (Here $|X|$ stands for the cardinality of the set X.) The construction will be done by induction with respect to k.

Note that for $k = 0$, the equation (8) is already satisfied for the constant function $f \equiv 0$, the only $f\colon n \to 2$ with $|f^{-1}(1)| = 0$. This gives the starting point for our induction. Notice also that if $k = n$, then the equation (8) with the constant function $f \equiv 1$ implies that

$$G \cup \bigcup_{j=1}^{n} h_j(G) = \mathbb{R}^n.$$

Thus it remains to perform the inductive step.

Assume that $k \in n$ and that the homeomorphisms $g^i_j\colon \mathbb{R} \to \mathbb{R}$ have been defined for every $i \in n$ and $j \in \{1, \dots, k\}$ in such a way that (8) is satisfied for every $f\colon n \to 2$ with $|f^{-1}(1)| = k$. We are going to define g^i_{k+1} for every $i \in n$ so that the equation (8) with k replaced by $k+1$ is satisfied for every $f\colon n \to 2$ with $|f^{-1}(1)| = k+1$.

For every $i \in n$, let F_i be the set of all functions $f\colon n \to 2$ such that

$$|f^{-1}(1)| = k \quad \text{and} \quad f(i) = 0.$$

Fix $i \in n$. It follows from Lemma 4.3 that for every

$$b = \langle b_0, \dots, b_{n-1}\rangle \in B_0 \times \dots \times B_{n-1},$$

and every $f \in F_i$ there is a G_δ-set $K^{f,b}_i \subseteq \mathbb{R}$ such that

$$\prod_{t\in n}(\{b_t\} \vee_f \mathbb{R} \vee_i K^{f,b}_i) \subseteq G \cup \bigcup_{j=1}^{k} h_j(G).$$

Notice also that, by (8), $B_i \subseteq K^{f,b}_i$. So, $K^{f,b}_i$ is a dense G_δ-set. Thus, the set

$$K_i = \bigcap\{K^{f,b}_i\colon\ f \in F_i \text{ and } b \in B_0 \times \dots \times B_{n-1}\}$$

is a dense G_δ-set with $B_i \subseteq K_i$ and

$$\prod_{t\in n}(B_t \vee_f \mathbb{R} \vee_i K_i) \subseteq G \cup \bigcup_{j=1}^{k} h_j(G) \tag{9}$$

for every $f \in F_i$. In particular, $Z_i = \mathbb{R} \setminus K_i$ is a meagre set with $B_i \cap Z_i = \emptyset$. By Lemma 4.2, there is a homeomorphism $g^i_{k+1} \colon \mathbb{R} \to \mathbb{R}$ such that $Z_i \subseteq g^i_{k+1}(Y_i)$ and $g^i_{k+1}(B_i) = B_i$.

Let $h_{k+1} = g^0_{k+1} \times \ldots \times g^{n-1}_{k+1}$. We claim that

$$\prod_{i\in n}(B_i \vee_f \mathbb{R}) \subseteq G \cup \bigcup_{j=1}^{k+1} h_j(G)$$

for every $f \colon n \to 2$ with $|f^{-1}(1)| = k+1$. Indeed, let $f \colon n \to 2$ be any function satisfying $|f^{-1}(1)| = k+1$ and pick

$$x \in \prod_{i\in n}(B_i \vee_f \mathbb{R}).$$

We will show that

$$x \in G \cup \bigcup_{j=1}^{k+1} h_j(G).$$

If there is $i \in f^{-1}(1)$ such that

$$x \in \prod_{t\in n}(B_t \vee_f \mathbb{R} \vee_i K_i),$$

then it follows from (9) that

$$x \in G \cup \bigcup_{j=1}^{k} h_j(G),$$

so we can assume that, for every $i \in f^{-1}(1)$, we have

$$x \notin \prod_{t\in n}(B_t \vee_f \mathbb{R} \vee_i K_i).$$

Then

$$x \in \prod_{i\in n}(B_i \vee_f Z_i) \subseteq \prod_{i\in n}(B_i \vee_f g^i_{k+1}(Y_i)).$$

Since $g^i_{k+1}(B_i) = B_i$ and

$$\prod_{i\in n}(B_i \vee_f Y_i) \subseteq G$$

for every $i \in n$, it follows that

$$\prod_{i\in n}(B_i \vee_f g^i_{k+1}(Y_i)) = h_{k+1}\left(\prod_{i\in n}(B_i \vee_f Y_i)\right) \subseteq h_{k+1}(G).$$

Therefore $x \in h_{k+1}(G)$ and so the proof is complete.

