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Algebraic properties of the class of Sierpiński–Zygmund functions ☆

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Abstract

Sums, products and compositions with Sierpiński–Zygmund functions are investigated. Moreover, cardinal invariants connected with those operations are defined and studied. © 1997 Elsevier Science B.V.

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1. Preliminaries

Let us establish some terminology to be used. No distinction is made between a function and its graph. The family of all functions from a set X into Y will be denoted by Y^X. Symbol $\operatorname{card}(X)$ will stand for the cardinality of a set X. The cardinality of the set $\mathbb{R}$ of real numbers is denoted by $\mathfrak{c}$. Symbol $[X]^\kappa$ denotes the family of all subsets Y of X with $\operatorname{card}(Y) = \kappa$. Similarly we define $[X]^{<\kappa}$ and $[X]^{\leqslant\kappa}$. For a cardinal number κ we will write $\operatorname{cf}(\kappa)$ for the cofinality of κ. Recall that a cardinal number κ is regular, if $\kappa = \operatorname{cf}(\kappa)$. For $A \subset \mathbb{R}$ its characteristic function is denoted by χ_A. If A is a planar set, we denote its x-projection by $\operatorname{dom}(A)$ and y-projection by $\operatorname{rng}(A)$. For $f, g \in \mathbb{R}^{\mathbb{R}}$ the notation $[f = g]$ means the set $\{x \in \mathbb{R}\colon\ f(x) = g(x)\}$. Likewise for $[f > g]$, $[f \neq g]$, etc.

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For $X \subset \mathbb{R}$ we say that a function $f: X \to \mathbb{R}$ is of *Sierpiński–Zygmund type* (shortly, an SZ-function), if its restriction $f \restriction M$ is discontinuous for any set $M \subset X$ with $\mathrm{card}(M) = \mathbf{c}$ [15]. The family of all SZ-functions from $\mathbb{R}$ to $\mathbb{R}$ will be denoted by SZ. The symbol $\mathcal{C}$ will stand for the family of all continuous functions $f: \mathbb{R} \to \mathbb{R}$, and $\mathcal{C}_{G_\delta}$ for the family of all continuous functions defined on G_δ-sets $X \subset \mathbb{R}$ with $\mathrm{card}(X) = \mathbf{c}$. Recall also that a function $f \in \mathbb{R}^\mathbb{R}$ is an SZ-function if and only if $\mathrm{card}([f = g]) < \mathbf{c}$ for every $g \in \mathcal{C}_{G_\delta}$ [15]. We will sometimes abuse this notation by writing $f \in SZ$ and $f \in \mathcal{C}$ for partial functions $f: X \to \mathbb{R}$ with $X \subsetneq \mathbb{R}$.

The following fact can be proved by a slight modification of the original proof of Sierpiński and Zygmund [15].

Proposition 1.1. *For every family $\{Y_x : x \in \mathbb{R}\}$ of subsets of $\mathbb{R}$ of cardinality $\mathbf{c}$ there exists an SZ-function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) \in Y_x$ for every $x \in \mathbb{R}$.*

In particular, $\mathrm{card}(SZ) = 2^\mathbf{c}$.

For every cardinal κ and a partially ordered set (shortly poset) $\mathbb{P}$ we shall consider the following statements. (See [3]. Compare also [7,9,10,16].)

$\mathbf{MA}_\kappa(\mathbb{P})$ (κ-Martin's Axiom for $\mathbb{P}$). For any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $\mathrm{card}(\mathcal{D}) < \kappa$ there exists a $\mathcal{D}$-generic filter G in $\mathbb{P}$, i.e., such that $D \cap G \neq \emptyset$ for every $D \in \mathcal{D}$.

$\mathbf{Lus}_\kappa(\mathbb{P})$. There exists a sequence $\langle G_\alpha \colon \alpha < \kappa \rangle$ of $\mathbb{P}$-filters, called a κ-Lusin sequence, such that $\mathrm{card}(\{\alpha < \kappa \colon G_\alpha \cap D = \emptyset\}) < \kappa$ for every dense set $D \subset \mathbb{P}$.

2. Sums

Theorem 2.1. *For every family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ with $\mathrm{card}(\mathcal{F}) \leqslant \mathbf{c}$ there exists an $h \in \mathbb{R}^\mathbb{R}$ such that $h + f \in SZ$ for each $f \in \mathcal{F}$.*

Proof. Let $\{g_\alpha \colon \alpha < \mathbf{c}\} = \mathcal{C}_{G_\delta}$, $\{x_\alpha \colon \alpha < \mathbf{c}\} = \mathbb{R}$, and $\{f_\alpha \colon \alpha < \mathbf{c}\} = \mathcal{F}$. For every $\alpha < \mathbf{c}$ choose $h(x_\alpha) \in \mathbb{R} \setminus \{g_\gamma(x_\alpha) - f_\beta(x_\alpha) \colon \beta, \gamma \leqslant \alpha\}$. Such a function h satisfies the following condition:

$$(\forall \beta < \mathbf{c})\ (\forall \gamma < \mathbf{c})\ \ [h + f_\beta = g_\gamma] \subset \{x_\alpha \colon \alpha < \max(\beta, \gamma)\},$$

so $\mathrm{card}((h + f_\beta) \cap g_\gamma) < \mathbf{c}$ for all $\beta, \gamma < \mathbf{c}$. □

Corollary 2.2. *Every real function f can be expressed as the sum of two SZ-functions.*

Proof. Use Theorem 2.1 with $\mathcal{F} = \{0, f\}$. □

The following cardinal function has been defined in [11] for $\mathcal{G} \subset \mathbb{R}^\mathbb{R}$. (Compare also [3,4].)

$$a(\mathcal{G}) = \min\bigl(\{\mathrm{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathbb{R}^{\mathbb{R}}\ \&\ \neg\exists h \in \mathbb{R}^{\mathbb{R}}\ \forall f \in \mathcal{F}\ h+f \in \mathcal{G}\} \cup \{(2^{\mathbf{c}})^+\}\bigr)$$
$$= \min\bigl(\{\mathrm{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathbb{R}^{\mathbb{R}}\ \&\ \forall h \in \mathbb{R}^{\mathbb{R}}\ \exists f \in \mathcal{F}\ h+f \notin \mathcal{G}\} \cup \{(2^{\mathbf{c}})^+\}\bigr).$$

Evidently, there is no $h \in \mathbb{R}^{\mathbb{R}}$ such that $h + f \in SZ$ for all $f \in \mathbb{R}^{\mathbb{R}}$. Therefore Theorem 2.1 yields to the following corollary.

Corollary 2.3. $\mathbf{c} < a(SZ) \leqslant 2^{\mathbf{c}}$.

Hence, if $\mathbf{c}^+ = 2^{\mathbf{c}}$, then $a(SZ) = 2^{\mathbf{c}}$. However, it is interesting whether or not anything more can be said about the cardinal $a(SZ)$. (The analogous problem for the classes AC of almost continuous functions and $\mathcal{D}$ of Darboux functions is considered in [3].) To address this question we need the following partially ordered sets $\langle \mathbb{P}, \leqslant\rangle$ and $\langle \mathbb{P}^*, \leqslant\rangle$.

$$\mathbb{P} = \{p \in \mathbb{R}^X\colon\ X \subseteq \mathbb{R}\ \&\ \mathrm{card}(X) < \mathbf{c}\},$$

i.e., $\mathbb{P}$ is the set of all partial functions from $\mathbb{R}$ to $\mathbb{R}$ of cardinality less than $\mathbf{c}$. We put $p \leqslant q$ if and only if $p \supseteq q$, i.e., when p extends q as a partial function.

$$\mathbb{P}^* = \{\langle p, E\rangle\colon\ p \in \mathbb{P}\ \&\ E \subseteq \mathbb{R}^{\mathbb{R}}\ \&\ \mathrm{card}(E) < \mathbf{c}\}.$$

The ordering on $\mathbb{P}^*$ is defined by

$$\langle p, E\rangle \leqslant \langle q, F\rangle \quad \text{iff} \quad p \supseteq q \text{ and } E \supseteq F$$
$$\text{and } \forall x \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)\ \ \forall f \in F\ p(x) \neq f(x).$$

The following theorem can be found in [3, Theorem 3.7].

Theorem 2.4. *Let $\lambda \geqslant \kappa \geqslant \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with* ZFC + CH *that $2^{\mathbf{c}} = \lambda$ and $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ holds.*

We will prove the following theorem.

Theorem 2.5. *If $\kappa > \mathbf{c}$ is a regular cardinal then $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ implies that $a(SZ) = \kappa$.*

This and Theorem 2.4 will immediately imply the following corollary.

Corollary 2.6. *Let $\lambda \geqslant \kappa \geqslant \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with* ZFC + CH *that $2^{\mathbf{c}} = \lambda$ and $a(SZ) = \kappa$.*

The proof of Theorem 2.5 will be split into three lemmas.

Lemma 2.7.
(i) $\mathrm{Lus}_\kappa(\mathbb{P}^*) \Rightarrow \mathrm{Lus}_\kappa(\mathbb{P})$.
(ii) *For any regular κ we have $\mathrm{Lus}_\kappa(\mathbb{P}^*) \Rightarrow \mathrm{MA}_\kappa(\mathbb{P}^*)$.*

Proof. The proof is implicitly contained in the proof of [3, Lemma 3.6]. Let $\langle G_\alpha\colon\ \alpha < \kappa\rangle$ be a κ-Lusin sequence for $\mathbb{P}^*$.

(i) follows from the fact that in some sense $\mathbb{P}$ is "living inside" of $\mathbb{P}^*$. To see it, let $r:\mathbb{R}\to\mathbb{R}$ be a map with of $\operatorname{card}(r^{-1}(y))=\mathbf{c}$ for every $y\in\mathbb{R}$. Define $\pi:\mathbb{P}^*\to\mathbb{P}$ by

$$\pi(p,F)=r\circ p.$$

Notice that if $\langle p,E\rangle\leqslant\langle q,F\rangle$ then $\pi(p,E)\leqslant\pi(q,F)$. This implies that $\pi[G]$ is a $\mathbb{P}$-filter for any $\mathbb{P}^*$-filter G. Furthermore, we claim that if $D\subseteq\mathbb{P}$ is dense, then $\pi^{-1}(D)$ is dense in $\mathbb{P}^*$. To see this, let $\langle p,F\rangle\in\mathbb{P}^*$ be arbitrary. Since D is dense, there exists $q\leqslant\pi(p,F)$ with $q\in D$. Now, find $s\in\mathbb{P}$ extending p such that $r\circ s=q\supseteq r\circ p$ and $s(x)\neq f(x)$ for every $x\in\operatorname{dom}(s)\setminus\operatorname{dom}(p)$ and $f\in F$. This can be done by choosing

$$s(x)\in r^{-1}\bigl(q(x)\bigr)\setminus\bigl\{f(x)\colon\ f\in F\bigr\}$$

for every $x\in\operatorname{dom}(q)\setminus\operatorname{dom}(p)$. Then, $\langle s,F\rangle\leqslant\langle p,F\rangle$ and $\langle s,F\rangle\in\pi^{-1}(q)\subseteq\pi^{-1}(D)$.

Now, $\langle\pi[G_\alpha]\colon\ \alpha<\kappa\rangle$ is a κ-Lusin sequence for $\mathbb{P}$ since for every dense $D\subseteq\mathbb{P}$,

$$\begin{aligned}\bigl\{\alpha<\kappa\colon\ \pi[G_\alpha]\cap D=\emptyset\bigr\}&=\bigl\{\alpha<\kappa\colon\ \pi[G_\alpha]\cap\pi\bigl[\pi^{-1}(D)\bigr]=\emptyset\bigr\}\\&\subseteq\bigl\{\alpha<\kappa\colon\ G_\alpha\cap\pi^{-1}(D)=\emptyset\bigr\}.\end{aligned}$$

To see (ii) take a family $\mathcal{D}$ of dense subsets of $\mathbb{P}^*$ of cardinality less than κ. By the regularity of κ, there exists $\alpha<\kappa$ such that G_α meets every element of $\mathcal{D}$. □

Lemma 2.8. *Assume that κ is a regular cardinal and $\kappa>\mathbf{c}$. Then $\mathrm{Lus}_\kappa(\mathbb{P})$ implies that $a(SZ)\leqslant\kappa$.*

Proof. Let $\langle G_\alpha\colon\ \alpha<\kappa\rangle$ be a κ-Lusin sequence of $\mathbb{P}$-filters and let

$$f_\alpha=\bigcup G_\alpha.$$

Then f_α is a partial function from $\mathbb{R}$ into $\mathbb{R}$. Let

$$D_x=\bigl\{p\in\mathbb{P}\colon\ x\in\operatorname{dom}(p)\bigr\}.$$

It is easy to see that each D_x is dense in $\mathbb{P}$. Hence, since $\mathbf{c}<\kappa$ and κ is regular, we may assume that each f_α is a total function.

Now, let $\{x_\xi\colon\ \xi<\mathbf{c}\}=\mathbb{R}$. For each $\xi<\mathbf{c}$, $g\in\mathcal{C}_{G_\delta}$, and $h\in\mathbb{R}^{\mathbb{R}}$ define

$$D_\xi(g,h)=\bigl\{p\in\mathbb{P}\colon\ (\exists\eta\geqslant\xi)\bigl(x_\eta\in\operatorname{dom}(p)\cap\operatorname{dom}(g)\ \&\ (h+p)(x_\eta)=g(x_\eta)\bigr)\bigr\}.$$

Note that $D_\xi(g,h)$ is dense in $\mathbb{P}$, since for any $p\in\mathbb{P}$ there is $\eta\geqslant\xi$ with

$$x_\eta\in\operatorname{dom}(g)\setminus\operatorname{dom}(p).$$

Then

$$p\cup\bigl\{\bigl\langle x_\eta,g(x_\eta)-h(x_\eta)\bigr\rangle\bigr\}\in D_\xi(g,h)$$

extends p. By the regularity of κ, for any $h\in\mathbb{R}^{\mathbb{R}}$ there exists $\alpha<\kappa$ such that G_α intersects every set $D_\xi(g,h)$ with $\xi<\mathbf{c}$ and $g\in\mathcal{C}_{G_\delta}$, and so, $\operatorname{card}((h+f_\alpha)\cap g)=\mathbf{c}$.

Thus, for every $h\in\mathbb{R}^{\mathbb{R}}$ there exists $\alpha<\kappa$ such that $h+f_\alpha\notin SZ$, i.e., the family $\mathcal{F}=\{f_\alpha\colon\ \alpha<\kappa\}$ shows that $a(SZ)\leqslant\kappa$ as was to be shown. □

Lemma 2.9. *If $\kappa > \mathfrak{c}$ then $\mathrm{MA}_\kappa(\mathbb{P}^*)$ implies that $a(SZ) \geqslant \kappa$.*

Proof. Let $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ be such that $\mathrm{card}(\mathcal{F}) < \kappa$. We will find $h \in \mathbb{R}^\mathbb{R}$ such that $h+f \in SZ$ for every $f \in \mathcal{F}$.

Notice that for any $x \in \mathbb{R}$ the set

$$D_x = \{\langle p, E\rangle \in \mathbb{P}^*:\ x \in \mathrm{dom}(p)\}$$

is dense in $\mathbb{P}^*$. Indeed, let $\langle q, F\rangle$ be an arbitrary element of $\mathbb{P}^*$ and suppose it is not already an element of D_x. The set $Q = \{f(x):\ f \in F\}$ has cardinality less than $\mathfrak{c}$, so there exists $y \in \mathbb{R} \setminus Q$. Let $p = q \cup \{\langle x, y\rangle\}$. Then $\langle p, F\rangle \leqslant \langle q, F\rangle$ and $\langle p, F\rangle \in D_x$. Therefore $h = \bigcup\{p:\ (\exists E)\,(\langle p, E\rangle \in G)\}$ is a function from $\mathbb{R}$ into $\mathbb{R}$ for any $\mathbb{P}^*$-filter G intersecting all sets D_x.

Note also, that for $f \in \mathbb{R}^\mathbb{R}$ the set

$$E_f = \{\langle p, E\rangle \in \mathbb{P}^*:\ f \in E\}$$

is dense in $\mathbb{P}^*$ since $\langle p, E \cup \{f\}\rangle \in E_f$ extends $\langle p, E\rangle$.

Let

$$\mathcal{D} = \{D_x:\ x \in \mathbb{R}\} \cup \{E_{\bar{g}-f}:\ f \in \mathcal{F}\ \&\ g \in \mathcal{C}_{G_\delta}\},$$

where $\bar{g} \in \mathbb{R}^\mathbb{R}$ extends $g \in \mathcal{C}_{G_\delta}$ by associating 0 at all undefined places. Then, $\mathcal{D}$ is a family of less than κ many dense subsets of $\mathbb{P}^*$. Let G be a $\mathcal{D}$-generic filter in $\mathbb{P}^*$ and let $h = \bigcup\{p:\ (\exists E)(\langle p, E\rangle \in G)\}$. We have to show that $h + f \in SZ$ for every $f \in \mathcal{F}$.

So, let $f \in \mathcal{F}$ and $g \in \mathcal{C}_{G_\delta}$. Then there exists $\langle p, E\rangle \in G \cap E_{\bar{g}-f}$. So, by the definition of order on $\mathbb{P}$ it is easy to see that

$$\{x \in \mathbb{R}:\ (f+h)(x) = g(x)\} \subseteq \{x \in \mathbb{R}:\ h(x) = \bar{g}(x) - f(x)\} \subseteq \mathrm{dom}(p).$$

Thus, $h + f \in SZ$ for every $f \in \mathcal{F}$. $\square$

Application of Lemmas 2.7, 2.8 and 2.9 finishes the proof of Theorem 2.5.

In [3] it has been proved that $a(\mathcal{D}) = a(AC) = e_\mathfrak{c}$ and that this number has cofinality greater than continuum $\mathfrak{c}$, where

$$e_\kappa = \min\{\mathrm{card}(F):\ F \subseteq \kappa^\kappa\ \&\ \forall h \in \kappa^\kappa\ \exists f \in F\ \mathrm{card}(f \cap h) < \kappa\}.$$

Next, we will compare $a(SZ)$ with $a(\mathcal{D})$, and give a characterization of $a(SZ)$ similar to that of $e_\mathfrak{c}$. We will also address an issue of the cofinality of $a(SZ)$.

Since for a regular $\kappa > \mathfrak{c}$ an axiom $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ implies $a(\mathcal{D}) = \kappa$ [3, Section 3] we can conclude the following fact.

Corollary 2.10. *Let $\lambda \geqslant \kappa \geqslant \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with* ZFC + CH *that $2^\mathfrak{c} = \lambda$ and $a(\mathcal{D}) = a(SZ) = \kappa$.*

Note also the following strengthening of [3, Theorem 3.3].

Theorem 2.11. *Let $\lambda \geqslant \omega_2$ be a cardinal such that* $\mathrm{cf}(\lambda) > \omega_1$. *Then it is relatively consistent with* ZFC + CH *that* $2^{\mathbf{c}} = \lambda$ *and* $\mathrm{Lus}_\kappa(\mathbb{P})$ *holds for every regular* $\kappa > \mathbf{c}$, $\kappa \leqslant 2^{\mathbf{c}}$.