5. *Proofs of Propositions* 2.7 *and* 2.8

In the proof that follows we will need some additional definitions and results from dimension theory. (See, for example, [**10**].)

Given $X \subseteq \mathbb{R}^n$ and an integer $m \geqslant 1$, we say that X is an *m-dimensional Cantor manifold* if X is compact, $\operatorname{ind} X = m$, and for every $Y \subseteq X$ with $\operatorname{ind} Y \leqslant m-2$, the set $X \setminus Y$ is connected. Note that an m-dimensional Cantor manifold X is connected and for every $p \in X$,

$$\operatorname{ind}_p X = m,$$

that is, there exists an open neighbourhood W of p such that $\operatorname{ind} \operatorname{bd}_X U = m - 1$ for any open neighbourhood $U \subseteq W$ of p.

Given $X \subseteq \mathbb{R}^n$ and $p, q \in \mathbb{R}^n \setminus X$, we say that X *separates* p and q if they are in distinct components of $\mathbb{R}^n \setminus X$.

The following lemmas are proved in [**10**].

LEMMA 5.1. *For any compact $Y \subseteq \mathbb{R}^n$ with $\operatorname{ind} Y \geqslant m$ there exists an m-dimensional Cantor manifold $X \subseteq Y$.*

LEMMA 5.2. *If $X \subseteq \mathbb{R}^n$ is a compact set that separates p and q, and no proper closed subset of X does so, then X is an $(n-1)$-dimensional Cantor manifold.*

Using Zorn's Lemma one can easily prove the following lemma.

LEMMA 5.3. *If $X \subseteq \mathbb{R}^n$ is a compact set that separates p and q, then there is a compact $X' \subseteq X$ that separates p and q and no proper closed subset of X' does so.*

Given a subset U of $\mathbb{R}^n$, we say that U is a *quasiball* if U is a bounded and connected open set, and $\operatorname{bd} U$ is an $(n-1)$-dimensional Cantor manifold. The open ball in $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$ will be denoted by $B^n(x, \varepsilon)$.

LEMMA 5.4. *If V is an open set and*

$$x \in V \subseteq B^n(x, \delta)$$

for some $x \in \mathbb{R}^n$ and $\delta > 0$, then there is a quasiball $U \subseteq B^n(x, \delta)$ containing x with $\operatorname{bd} U \subseteq \operatorname{bd} V$.

Proof. Let y be an element of the unbounded component of $\mathbb{R}^n \setminus \operatorname{cl} V$. Since V is bounded, $\operatorname{bd} V$ is compact, so it follows from Lemmas 5.3 and 5.2 that there is an $(n-1)$-dimensional Cantor manifold $X \subseteq \operatorname{bd} V$ that separates x from y. Let U be the component of $\mathbb{R}^n \setminus X$ containing x. It is clear that U satisfies the requirements.

COROLLARY 5.5. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Then for any $x \in \mathbb{R}^n$, any $\varepsilon > 0$ and any open set W in $\mathbb{R}^n$ containing x, there is a quasiball $U \subseteq W$ containing x such that $|f(x) - f(y)| < \varepsilon$ for any $y \in \operatorname{bd} U$.*

Proof. Let $\delta > 0$ be such that $B^n(x, \delta) \subset W$. Since f is peripherally continuous, there is an open neighbourhood $V \subset \operatorname{bd} V \subset B^n(x, \delta)$ of x such that $|f(x) - f(y)| < \varepsilon$ for any $y \in \operatorname{bd} U$. Then U from Lemma 5.4 satisfies the requirements.

Given open sets U and W in $\mathbb{R}^n$, we say that U and W are *independent* if all the intersections $U \cap W$, $U \cap W^c$, $U^c \cap W$, and $U^c \cap W^c$ are non-empty, where U^c

and W^c are the complements of the closures of U and W, respectively. Given $x \in \mathbb{R}^n$, a *half-line starting at* x is a set A of the form

$$A = \{x + \alpha z \colon \alpha \geqslant 0\}$$

for some non-zero $z \in \mathbb{R}^n$.

LEMMA 5.6. *If* U *and* W *are independent quasiballs, then* $\operatorname{bd} U \cap \operatorname{bd} W \neq \emptyset$.

Proof. Let U^c and W^c be the complements of the closures of U and W respectively. Since $W \cap U$ and $W \cap U^c$ are non-empty and W is connected, it follows that $W \cap \operatorname{bd} U$ is non-empty. Similarly, $U \cap \operatorname{bd} W$ is non-empty.