Proof. The proof is identical to that of [3, Theorem 3.3]. □

Now, recall also that $\mathrm{Lus}_\kappa(\mathbb{P})$ implies $a(\mathcal{D}) \geqslant \kappa$ for every regular $\kappa > \mathbf{c}$ [3]. Thus, in a model of Theorem 2.11 we have $a(\mathcal{D}) = 2^{\mathbf{c}} = \lambda$. On the other hand in this model we have $\mathrm{Lus}_{\mathbf{c}^+}(\mathbb{P})$. So, by Lemma 2.8 and Corollary 2.3, $a(SZ) = \mathbf{c}^+$. In particular, we obtain the following corollary.

Corollary 2.12. *Let $\lambda > \omega_2$ be a cardinal such that* $\mathrm{cf}(\lambda) > \omega_1$. *Then it is relatively consistent with* ZFC + CH *that $2^{\mathbf{c}} = \lambda$ is true, and* $a(SZ) = \mathbf{c}^+ < 2^{\mathbf{c}} = a(\mathcal{D})$.

The following remains an open problem.

Problem 2.13. Is it consistent that $a(SZ) > a(\mathcal{D})$?

For an infinite cardinal κ define

$$d_\kappa = \min\bigl\{\mathrm{card}(F)\colon\ F \subseteq \kappa^\kappa \ \&\ \forall h \in \kappa^\kappa\ \exists f \in F\ \mathrm{card}(f \cap h) = \kappa\bigr\}.$$

Notice that $d_\kappa > \kappa$.

Theorem 2.14. $a(SZ) = d_{\mathbf{c}}$.

Proof. To see that $d_{\mathbf{c}} \leqslant a(SZ)$ choose $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ with $\mathrm{card}(\mathcal{F}) < d_{\mathbf{c}}$ and define

$$\overline{\mathcal{F}} = \bigl\{\bar{g} - f\colon\ f \in \mathcal{F} \ \&\ g \in \mathcal{C}_{G_\delta}\bigr\},$$

where $\bar{g} \in \mathbb{R}^{\mathbb{R}}$ extends g by associating 0 at all undefined places. Then,

$$\mathrm{card}\bigl(\overline{\mathcal{F}}\bigr) \leqslant \mathrm{card}(\mathcal{F}) \cdot \mathbf{c} < d_{\mathbf{c}}.$$

So, there exists an $h \in \mathbb{R}^{\mathbb{R}}$ such that $\mathrm{card}(h \cap \bar{f}) < \mathbf{c}$ for every $\bar{f} \in \overline{\mathcal{F}}$. Hence, for every $f \in \mathcal{F}$ and $g \in \mathcal{C}_{G_\delta}$

$$\mathrm{card}\bigl((h + f) \cap g\bigr) \leqslant \mathrm{card}\bigl((h + f) \cap \bar{g}\bigr) = \mathrm{card}\bigl(h \cap (\bar{g} - f)\bigr) < \mathbf{c}$$

since $\bar{g} - f \in \overline{\mathcal{F}}$. So, $h + f \in SZ$ every $f \in \mathcal{F}$, and $d_{\mathbf{c}} \leqslant a(SZ)$.

To see that $a(SZ) \leqslant d_{\mathbf{c}}$ choose $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ with $\mathrm{card}(\mathcal{F}) < a(SZ)$ and let $-\mathcal{F} = \{-f\colon\ f \in \mathcal{F}\}$. Using the definition of $a(SZ)$ to $-\mathcal{F}$ we can find $h \in \mathbb{R}^{\mathbb{R}}$ such that $h - f \in SZ$ for every $f \in \mathcal{F}$. In particular, for $g_0 \equiv 0$ we have

$$\mathrm{card}(h \cap f) = \mathrm{card}\bigl(h \cap (f + g_0)\bigr) = \mathrm{card}\bigl((h - f) \cap g_0\bigr) < \mathbf{c}$$

for every $f \in \mathcal{F}$. So, $a(SZ) \leqslant d_{\mathbf{c}}$. □

To address the problem of cofinality of $a(SZ)$ we need the following theorem, where $\kappa^{<\kappa}$ is the supremum of all cardinals κ^λ with $\lambda < \kappa$.

Theorem 2.15. *If $\kappa \geqslant \omega$ is a cardinal number such that $\kappa^{<\kappa} = \kappa$ then* $\mathrm{cf}(d_\kappa) > \kappa$.

Proof. Let T be the set of all functions from some $\xi < \kappa$ into κ, i.e., $T = \bigcup_{\xi<\kappa} \kappa^\xi$. Thus, by our assumption, $\mathrm{card}(T) = \kappa$. Let $\langle F_\xi \subset T^\kappa\colon \xi < \kappa\rangle$ be an increasing sequence such that $\mathrm{card}(F_\xi) < d_\kappa$ for every $\xi < \kappa$. We shall show that the cardinality of $F = \bigcup_{\xi<\kappa} F_\xi$ is less than d_κ by finding $h \in T^\kappa$ such that $\mathrm{card}(h \cap f) < \kappa$ for every $f \in F$. This will finish the proof.

For $\xi < \kappa$ define

$$\overline{F}_\xi = \left\{\bar{f} \in \left(\kappa^\xi\right)^\kappa\colon\ (\exists f \in F_\xi)(\forall\alpha < \kappa)\left(\bar{f}(\alpha) = f(\alpha)\restriction^\star\xi\right)\right\},$$

where $[f(\alpha)\restriction^\star\xi](\zeta) = f(\alpha)(\zeta)$ if $\zeta \in \mathrm{dom}(f(\alpha))$ and $[f(\alpha)\restriction^\star\xi](\zeta) = 0$ otherwise. Thus, $\mathrm{card}(\overline{F}_\xi) \leqslant \mathrm{card}(F_\xi) < d_\kappa$ for every $\xi < \kappa$.

By induction on $\xi < \kappa$ we will define a sequence $\langle h_\xi \in (\kappa^\xi)^\kappa\colon \xi < \kappa\rangle$ such that

(i) $h_\zeta(\alpha) \subset h_\xi(\alpha)$ for every $\alpha < \kappa$ and $\zeta < \xi < \kappa$.

(ii) $\mathrm{card}(h_\xi \cap \bar{f}) < \kappa$ for every $\bar{f} \in \overline{F}_\xi$ and every successor ordinal $\xi < \kappa$.

So assume that for some $\xi < \kappa$ the sequence $\langle h_\zeta\colon \zeta < \xi\rangle$ is already constructed. If ξ is a limit ordinal put $h_\xi(\alpha) = \bigcup_{\zeta<\xi} h_\zeta(\alpha)$ for every $\alpha < \kappa$. Then (i) is clearly satisfied, and (ii) does not apply.

If $\xi = \eta + 1$ is a successor ordinal, then the space

$$H_\xi = \left\{h \in \left(\kappa^\xi\right)^\kappa\colon\ (\forall\alpha < \kappa)\left(h_\eta(\alpha) \subset h(\alpha)\right)\right\}$$

is naturally isomorphic to κ^κ by an isomorphism $i\colon H_\xi \to \kappa^\kappa$, $i(h)(\alpha) = h(\alpha)(\eta)$ for $h \in H_\xi$ and $\alpha < \kappa$. Moreover, $\mathrm{card}(\overline{F}_\xi \cap H_\xi) \leqslant \mathrm{card}(\overline{F}_\xi) < d_\kappa$. So, there exists $h_\xi \in H_\xi \subset (\kappa^\xi)^\kappa$ satisfying (ii), while (i) is satisfied by any $h \in H_\xi$. The construction is completed.

To finish the proof define $h\colon \kappa \to T$ by $h(\xi) = h_\xi(\xi)$. We will show that $\mathrm{card}(h \cap f) < \kappa$ for every $f \in F$.

So, let $f \in F$. Then, there exists a successor ordinal number $\xi < \kappa$ such that $f \in F_\xi$. Let $\bar{f} \in \overline{F}_\xi$ be such that $\bar{f}(\alpha) = f(\alpha)\restriction^\star\xi$ for every $\alpha < \kappa$. Then

$$\begin{aligned}\left\{\alpha < \kappa\colon\ h(\alpha) = f(\alpha)\right\} &\subset \xi \cup \left\{\alpha < \kappa\colon\ h(\alpha) \supset \bar{f}(\alpha)\right\}\\ &= \xi \cup \left\{\alpha < \kappa\colon\ h_\xi(\alpha) = \bar{f}(\alpha)\right\}\end{aligned}$$

and, by (ii), this last set has cardinality less than κ. So $\mathrm{card}(h \cap f) < \kappa$. $\square$

From Theorems 2.14 and 2.15 we obtain the following corollary. (Note that $\mathbf{c}^{<\mathbf{c}}$ is the supremum of all cardinals 2^λ with $\lambda < \mathbf{c}$.)

Corollary 2.16. *If* $\mathbf{c}^{<\mathbf{c}} = \mathbf{c}$ *then* $\mathrm{cf}(a(SZ)) > \mathbf{c}$.

The following remains an open problem.

Problem 2.17. Can $a(SZ)$ be a singular cardinal?

Since $a(SZ) = d_{\mathbf{c}}$ and $a(\mathcal{D}) = e_{\mathbf{c}}$, Problems 2.13 and 2.17 can be rephrased as follows.

($\star$) Let $\kappa = \mathbf{c}$. Is it consistent that $d_\kappa > e_\kappa$? Can d_κ be singular?

Notice that for $\kappa = \omega$ the answer for these problems is well known, since $d_\omega =$ non(meager) is the minimum cardinality of a nonmeager subset of $\mathbb{R}$, and $e_\omega =$ cov(meager) is the minimum cardinality of a family of meager subset of $\mathbb{R}$ whose union is equal to $\mathbb{R}$. (See [2].) Thus, for $\kappa = \omega$ the answer for both questions is positive. (Compare also [8] for some results concerning e_κ for $\kappa > \omega$.)

Next, let $\mathcal{M}_a(SZ)$ denote the *maximal additive family* for the class SZ, i.e.,

$$\mathcal{M}_a(SZ) = \{f \in \mathbb{R}^{\mathbb{R}}\colon\ f + h \in SZ \text{ for each } h \in SZ\}.$$

To describe the structure of $\mathcal{M}_a(SZ)$ we need the following easy lemma.

Lemma 2.18. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be an SZ-function. Then there exists an SZ-extension of f, i.e., an $f^* \in \mathbb{R}^{\mathbb{R}}$ that $f^* \in SZ$ and $f^* \restriction X = f$.*

Proof. Obviously for each $h : \mathbb{R} \to \mathbb{R}$, $h \in SZ$ if and only if $h\restriction(\mathbb{R}\setminus X) \in SZ$ and $h\restriction X \in SZ$. Moreover, we can use the Sierpiński–Zygmund's method to obtain an SZ-function defined on any subset of $\mathbb{R}$. Therefore it is enough to construct an SZ-function $g : \mathbb{R}\setminus X \to \mathbb{R}$ and put $f^* = f \cup g$. $\square$

Theorem 2.19. *For every function $f \in \mathbb{R}^{\mathbb{R}}$ the following conditions are equivalent*:
 (i) $f \in \mathcal{M}_a(SZ)$;
 (ii) *for each $X \in [\mathbb{R}]^{\mathbf{c}}$ there exists a $Y \in [X]^{\mathbf{c}}$ such that $f\restriction Y \in \mathcal{C}$.*

Proof. (ii) $\Rightarrow$ (i). Suppose that f satisfies the condition (ii) and $h + f \notin SZ$ for some $h \in SZ$. Then $(h + f)\restriction X \in \mathcal{C}$ for some set $X \in [\mathbb{R}]^{\mathbf{c}}$. Let $Y \in [X]^{\mathbf{c}}$ be a set such that $f\restriction Y \in \mathcal{C}$. Then $h\restriction Y \in \mathcal{C}$, in contradiction with $h \in SZ$.

(i) $\Rightarrow$ (ii). Suppose that f does not fulfill the condition (ii). Then there exists $X \in [\mathbb{R}]^{\mathbf{c}}$ such that $f\restriction Y \notin \mathcal{C}$ for each $Y \in [X]^{\mathbf{c}}$, i.e., $f\restriction X \in SZ$. Let $f^* \in \mathbb{R}^{\mathbb{R}}$ be an SZ-extension of f. Then $-f^* \in SZ$ and $(f - f^*)\restriction X \in \mathcal{C}$, so $f \notin \mathcal{M}_a(SZ)$. $\square$

Remark. U. Darji proved under CH that a Borel function f satisfies the the condition (ii) if and only if it is countably continuous [6, Theorem 10]. In the same way one can prove that (ii) implies the following condition:

(iii) f is the union of less than $\mathbf{c}$ many continuous functions;

and, assuming regularity of $\mathbf{c}$, that (iii) implies (ii).

Proof. (ii) $\Rightarrow$ (iii). Let $\{g_\alpha\colon\ \alpha < \mathbf{c}\} = \mathcal{C}_{G_\delta}$. Suppose that f is not the union of less than $\mathbf{c}$ many continuous functions. Then card(dom($f \setminus \bigcup_{\beta<\alpha} g_\alpha$)) $= \mathbf{c}$ for each $\alpha < \mathbf{c}$. For every $\alpha < \mathbf{c}$ choose $x_\alpha \in$ dom($f \setminus \bigcup_{\beta<\alpha} g_\alpha$) $\setminus \{x_\beta\colon\ \beta < \alpha\}$ and set $X = \{x_\alpha\colon\ \alpha < \mathbf{c}\}$. By (ii), there exists $Y \in [X]^{\mathbf{c}}$ such that $f\restriction Y$ is continuous. Therefore $f\restriction Y = g_\alpha\restriction Y$ for some $\alpha < \mathbf{c}$, so card($f \cap g_\alpha$) $= \mathbf{c}$, contrary to the construction of X.

Now assume that $\mathbf{c}$ is a regular cardinal and f satisfies (iii). Then $f = \bigcup_{\alpha<\kappa} f\restriction X_\alpha$ for some $\kappa < \mathbf{c}$ and all functions $f\restriction X_\alpha$ are continuous. Fix $X \in [\mathbb{R}]^{\mathbf{c}}$. By the regularity of $\mathbf{c}$, $\mathrm{card}(X \cap X_\alpha) = \mathbf{c}$ for some $\alpha < \kappa$ and, for $Y = X \cap X_\alpha$, $f\restriction Y$ is continuous. □

It is also worth to notice in this context that if $f : X \to \mathbb{R}$ is SZ for some $X \subset \mathbb{R}$ then for every $Y \in [X]^{\mathbf{c}}$ its restriction $f\restriction Y$ is not countably (even $\kappa < \mathrm{cf}(\mathbf{c})$) continuous.

3. Products

In this section we will examine for which functions $f \in \mathbb{R}^{\mathbb{R}}$ there exists $h \in \mathbb{R}^{\mathbb{R}}$ such that $hf \in SZ$.

First note that if $\mathrm{card}([f = 0]) = \mathbf{c}$ then $hf \in SZ$ for no $h : \mathbb{R} \to \mathbb{R}$. Thus, we will restrict our attention to the family

$$\mathcal{R}_0 = \{f \in \mathbb{R}^{\mathbb{R}}:\ \mathrm{card}([f = 0]) < \mathbf{c}\}.$$

Theorem 3.1. *For every family $\mathcal{F} \subset \mathcal{R}_0$ with $\mathrm{card}(\mathcal{F}) \leqslant \mathbf{c}$ there exists an $h : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $hf \in SZ$ for each $f \in \mathcal{F}$.*

Proof. Let $\{g_\alpha:\ \alpha < \mathbf{c}\} = \mathcal{C}_{G_\delta}$, $\{x_\alpha:\ \alpha < \mathbf{c}\} = \mathbb{R}$, and $\{f_\alpha:\ \alpha < \mathbf{c}\} = \mathcal{F}$. For $\alpha < \mathbf{c}$ choose

$$h(x_\alpha) \in \mathbb{R} \setminus \left(\{0\} \cup \left\{\frac{g_\gamma(x_\alpha)}{f_\beta(x_\alpha)}:\ \beta, \gamma \leqslant \alpha\ \&\ f_\beta(x_\alpha) \neq 0\right\}\right).$$

Such a function h satisfies the following condition:

$$(\forall \beta < \mathbf{c})\ (\forall \gamma < \mathbf{c})\ \ [hf_\beta = g_\gamma] \subset [f_\beta = 0] \cup \{x_\alpha:\ \alpha < \max(\beta, \gamma)\},$$

so $\mathrm{card}((hf_\beta) \cap g_\gamma) < \mathbf{c}$ for all $\beta, \gamma < \mathbf{c}$. □

Corollary 3.2. *For every function $f \in \mathbb{R}^{\mathbb{R}}$ the following conditions are equivalent:*

(i) $\mathrm{card}([f = 0]) < \mathbf{c}$,

(ii) *f is the product of two SZ-functions.*

Let $m(SZ)$ denote the least cardinal κ for which there exists a family $\mathcal{F} \subset \mathcal{R}_0$ such that $\mathrm{card}(\mathcal{F}) = \kappa$ and for every $h : \mathbb{R} \to \mathbb{R}$ there exists $f \in \mathcal{F}$ with $hf \notin SZ$. (Note that this definition is different from the definition of the cardinal function m defined in [11]; cf. [13].)

Theorem 3.3. $a(SZ) = m(SZ)$.

Proof. "$a(SZ) \leqslant m(SZ)$". Assume that $\mathcal{F} \subset \mathcal{R}_0$ is a family of functions such that $\mathrm{card}(\mathcal{F}) < a(SZ)$. For every $f \in \mathcal{F}$ let $\tilde{f}$ be the function defined by

$$\tilde{f}(x) = \begin{cases} |f(x)| & \text{if } f(x) \neq 0, \\ 1 & \text{if } f(x) = 0. \end{cases}$$

Note that $\text{card}(\{\tilde{f}\colon f \in \mathcal{F}\}) \leqslant \text{card}(\mathcal{F}) < a(SZ)$, so there exists $h:\mathbb{R} \to \mathbb{R}$ such that $h + \ln(\tilde{f}) \in SZ$ for each $f \in \mathcal{F}$. Therefore $\exp(h + \ln(\tilde{f})) \in SZ$, so $\exp(h)\tilde{f} \in SZ$ for $f \in \mathcal{F}$. We shall verify that $\exp(h)f \in SZ$ for every $f \in \mathcal{F}$. Suppose that $\exp(h)f \restriction X \in \mathcal{C}$ for some $X \subset \mathbb{R}$. Let $X_- = X \cap [f < 0]$, $X_+ = X \cap [f > 0]$ and $X_0 = X \cap [f = 0]$. Note that $\text{card}(X_0) < \mathfrak{c}$. Also, $\text{card}(X_+) < \mathfrak{c}$, since $\exp(h)\tilde{f} \restriction X_+ = \exp(h)f \restriction X_+ \in \mathcal{C}$. Similarly, $\text{card}(X_-) < \mathfrak{c}$, since $\exp(h)\tilde{f} \restriction X_- = -\exp(h)f \restriction X_- \in \mathcal{C}$. Thus $\text{card}(X) < \mathfrak{c}$ and consequently, $\exp(h)f \in SZ$.