Since U is bounded, any half-line starting at a point in U intersects $\operatorname{bd} U$. The analogous statement holds for W. Let $x \in U \cap W$ and A be a half-line starting at x. Since $\operatorname{bd} U \cup \operatorname{bd} W$ is compact, there is

$$y \in A \cap (\operatorname{bd} U \cup \operatorname{bd} W)$$

such that the half-line B starting at y that is a subset of A does not intersect $\operatorname{bd} U \cup \operatorname{bd} W$ except at y. Without loss of generality, we can assume that $y \in \operatorname{bd} U \setminus \operatorname{bd} W$. Then B does not intersect $\operatorname{bd} W$, which implies that $B \cap W = \emptyset$. Therefore

$$y \in W^c \cap \operatorname{bd} U,$$

which implies that $W^c \cap \operatorname{bd} U$ is non-empty. Since $W \cap \operatorname{bd} U$ is also non-empty and $\operatorname{bd} U$ is connected, we conclude that $\operatorname{bd} U \cap \operatorname{bd} W$ is non-empty.

For $n \in \omega$ let ω^n be the set of all sequences of elements of ω of length n, and let

$$\omega^{<\omega} = \bigcup_{n \in \omega} \omega^n.$$

Note that $\omega^0 = \{\emptyset\}$. For $s \in \omega^{<\omega}$ and $j \in \omega$ let $s * j$ be the *concatenation* of s and j, that is,

$$s * j = \langle s_0, s_1, \ldots, s_{n-1}, j \rangle,$$

where $s = \langle s_0, s_1, \ldots, s_{n-1} \rangle$. Given $T \subseteq \omega^{<\omega}$ and $n \in \omega$, let

$$T_n = T \cap \omega^n.$$

Given $s \in T_n$ and $t \in T_{n+1}$ such that there is $j \in \omega$ with $t = s * j$, we say that s is the *father* of t and that t is a *son* of s. A non-empty subset T of $\omega^{<\omega}$ is a *tree* if for every $s \in T \setminus \{\emptyset\}$ the father of s belongs to T and every element of T has at least one son in T. We say that the tree T is *finitely branching* if T_n is finite for every $n \in \omega$.

Let T be a finitely branching tree and $f \colon \mathbb{R}^m \to \mathbb{R}$ be a peripherally continuous function. A family

$$\mathcal{U} = \{U_s \colon s \in T\}$$

of quasiballs in $\mathbb{R}^m$ will be called a *good* T*-family of quasiballs for* f if there are a function $\eta \colon T \to \mathbb{R}^m$ and two sequences $\langle q_n \colon n \in \omega \rangle$ and $\langle r_n \colon n \in \omega \rangle$ of positive real numbers such that the series $\sum_{n=0}^{\infty} q_n$ and $\sum_{n=0}^{\infty} r_n$ converge and the following conditions are satisfied for any $n \in \omega$, $s \in T_n$ and any son t of s:

(i) $\eta(s) \in U_s$;

(ii) the distance from $\eta(s)$ to any element of U_s is at most q_n;

(iii) $|f(x) - f(\eta(s))| \leqslant r_n$ for any $x \in \operatorname{bd} U_s$;

(iv) $\eta(t) \in \operatorname{bd} U_s$;

(v) the quasiballs U_s and U_t are independent.

For any $\gamma \in \omega^\omega$ let γ_n be the initial segment of γ of length n. Assume that $\mathcal{U} = \{U_s\colon s \in T\}$ is a good T-family of quasiballs for f and that $\eta\colon T \to \mathbb{R}^m$, $\langle q_n\colon n \in \omega\rangle$ and $\langle r_n\colon n \in \omega\rangle$ satisfy conditions (i)–(v). Define

$$T^* = \{\gamma \in \omega^\omega\colon \gamma_n \in T\}.$$

Given $\gamma \in T^*$, we say that $x \in \mathbb{R}^m$ is a *γ-limit* of $\mathcal{U}$ if for every open neighbourhood V of x in $\mathbb{R}^m$ there is $k \in \omega$ with

$$U_{\gamma_n} \cap V \neq \emptyset$$

for every $n \geqslant k$. It follows from condition (ii) that for every $\gamma \in T^*$ there is exactly one γ-limit x_γ of $\mathcal{U}$. Define

$$L_{\mathcal{U}} = \{x_\gamma \in \mathbb{R}^m\colon \gamma \in T^*\}$$

to be the set of all limit points of $\mathcal{U}$.