"$m(SZ) \leqslant a(SZ)$". Now assume that $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is a family of functions such that $\text{card}(\mathcal{F}) < m(SZ)$. Let $h \in \mathbb{R}^{\mathbb{R}}$ be a function such that $\exp(f)h \in SZ$ and $-\exp(f)h \in SZ$ for all $f \in \mathcal{F}$. Obviously, we can ensure that $h \in SZ$ by adding the constant function 0 to $\mathcal{F}$. Let $\tilde{h}$ be defined as above. Then $\text{rng}(\tilde{h}) \subset (0, \infty)$ and $\exp(f)\tilde{h} \in SZ$ for each $f \in \mathcal{F}$. Indeed, suppose that $\exp(f)\tilde{h} \restriction X \in \mathcal{C}$ for some $X \subset \mathbb{R}$ and $f \in \mathcal{F}$. Then $X = X_- \cup X_0 \cup X_+$, where $X_- = X \cap [h < 0]$, $X_+ = X \cap [h > 0]$ and $X_0 = X \cap [h = 0]$. Of course, $\text{card}(X_0) < \mathfrak{c}$. Moreover, $\exp(f)\tilde{h} \restriction X_+ = \exp(f)h \restriction X_+ \in \mathcal{C}$ and $\exp(f)\tilde{h} \restriction X_- = -\exp(f)h \restriction X_- \in \mathcal{C}$, so $\text{card}(X_+) < \mathfrak{c}$ and $\text{card}(X_-) < \mathfrak{c}$. Hence $\text{card}(X) < \mathfrak{c}$.

Therefore $\ln(\exp(f)\tilde{h}) \in SZ$, so $\ln(\tilde{h}) + f \in SZ$ for each $f \in \mathcal{F}$. $\square$

Let $\mathcal{M}_m(SZ)$ denote the *maximal multiplicative family* for the class SZ, i.e.,

$$\mathcal{M}_m(SZ) = \{f \in \mathbb{R}^{\mathbb{R}}\colon\ fh \in SZ \text{ for each } h \in SZ\}.$$

Theorem 3.4. *For every function $f \in \mathbb{R}^{\mathbb{R}}$ the following conditions are equivalent*:

(i) $f \in \mathcal{M}_m(SZ)$;

(ii) $\text{card}([f = 0]) < \mathfrak{c}$ *and for each $X \in [\mathbb{R}]^{\mathfrak{c}}$ there exists a $Y \in [X]^{\mathfrak{c}}$ such that $f \restriction Y \in \mathcal{C}$.*

Proof. (ii) $\Rightarrow$ (i). Suppose that f satisfies the condition (ii) and $hf \notin SZ$ for some $h \in SZ$. Then $hf \restriction X \in \mathcal{C}$ for some set $X \in [\mathbb{R}]^{\mathfrak{c}}$. Let $Y \in [X \setminus [f = 0]]^{\mathfrak{c}}$ be a set such that $f \restriction Y \in \mathcal{C}$. Then $h \restriction Y = (hf)/f \restriction Y \in \mathcal{C}$, in contradiction with $h \in SZ$.

(i) $\Rightarrow$ (ii). Assume that $f \in \mathcal{M}_m(SZ)$. Note that $\text{card}([f = 0]) < \mathfrak{c}$. Fix $X \in [\mathbb{R}]^{\mathfrak{c}}$ and set $X_0 = X \setminus [f = 0]$. Obviously, $\text{card}(X_0) = \mathfrak{c}$. Suppose that $f \restriction Y \in \mathcal{C}$ for no $Y \in [X_0]^{\mathfrak{c}}$, i.e., $f \restriction X_0 \in SZ$. Then $(1/f) \restriction X_0 \in SZ$ and there exists an SZ-extension $f^* \in \mathbb{R}^{\mathbb{R}}$ of the function $(1/f) \restriction X_0$. Then $(f^* f) \restriction X_0 \in \mathcal{C}$, a contradiction. Hence there exists $Y \in [X]^{\mathfrak{c}}$ such that $f \restriction Y \in \mathcal{C}$. $\square$

4. Compositions

Let

$$\mathcal{M}_{\text{out}}(SZ) = \{f \in \mathbb{R}^{\mathbb{R}}\colon\ f \circ h \in SZ \text{ for each } h \in SZ\},$$

$$\mathcal{M}_{\text{in}}(SZ) = \{f \in \mathbb{R}^{\mathbb{R}}\colon\ h \circ f \in SZ \text{ for each } h \in SZ\}.$$

Theorem 4.1. *Assume that* $\mathfrak{c}$ *is a regular cardinal. Then for every function* $f \in \mathbb{R}^{\mathbb{R}}$ *the following conditions are equivalent*:

(i) $f \in \mathcal{M}_{\text{out}}(SZ)$;

(ii) $\text{card}(f^{-1}(y)) < \mathfrak{c}$ *for each* $y \in \mathbb{R}$, *and every choice function* $g : \text{rng}(f) \to \mathbb{R}$, $g(y) \in f^{-1}(y)$, *satisfies the following condition*

$$\textit{for each } X \in [\text{rng}(f)]^{\mathfrak{c}} \textit{ there exists a } Y \in [X]^{\mathfrak{c}} \textit{ such that } g \restriction Y \in \mathcal{C}; \qquad (*)$$

(iii) $f \in \mathcal{M}_{\text{in}}(SZ)$.

Proof. (i) $\Rightarrow$ (ii). Fix $f \in \mathcal{M}_{\text{out}}(SZ)$. Suppose that $\text{card}(f^{-1}(y)) = \mathfrak{c}$ for some $y \in \mathbb{R}$. By Proposition 1.1 we can choose an SZ-function $g \in \mathbb{R}^{\mathbb{R}}$ with $\text{rng}(g) \subset f^{-1}(y)$. Then $f \circ g \in \mathcal{C}$, a contradiction.

Suppose that there exists a choice function $g : \text{rng}(f) \to \mathbb{R}$, $g(y) \in f^{-1}(y)$, without the property $(*)$, i.e., that there exist $X \in [\text{rng}(f)]^{\mathfrak{c}}$ and $g \in \mathbb{R}^X$ such that $g \in SZ$ and $f \circ g = \text{id}_X$. Let $g^* \in \mathbb{R}^{\mathbb{R}}$ be an SZ-extension of g. Then $f \circ g^* \restriction X \in \mathcal{C}$, so $f \circ g^* \notin SZ$ and consequently, $f \notin \mathcal{M}_{\text{out}}(SZ)$, a contradiction.

(ii) $\Rightarrow$ (i). Suppose that $f \circ h \notin SZ$ for some SZ-function $h \in \mathbb{R}^{\mathbb{R}}$. Then there exists $X \in [\mathbb{R}]^{\mathfrak{c}}$ such that $f \circ h \restriction X \in \mathcal{C}$. Note that $\text{card}(\text{rng}(f \circ h \restriction X)) = \mathfrak{c}$. Indeed, otherwise, by regularity of $\mathfrak{c}$, $f \circ h$ is constant on some set $X_0 \in [X]^{\mathfrak{c}}$ and because $\text{card}(f^{-1}(y)) < \mathfrak{c}$ for each y, h is constant on some set $X_1 \in [X_0]^{\mathfrak{c}}$, a contradiction. Let $g : \text{rng}(f) \to \mathbb{R}$, $g(y) \in f^{-1}(y)$, be a choice function such that $g(t) \in \text{rng}(h \restriction X)$ for $t \in \text{rng}(f \circ h \restriction X)$. Let $g \restriction Y \in \mathcal{C}$ for $Y \in [\text{rng}(f \circ h \restriction X)]^{\mathfrak{c}}$. Then $X_0 = (f \circ h)^{-1}(Y) \cap X \in [X]^{\mathfrak{c}}$ and $h \restriction X_0 = g \circ (f \circ h \restriction X_0) \in \mathcal{C}$, a contradiction.

(iii) $\Rightarrow$ (ii). Fix $f \in \mathcal{M}_{\text{in}}(SZ)$. Obviously, $\text{card}(f^{-1}(y)) < \mathfrak{c}$ for every $y \in \mathbb{R}$. Suppose that $g : \text{rng}(f) \to \mathbb{R}$, $g(y) \in f^{-1}(y)$, is a choice function without the property $(*)$, i.e., that there exists $X \in [\text{rng}(f)]^{\mathfrak{c}}$ such that $g \restriction X \in SZ$. Let $g^* \in \mathbb{R}^{\mathbb{R}}$ be an SZ-extension of $g \restriction X$. Then $g^* \circ f \restriction (\text{rng}(g|X)) = \text{id}_{\text{rng}(g \restriction X)}$. But g is one-to-one. So, $\text{card}(\text{rng}(g \restriction X)) = \mathfrak{c}$ and $g^* \circ f \notin SZ$. A contradiction with $f \in \mathcal{M}_{\text{in}}(SZ)$.

(ii) $\Rightarrow$ (iii). Suppose that $h \circ f \notin SZ$ for some $h \in SZ$. Then $h \circ f \restriction X \in \mathcal{C}$ for some $X \in [\mathbb{R}]^{\mathfrak{c}}$. Note that $\text{card}(\text{rng}(f \restriction X)) = \mathfrak{c}$ since $\text{card}(f^{-1}(y)) < \mathfrak{c}$ for each $y \in \mathbb{R}$ and $\mathfrak{c}$ is regular. Let $g : \text{rng}(f) \to \mathbb{R}$, $g(y) \in f^{-1}(y)$, be a choice function such that $g(y) \in X$ for $y \in \text{rng}(f \restriction X)$ and let $Y \in [\text{rng}(f \restriction X)]^{\mathfrak{c}}$ be such that $g \restriction Y \in \mathcal{C}$. Then $h \restriction Y = (h \circ f) \circ g \restriction Y \in \mathcal{C}$, a contradiction. $\square$

Notice that in the proofs of implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) we did not use the assumption that $\mathfrak{c}$ is regular. Moreover, in the above proof of implication (ii) $\Rightarrow$ (iii) we do not have to use the assumption of regularity of $\mathfrak{c}$ if we additionally assume that f is one-to-one. (Or even only that $\sup\{\text{card}(f^{-1}(y)) \colon y \in \mathbb{R}\} < \mathfrak{c}$.) This implies the following two corollaries.

Corollary 4.2. *If* $\mathfrak{c}$ *is regular then* $\mathcal{M}_{\text{out}}(SZ) = \mathcal{M}_{\text{in}}(SZ)$.

Corollary 4.3. *If a one-to-one function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies condition* (ii) *from Theorem* 4.1 *then* $f \in \mathcal{M}_{\text{in}}(SZ)$.

The next result, being a version of Sierpiński–Zygmund theorem, will be used to show that Corollary 4.2 is false when $\mathbf{c}$ is singular.

Theorem 4.4. *Suppose that $\kappa \leqslant \mathbf{c}$ is a cardinal such that $\mathrm{cf}(\kappa) = \mathrm{cf}(\mathbf{c})$. Then for every $X \in [\mathbb{R}]^\kappa$ there exists $f : X \to \mathbb{R}$ such that $\mathrm{card}(\mathrm{rng} f) = \mathrm{cf}(\mathbf{c})$ and $f \restriction X_0$ is continuous for no $X_0 \in [X]^\kappa$.*

Proof. Let $\{\lambda_\xi\colon \xi < \mathrm{cf}(\mathbf{c})\}$ and $\{\mu_\xi\colon \xi < \mathrm{cf}(\mathbf{c})\}$ be increasing sequences of ordinal numbers such that $\kappa = \bigcup_{\xi<\mathrm{cf}(\mathbf{c})} \lambda_\xi$ and $\mathbf{c} = \bigcup_{\xi<\mathrm{cf}(\mathbf{c})} \mu_\xi$ and let $X = \{x_\xi\colon \xi < \kappa\}$. Choose a partition $\{X_\xi\colon \xi < \mathrm{cf}(\mathbf{c})\}$ of X such that $\mathrm{card}(X_\xi) = \mathrm{card}(\lambda_\xi)$ for every $\xi < \mathrm{cf}(\mathbf{c})$ and let $\{g_\xi\colon \xi < \mathbf{c}\}$ be an enumeration of $\mathcal{C}_{G_\delta}$. By induction on $\xi < \kappa$ define a sequence $\langle y_\xi \in \mathbb{R}\colon \xi < \mathrm{cf}(\mathbf{c})\rangle$ such that for every $\xi < \kappa$

$$y_\xi \in \mathbb{R} \setminus \{g_\eta(x)\colon \eta < \mu_\xi \ \&\ x \in X_\xi\}.$$

Now, define h by putting $h(x) = y_\xi$ for $x \in X_\xi$ and $\xi < \mathrm{cf}(\mathbf{c})$. It is easy to see that $\mathrm{rng}(h) = \{y_\xi\colon \xi < \mathrm{cf}(\mathbf{c})\}$. Also, if $g = g_\eta \in \mathcal{C}_{G_\delta}$ and $\eta < \mu_\xi$ then $[h = g] \subset \bigcup_{\zeta\leqslant\xi} X_\zeta$. Thus, $\mathrm{card}([h = g]) < \kappa$ and, as in Sierpiński–Zygmund's proof, we conclude that $h \restriction X_0$ is continuous for no $X_0 \in [X]^\kappa$. □

Corollary 4.5. *There exists an SZ function $h : \mathbb{R} \to \mathbb{R}$ with $\mathrm{card}(\mathrm{rng}(h)) = \mathrm{cf}(\mathbf{c})$.*

Problem 4.6. Does there exist an SZ function $h : \mathbb{R} \to Y$ for every $Y \in [\mathbb{R}]^{\mathrm{cf}(\mathbf{c})}$?

Corollary 4.7. *If $\mathbf{c}$ is singular then $\mathcal{M}_{\mathrm{in}}(SZ) \not\subset \mathcal{M}_{\mathrm{out}}(SZ)$.*

Proof. Let h be as in Corollary 4.5. Fix $x_0 \in \mathrm{rng}(h)$ and define a function f by putting $f(x) = x_0$ for $x \in \mathrm{rng}(h)$ and $f(x) = x$ otherwise. Notice that $f \in \mathcal{M}_{\mathrm{in}}(SZ)$. Indeed, consider $g \in SZ$. In order to show that $g \circ f \in SZ$ by way of contradiction suppose that there is an $X \in [\mathbb{R}]^{\mathbf{c}}$ such that $g \circ f \restriction X$ is continuous. But $\mathrm{card}(X \setminus \mathrm{rng}(h)) = \mathbf{c}$, since $\mathrm{cf}(\mathbf{c}) < \mathbf{c}$. Moreover, $f(x) = x$ for every $x \in X \setminus \mathrm{rng}(h)$. So, $g \restriction X \setminus \mathrm{rng}(h) = g \circ f \restriction X \setminus \mathrm{rng}(h)$ is continuous on a set of cardinality $\mathbf{c}$, contradicting $g \in SZ$.

On the other hand, $f \circ h$ is constant, so $f \circ h \notin SZ$, while $h \in SZ$. Thus, $f \notin \mathcal{M}_{\mathrm{out}}(SZ)$. □

Problem 4.8. Can inclusion $\mathcal{M}_{\mathrm{out}}(SZ) \subset \mathcal{M}_{\mathrm{in}}(SZ)$ be proved without the assumption that $\mathbf{c}$ is regular?

4.1. Compositions with SZ-functions from the left

Theorem 4.9. *For each $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:*

(i) *there exists $h \in SZ \cap \mathbb{R}^\mathbb{R}$ such that $h \circ f \in SZ$;*
(ii) *there exists $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f \in SZ$;*
(iii) $\mathrm{card}(f^{-1}(y)) < \mathbf{c}$ *for each $y \in \mathbb{R}$.*

Proof. (i) $\Rightarrow$(ii). Obvious.

(ii) $\Rightarrow$ (iii). Suppose that $\mathrm{card}(f^{-1}(y_0)) = \mathbf{c}$ for some $y_0 \in \mathbb{R}$. Then $h \circ f$ is constant on $f^{-1}(y_0)$, a contradiction.

(iii) $\Rightarrow$ (i). First notice that there exists $\mathcal{E} \subseteq \mathbf{c}$ and a one-to-one enumeration $\{y_\alpha\colon \alpha \in \mathcal{E}\}$ of $\mathbb{R}$ such that

$$\mathrm{card}\big(f^{-1}(y_\alpha)\big) \leqslant \mathrm{card}(\alpha) \quad \text{for every } \alpha \in \mathcal{E}. \tag{$\star$}$$

To see it, let $\{y_\alpha\colon \alpha < \mathbf{c}\}$ be an enumeration of $\mathbb{R}$ with each number appearing $\mathbf{c}$ many times. For $y \in \mathbb{R}$ let $\alpha(y) = \min\{\alpha < \mathbf{c}\colon y_\alpha = y \,\&\, \mathrm{card}(f^{-1}(y)) \leqslant \mathrm{card}(\alpha)\}$ and put $\mathcal{E} = \{\alpha(y)\colon y \in \mathbb{R}\}$. Then $\{y_\alpha\colon \alpha \in \mathcal{E}\}$ has the desired properties.

Next, let $\{g_\xi\colon \xi < \mathbf{c}\} = \mathcal{C}_{G_\delta}$ and let $\{\alpha_\xi\colon \xi < \mathbf{c}\}$ be an increasing enumeration of $\mathcal{E}$. Then $\{y_{\alpha_\xi}\colon \xi < \mathbf{c}\}$ is a one-to-one enumeration of $\mathbb{R}$. For each $\xi < \mathbf{c}$ choose

$$h(y_{\alpha_\xi}) \in \mathbb{R} \setminus \Big(\{g_\zeta(y_{\alpha_\xi})\colon \zeta < \xi\} \cup \bigcup\{g_\zeta[f^{-1}(y_{\alpha_\xi})]\colon \zeta < \xi\}\Big).$$

Such a choice can be made, since the set $\bigcup\{g_\zeta[f^{-1}(y_{\alpha_\xi})]\colon \zeta < \xi\}$ is a union of $\mathrm{card}(\xi) < \mathbf{c}$ many sets, each set of cardinality $\leqslant \mathrm{card}(\alpha_\xi) < \mathbf{c}$.

It is clear that $h \in SZ$. To verify that $h \circ f \in SZ$ fix $\zeta < \mathbf{c}$. Observe that

$$[h \circ f = g_\zeta] \subseteq \bigcup_{\xi \leqslant \zeta} f^{-1}(y_{\alpha_\xi}).$$

Indeed, if $h \circ f(x) = g_\zeta(x)$ and $f(x) = y_{\alpha_\xi}$ some $\xi < \mathbf{c}$ then $h(y_{\alpha_\xi}) \in g_\zeta[f^{-1}(y_{\alpha_\xi})]$. So $\xi \leqslant \zeta$ and $x \in \bigcup_{\xi \leqslant \zeta} f^{-1}(y_{\alpha_\xi})$. Thus, by ($\star$),

$$\mathrm{card}\big((h \circ f) \cap g_\zeta\big) \leqslant \mathrm{card}\Bigg(\bigcup_{\xi \leqslant \zeta} f^{-1}(y_{\alpha_\xi})\Bigg) \leqslant \mathrm{card}(\zeta) \cdot \mathrm{card}(\alpha_\zeta) < \mathbf{c}. \qquad \square$$

Theorem 4.9 justifies restriction of our attention only to the functions from a family

$$\mathcal{R}_1 = \{f \in \mathbb{R}^\mathbb{R}\colon \mathrm{card}\big(f^{-1}(y)\big) < \mathbf{c} \text{ for every } y \in \mathbb{R}\}$$

and definition

$$\begin{aligned} c_{\mathrm{out}}(SZ) &= \min\big(\{\mathrm{card}(\mathcal{F})\colon \mathcal{F} \subset \mathcal{R}_1 \ \&\ \neg\exists h \in \mathbb{R}^\mathbb{R}\ \forall f \in \mathcal{F}\ h \circ f \in SZ\} \cup \{(2^\mathbf{c})^+\}\big) \\ &= \min\big(\{\mathrm{card}(\mathcal{F})\colon \mathcal{F} \subset \mathcal{R}_1 \ \&\ \forall h \in \mathbb{R}^\mathbb{R}\ \exists f \in \mathcal{F}\ h \circ f \notin SZ\} \cup \{(2^\mathbf{c})^+\}\big). \end{aligned}$$

Note that $SZ \subseteq \mathcal{R}_1$, so $\mathrm{card}(\mathcal{R}_1) = 2^\mathbf{c}$.