LEMMA 5.7. *Let T be a finitely branching tree and $f\colon \mathbb{R}^m \to \mathbb{R}$ be a peripherally continuous function. If $\mathcal{U} = \{U_s\colon s \in T\}$ is a good T-family of quasiballs for f, then the restriction of f to $L_{\mathcal{U}}$ is continuous.*

Proof. Let $\eta\colon T \to \mathbb{R}^m$, $\langle q_n\colon n \in \omega\rangle$ and $\langle r_n\colon n \in \omega\rangle$ satisfy conditions (i)–(v). Given $\gamma \in T^*$ and $t \in \omega$, let

$$B_{\gamma,t} = \bigcup_{n=t}^{\infty} \operatorname{bd} U_{\gamma_n}.$$

Let $t \in \omega$. It follows from condition (v) and Lemma 5.6 that the set $B_{\gamma,t}$ is connected. Since the γ-limit x_γ of $\mathcal{U}$ belongs to $\operatorname{cl} B_{\gamma,t}$, the union $B_{\gamma,t} \cup \{x_\gamma\}$ is connected. Since f is a peripherally continuous function, it is also a Darboux function, which implies that the set $f(B_{\gamma,t} \cup \{x_\gamma\})$ is connected, and so $f(x_\gamma) \in \operatorname{cl} f(B_{\gamma,t})$. Since

$$|f(x_\gamma) - f(\eta(\gamma_t))| \leqslant |f(\eta(\gamma_t)) - f(\eta(\gamma_{t+1}))| + |f(\eta(\gamma_{t+1})) - f(\eta(\gamma_{t+2}))| + \dots,$$

it follows from (iii) that

$$|f(x_\gamma) - f(\eta(\gamma_t))| \leqslant \sum_{n=t}^{\infty} r_n, \tag{10}$$

for every $\gamma \in T^*$ and $t \in \omega$.

Now let $x \in L_{\mathcal{U}}$ and $\varepsilon > 0$. Since the series $\sum_{n=0}^{\infty} r_n$ converges, there exists $t \in \omega$ such that

$$\sum_{n=t}^{\infty} r_n < \tfrac{1}{2}\varepsilon. \tag{11}$$

For each $s \in T_t$ let

$$B_s = \{x_\gamma \colon \gamma_t = s\}.$$

It is clear that B_s is closed in $\mathbb{R}^n$ for every $s \in T_t$. Since the set T_t is finite, there is an open neighbourhood V of x such that

$$V \cap B_s = \emptyset,$$

for every $s \in T_t$ with $x \notin B_s$. It follows that for every $y \in V \cap L_{\mathcal{U}}$ there exists $s \in T_t$ with $x, y \in B_s$, which implies by (10) and (11) that

$$|f(x) - f(y)| \leqslant |f(x) - f(\eta(s))| + |f(y) - f(\eta(s))| < \varepsilon.$$

Therefore f is continuous at x and so the proof is complete.

Proof of Proposition 2.7. Let x_0, W and ε be as in the proposition. Let $\langle q_i \colon i \in \omega\rangle$ and $\langle r_i \colon i \in \omega\rangle$ be any sequences of positive real numbers such that

$$\sum_{i=0}^{\infty} r_i < \varepsilon \tag{12}$$

and

$$B^n\left(x_0, \sum_{i=0}^{\infty} q_i\right) \subseteq W. \tag{13}$$

We will define inductively a finitely branching tree T, a good T-family $\mathcal{U} = \{U_s \colon s \in T\}$ of quasiballs for f, and a function $\eta \colon T \to \mathbb{R}^n$ such that conditions (i)–(v) are satisfied, and moreover,

$$\operatorname{bd}\left(\bigcup_{i=0}^{m} \bigcup_{s \in T_i} U_s\right) \subseteq \bigcup_{t \in T_{m+1}} U_t. \tag{14}$$

Let $T_0 = \{\emptyset\}$ and $\eta(\emptyset) = x_0$. Since f is peripherally continuous, it follows from Corollary 5.5 that there is a $U_\emptyset$ such that conditions (i)–(iii) are satisfied. Suppose that $m \in \omega$, and that T_i and U_s have been defined for every $i \leqslant m$ and $s \in \bigcup_{i \leqslant m} T_i$. Let

$$C = \operatorname{bd}\left(\bigcup_{i=0}^{m} \bigcup_{s \in T_i} U_s\right). \tag{15}$$