Now, we have the following analog of Theorem 2.1.

Theorem 4.10. *If $\mathbf{c}$ is a regular cardinal then*

$$\mathbf{c} < c_{\mathrm{out}}(SZ) \leqslant 2^\mathbf{c}.$$

Proof. The inequality $\mathbf{c} < c_{\mathrm{out}}(SZ)$ is proved similarly as the implication (iii) $\Rightarrow$ (i) of Theorem 4.9. To see it, let $\mathcal{F} = \{f_\xi\colon \xi < \mathbf{c}\} \subseteq \mathcal{R}_1$, $\{g_\xi\colon \xi < \mathbf{c}\} = \mathcal{C}_{G_\delta}$ and $\{y_\xi\colon \xi < \mathbf{c}\}$ be a one-to-one enumeration of $\mathbb{R}$. For each $\xi < \mathbf{c}$ choose

$$h(y_\xi) \in \mathbb{R} \setminus \Big(\bigcup\{g_\zeta[f_\eta^{-1}(y_\xi)]\colon \zeta, \eta < \xi\}\Big).$$

The possibility of such a choice is guaranteed by the regularity of $\mathbf{c}$, since the set $\bigcup\{g_\zeta[f_\eta^{-1}(y_\xi)]\colon \zeta,\eta<\xi\}$ is a union of less than $\mathbf{c}$ many sets of cardinality less than $\mathbf{c}$. To see that $h\circ f_\eta\in SZ$ for every $\eta<\mathbf{c}$ it is enough to notice that

$$[h\circ f_\eta=g_\zeta]\subseteq\bigcup_{\xi\leqslant\max\{\zeta,\eta\}}f_\eta^{-1}(y_\xi)\quad\text{for every } \zeta<\mathbf{c}.$$

To prove the inequality $c_{\mathrm{out}}(SZ)\leqslant 2^{\mathbf{c}}$ take $\mathcal{F}=\mathcal{R}_1$ and $h\in\mathbb{R}^{\mathbb{R}}$. It is enough to find $f\in\mathcal{F}$ such that $h\circ f\notin SZ$.

By way of contradiction assume that $h\circ f\in SZ$ for every $f\in\mathcal{R}_1$. Then, $h=h\circ\mathrm{id}\in SZ$, since $\mathrm{id}\in\mathcal{R}_1$. In particular, $\mathrm{card}(\mathrm{rng}(h))=\mathbf{c}$, since otherwise h would be constant on a set of cardinality $\mathbf{c}$. So, there exists $f\in\mathcal{R}_1$ such that $f(y)\in h^{-1}(y)$ for every $y\in\mathrm{rng}(h)$. Then $h\circ f(y)=y$ for every $y\in\mathrm{rng}(h)$ and so $\mathrm{card}((h\circ f)\cap\mathrm{id})=\mathbf{c}$, a contradiction. □

The importance of the assumption of regularity of $\mathbf{c}$ in Theorem 4.10 is not clear. For an arbitrary value of $\mathbf{c}$, including the case when $\mathbf{c}$ is singular, we have only the following theorem.

Theorem 4.11. $\mathrm{cf}(\mathbf{c})\leqslant c_{\mathrm{out}}(SZ)\leqslant 2^{\mathrm{cf}(\mathbf{c})}=\mathbf{c}^{\mathrm{cf}(\mathbf{c})}$.

Proof. The proof of the inequality $\mathrm{cf}(\mathbf{c})\leqslant c_{\mathrm{out}}(SZ)$ is a simple modification of the proof of the implication (iii) $\Rightarrow$ (i) from Theorem 4.9. To see it, take $\mathcal{F}\subseteq\mathcal{R}_1$ with $\mathrm{card}(\mathcal{F})<\mathrm{cf}(\mathbf{c})$ and choose a one-to-one enumeration $\{y_\alpha\colon \alpha\in\mathcal{E}\}$ of $\mathbb{R}$, $\mathcal{E}\subseteq\mathbf{c}$, such that

$$\mathrm{card}\Big(\bigcup_{f\in\mathcal{F}}f^{-1}(y_\alpha)\Big)\leqslant\mathrm{card}(\alpha)\quad\text{for every }\alpha\in\mathcal{E}.\tag{$\star$}$$

Let $\{g_\xi\colon \xi<\mathbf{c}\}=\mathcal{C}_{G_\delta}$ and $\{\alpha_\xi\colon \xi<\mathbf{c}\}$ be as in Theorem 4.9 and for each $\xi<\mathbf{c}$ choose

$$h(y_{\alpha_\xi})\in\mathbb{R}\setminus\Big(\bigcup\Big\{g_\zeta\Big[\bigcup\{f^{-1}(y_{\alpha_\xi})\colon f\in\mathcal{F}\}\Big]\colon \zeta<\xi\Big\}\Big).$$

It is easy to see that for such defined h we have $h\circ f\in SZ$ for every $f\in\mathcal{F}$.

The other inequality for regular $\mathbf{c}$ follows from Theorem 4.10. So, assume that $\mathbf{c}$ is singular and let $\langle\lambda_\alpha\colon \alpha<\mathrm{cf}(\mathbf{c})\rangle$ be an increasing sequence of cardinals such that $\lambda_\alpha\nearrow\mathbf{c}$. Let S be the set of all one-to-one functions $s\colon\mathrm{cf}(\mathbf{c})\to\mathbb{R}$ and $g\colon\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\mathrm{card}(g^{-1}(y))=\mathbf{c}$ for every $y\in\mathbb{R}$. For every pair $s,t\in S$ choose: a sequence of sets $\langle X_\alpha^{st}\subset g^{-1}(s(\alpha))\colon \alpha<\mathrm{cf}(\mathbf{c})\rangle$ such that $\mathrm{card}(X_\alpha^{st})=\lambda_\alpha$ for each $\alpha<\mathrm{cf}(\mathbf{c})$, and a function $f_{st}\in\mathcal{R}_1$ such that $f_{st}(x)=t(\alpha)$ for every $x\in X_\alpha^{st}$ and $\alpha<\mathrm{cf}(\mathbf{c})$. Define

$$\mathcal{F}=\{\mathrm{id}\}\cup\{f_{st}\colon s,t\in S\}$$

and notice that $\mathrm{card}(\mathcal{F})=\mathbf{c}^{\mathrm{cf}(\mathbf{c})}$. It is enough to show that for every $h\colon\mathbb{R}\to\mathbb{R}$ there exists $f\in\mathcal{F}$ such that $h\circ f\notin SZ$.

By way of contradiction assume that $h \circ f \in SZ$ for every $f \in \mathcal{F}$. Then, $h = h \circ \mathrm{id} \in SZ$, since $\mathrm{id} \in \mathcal{F}$. In particular, $\mathrm{card}(\mathrm{rng}(h)) \geqslant \mathrm{cf}(\mathbf{c})$, since otherwise h would be constant on a set of cardinality $\mathbf{c}$. Choose $s, t \in S$ such that $s[\mathrm{cf}(\mathbf{c})] \subset \mathrm{rng}(h)$ and $t(\alpha) \in h^{-1}(s(\alpha))$ for every $\alpha < \mathrm{cf}(\mathbf{c})$. Then, for every $\alpha < \mathrm{cf}(\mathbf{c})$ and $x \in X_\alpha^{st}$ we have

$$h \circ f_{st}(x) = h \circ t(\alpha) = s(\alpha) = g(x).$$

Thus, $h \circ f_{st}$ equals to g on

$$X_{st} = \bigcup_{\alpha < \mathrm{cf}(\mathbf{c})} X_\alpha^{st}.$$

So $h \circ f_{st} \notin SZ$, since $\mathrm{card}(X_{st}) = \mathbf{c}$. □

By Theorem 4.11 we can restrict our attention in the definition of $c_{\mathrm{out}}(SZ)$ to functions h from SZ. This is the case, since we can always assume that the identity function id belong to $\mathcal{F}$. So, we have the following corollary.

Corollary 4.12.

$$c_{\mathrm{out}}(SZ) = \min\bigl(\{\mathrm{card}(\mathcal{F}) : \mathcal{F} \subset \mathcal{R}_1 \ \& \ \neg\exists h \in SZ \forall f \in \mathcal{F}\ h \circ f \in SZ\} \\ \cup \{(2^{\mathbf{c}})^+\}\bigr).$$

Despite of some knowledge of $\mathrm{cf}(\mathbf{c})$ for singular $\mathbf{c}$, given by Theorem 4.11, the following problem remains open.

Problem 4.13. Is the assumption of regularity of $\mathbf{c}$ important in Theorem 4.10?

On the other hand, the case when $\mathbf{c} = \kappa^+$ for some cardinal κ the number $c_{\mathrm{out}}(SZ)$ is pretty easily handled by our results from the previous sections and the following theorem.

Theorem 4.14. *If $\mathbf{c} = \kappa^+$ for some cardinal κ then $c_{\mathrm{out}}(SZ) = a(SZ)$.*

Proof. By Theorem 2.14 it is enough to show that $c_{\mathrm{out}}(SZ) = d_{\mathbf{c}}$.