Since the set T_m is finite, we have

$$C \subseteq \operatorname{bd}\left(\bigcup_{i<m} \bigcup_{s \in T_i} U_s\right) \cup \bigcup_{s \in T_m} \operatorname{bd} U_s,$$

and condition (14) of the inductive hypothesis implies that

$$\operatorname{bd}\left(\bigcup_{i<m} \bigcup_{s \in T_i} U_s\right) \subseteq \bigcup_{s \in T_m} U_s.$$

Therefore

$$C \subseteq \bigcup_{s \in T_m} \operatorname{bd} U_s.$$

Let $\{C_s \colon s \in T_m\}$ be a partition of C such that $C_s \subseteq \operatorname{bd} U_s$ for every $s \in T_m$. Since f is peripherally continuous, it follows from Corollary 5.5 that for every $y \in C$

there is a quasiball B_y containing y such that the distance from y to any element of B_y is at most q_n and $|f(x) - f(y)| \leqslant r_n$ for any $x \in \operatorname{bd} B_y$. Moreover B_y can be chosen so that B_y and U_s are independent if $y \in C_s$. Since C is compact, there is a finite subset Y of C such that

$$C \subseteq \bigcup_{y \in Y} B_y.$$

Let

$$Y_s = C_s \cap Y.$$

Then $\{Y_s\colon\ s \in T_m\}$ is a partition of Y. Define

$$T_{m+1} = \{s * j\colon\ s \in T_m \text{ and } j \in \{0, 1, \dots, |Y_s| - 1\}\}.$$

For $t \in T_{m+1}$ let $\eta(t)$ be such that if $s \in T_m$, then

$$\{\eta(t)\colon\ t \text{ is a son of } s\} = Y_s,$$

and let

$$U_t = B_{\eta(t)}.$$

This completes the definition of T, η and $\mathcal{U}$. Let

$$U = \bigcup_{s \in T} U_s.$$

Then

$$\operatorname{bd} U \subseteq L_{\mathcal{U}},$$

and so Lemma 5.7 implies that f is continuous on $\operatorname{bd} U$. Condition (13) implies that $U \subseteq W$ and condition (12) implies that $|f(x_0) - f(y)| < \varepsilon$; thus the proof is complete.

Proof of Proposition 2.8. Let T be the tree consisting of all finite zero-one sequences. We are going to define a good T-family $\mathcal{U} = \{U_s\colon\ s \in T\}$ of quasiballs for g. Let $\langle r_i\colon\ i \in \omega\rangle$ be a sequence of positive real numbers such that the series $\sum_{i=0}^{\infty} r_i$ converges. We shall define a sequence $\langle q_i\colon\ i \in \omega\rangle$ of positive real numbers with $\sum_{i=0}^{\infty} q_i < \infty$ and a function $\eta\colon\ T \to \mathbb{R}^n$ such that conditions (i)–(v) are satisfied. We will also define an auxiliary function $\eta'\colon\ T \to X$. The construction will be done by induction on $i < \omega$ in such a way that the following additional conditions hold:

(a) $\eta(\emptyset) \in X$ is arbitrary and $q_1 = q_0 < \frac{1}{2}\operatorname{diam}(X)$;

(b) $\eta(s * 0) = \eta(s * 1) = \eta'(s) \in \operatorname{bd} U_s \cap X$ for any $s \in T_i$;

(c) $q_i = \frac{1}{4}\min_{s \in T_{i-2}} |\eta'(s * 0) - \eta'(s * 1)|$ for $i > 1$;

(d) $\operatorname{cl} U_{s*1} \subseteq U_{s*0}$ for any $s \in T_i$.

To see that the construction can be made, notice that the choice of each U_s satisfying (i)–(iii), (v) and (d) can be guaranteed by Corollary 5.5. We can choose $\eta'(s) \in \operatorname{bd} U_s \cap X$, since $\operatorname{bd} U_s \cap X$ is non-empty as X is connected and U_s has the diameter smaller than X. So, (b) implies (iv). Also, $q_i > 0$, since the points $\eta'(s * 0)$ and $\eta'(s * 1)$ are different by (d). This completes the construction.

Let $P = L_{\mathcal{U}}$. It is clear that P is a closed subset of X, and it follows from (c) that $x_\gamma \neq x_\delta$ for distinct $\gamma, \delta \in T^*$. This implies that P is a perfect set. We conclude from Lemma 5.7 that the restriction of g to P is continuous, which completes the proof.

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Department of Mathematics
West Virginia University
P.O. Box 6310
Morgantown
West Virginia 26506-6310
U.S.A.

E-mail: kcies@vaxa.wvnet.edu
jerzy@math.wvu.edu