"$c_{\mathrm{out}}(SZ) \leqslant d_{\mathbf{c}}$". Let $\mathcal{N}$ stand for the set of irrational numbers and let $\mathcal{F} \subseteq \mathcal{N}^{\mathcal{N}}$ be such that $\mathrm{card}(\mathcal{F}) < c_{\mathrm{out}}(SZ)$. We will show that $\mathrm{card}(\mathcal{F}) < d_{\mathbf{c}}$ by finding $h : \mathcal{N} \to \mathcal{N}$ such that $\mathrm{card}(h \cap f) < \mathbf{c}$ for every $f \in \mathcal{F}$.

For $f \in \mathcal{F}$ define a partial function $\hat{f}^\star$ on a subset of $\mathcal{N}^2$ by putting

$$\hat{f}^\star(\langle x, f(x)\rangle) = x$$

for every $x \in \mathcal{N}$. Notice that $\hat{f}^\star$ is one-to-one on its domain. By identifying $\mathcal{N}^2$ with $\mathcal{N}$ via natural homeomorphism we can consider $\hat{f}^\star$ as a partial function on $\mathbb{R}$. Let $f^\star : \mathbb{R} \to \mathbb{R}$ be an extension of $\hat{f}^\star$ such that $f^\star \in \mathcal{R}_1$ and define $\widehat{\mathcal{F}} = \{\mathrm{id}\} \cup \{f^\star : f \in \mathcal{F}\}$. Since $\mathrm{card}(\widehat{\mathcal{F}}) \leqslant \mathrm{card}(\mathcal{F}) + 1 < c_{\mathrm{out}}(SZ)$ there exists an $\hat{h} \in \mathbb{R}^{\mathbb{R}}$ such that $\hat{h} \circ \hat{f} \in SZ$ for every $\hat{f} \in \widehat{\mathcal{F}}$. We will prove that for every $f \in \mathcal{F}$

$$\mathrm{card}\bigl(\{x \in \mathcal{N} : f(x) = \hat{h}(x)\}\bigr) < \mathbf{c}. \tag{1}$$

It is enough, since $\hat{h} = \hat{h} \circ \mathrm{id} \in SZ$ implies that $\hat{h}^{-1}(\mathbb{Q})$ has cardinality $< \mathbf{c}$, and so, there exists $h : \mathcal{N} \to \mathcal{N}$ such that $\mathrm{card}(\{x \in \mathcal{N} : \hat{h}(x) \neq h(x)\}) < \mathbf{c}$.

To see (1) let $f \in \mathcal{F}$ and let $x \in \mathcal{N}$ be such that $f(x) = \hat{h}(x)$. Then

$$\hat{h} \circ f^{\star}(\langle x, f(x)\rangle) = \hat{h}(x) = f(x) = \pi_2(\langle x, f(x)\rangle),$$

where $\pi_2 : \mathcal{N}^2 \to \mathcal{N}$ is the projection onto the second coordinate, thus continuous. So,

$$\mathrm{card}(\{x \in \mathcal{N}:\ f(x) = \hat{h}(x)\}) \leqslant \mathrm{card}([\hat{h} \circ f^{\star} = \pi_2]) < \mathbf{c}$$

since $\hat{h} \circ f^{\star} \in SZ$. This finishes the proof of "$c_{\mathrm{out}}(SZ) \leqslant d_{\mathbf{c}}$". (Notice, we do not use here even regularity of $\mathbf{c}$!)

"$d_{\mathbf{c}} \leqslant c_{\mathrm{out}}(SZ)$". Now assume that $\mathcal{F} \subset \mathcal{R}_1$ and $\mathrm{card}(\mathcal{F}) < d_{\mathbf{c}}$. For every $f \in \mathcal{F}$ choose the family $\{\hat{f}_\alpha:\ \alpha < \kappa\}$ such that $f^{-1}(y) = \{\hat{f}_\alpha(y) : \alpha < \kappa\}$ for each $y \in \mathrm{rng}(f)$, and define

$$\widehat{\mathcal{F}} = \{\bar{g} \circ \hat{f}_\alpha:\ g \in \mathcal{C}_{G_\delta}\ \&\ f \in \mathcal{F}\ \&\ \alpha < \kappa\},$$

where $\bar{g} \in \mathbb{R}^{\mathbb{R}}$ extends $g \in \mathcal{C}_{G_\delta}$ by associating 0 at all undefined places. Note that $\mathrm{card}(\widehat{\mathcal{F}}) \leqslant \mathrm{card}(\mathcal{F}) \cdot \mathbf{c} < d_{\mathbf{c}}$, hence there exists an $h \in \mathbb{R}^{\mathbb{R}}$ such that $\mathrm{card}(h \cap \hat{f}) < \mathbf{c}$ for each $\hat{f} \in \widehat{\mathcal{F}}$. We shall verify that $h \circ f \in SZ$ for every $f \in \mathcal{F}$. For this fix $g \in \mathcal{C}_{G_\delta}$ and observe that

$$\begin{aligned}\mathrm{card}((h \circ f) \cap g) &= \mathrm{card}(\{x:\ h \circ f(x) = g(x)\})\\ &= \mathrm{card}\Bigl(\bigcup_{\alpha<\kappa} \{\hat{f}_\alpha(y):\ y \in \mathrm{rng}(f)\ \&\ h(y) = g \circ \hat{f}_\alpha(y)\}\Bigr)\\ &= \sum_{\alpha<\kappa} \mathrm{card}(\{y:\ h(y) = g \circ \hat{f}_\alpha(y)\}) < \mathbf{c}.\end{aligned}$$

This finishes the proof of Theorem 4.14. □

Problem 4.15. Can Theorem 4.14 be proved for any value of $\mathbf{c}$? What about $\mathbf{c}$ being a regular limit cardinal?

Theorem 4.14 implies immediately the following corollary.

Corollary 4.16. *Let $\lambda \geqslant \kappa \geqslant \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with* ZFC *that the Continuum Hypothesis* ($\mathbf{c} = \aleph_1$) *is true, $2^{\mathbf{c}} = \lambda$, and $c_{\mathrm{out}}(SZ) = \kappa$.*

4.2. Compositions with SZ functions from the right

In this section we will examine for which functions $f \in \mathbb{R}^{\mathbb{R}}$ there exists an $h \in \mathbb{R}^{\mathbb{R}}$ such that $f \circ h \in SZ$. The class of all functions $f \in \mathbb{R}^{\mathbb{R}}$ having this property will be denoted by $\mathcal{R}_2$. Also, as in previous sections, we will define the cardinal $c_{\mathrm{in}}(SZ)$ analogous to $c_{\mathrm{out}}(SZ)$ restricting our attention to the maximal family for which such a definition has a sense, i.e., to $\mathcal{R}_2$. Thus, we define

$$\begin{aligned} c_{\mathrm{in}}(SZ) &= \min\bigl(\{\mathrm{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathcal{R}_2\ \&\ \neg\exists h \in \mathbb{R}^{\mathbb{R}}\ \forall f \in \mathcal{F}\ f \circ h \in SZ\} \cup \{(2^{\mathbf{c}})^+\}\bigr) \\ &= \min\bigl(\{\mathrm{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathcal{R}_2\ \&\ \forall h \in \mathbb{R}^{\mathbb{R}}\ \exists f \in \mathcal{F}\ f \circ h \notin SZ\} \cup \{(2^{\mathbf{c}})^+\}\bigr). \end{aligned}$$

The next theorem gives a characterization of the family $\mathcal{R}_2$ in case when $\mathbf{c}$ is regular.

Theorem 4.17. *Assume that $\mathbf{c}$ is a regular cardinal. For each $f:\mathbb{R} \to \mathbb{R}$ the following conditions are equivalent*:

(i) *there exists $h \in SZ \cap \mathbb{R}^{\mathbb{R}}$ such that $f \circ h \in SZ$*;

(ii) *there exists $h:\mathbb{R} \to \mathbb{R}$ such that $f \circ h \in SZ$*;

(iii) $\mathrm{card}(\mathrm{rng}(f)) = \mathbf{c}$.

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). Note that $\mathrm{card}(\mathrm{rng}(h)) = \mathbf{c}$. Indeed, otherwise, by regularity of $\mathbf{c}$, $\mathrm{card}(h^{-1}(y_0)) = \mathbf{c}$ for some $y_0 \in \mathbb{R}$ and then $f \circ h$ is constant on $h^{-1}(y_0)$ for any f, a contradiction. Next, by way of contradiction, suppose that $\mathrm{card}(\mathrm{rng}(f)) < \mathbf{c}$. Then, there exists a $y_0 \in \mathbb{R}$ such that $\mathrm{card}(f^{-1}(y_0) \cap \mathrm{rng}(h)) = \mathbf{c}$. Therefore,

$$\mathrm{card}\bigl((f \circ h)^{-1}(y_0)\bigr) = \mathbf{c},$$

a contradiction.

(iii) $\Rightarrow$ (i). Let $\{g_\alpha\colon\ \alpha < \mathbf{c}\} = \mathcal{C}_{G_\delta}$, and $\{x_\alpha\colon\ \alpha < \mathbf{c}\} = \mathbb{R}$. For every $\alpha < \mathbf{c}$ choose

$$h(x_\alpha) \in \mathbb{R} \setminus \Bigl(\{g_\beta(x_\alpha)\colon\ \beta \leqslant \alpha\} \cup \bigcup_{\beta \leqslant \alpha} f^{-1}\bigl(g_\beta(x_\alpha)\bigr)\Bigr).$$

The choice can be made since, by (iii),

$$\mathbb{R} \setminus \Bigl(\{g_\beta(x_\alpha)\colon\ \beta \leqslant \alpha\} \cup \bigcup_{\beta \leqslant \alpha} f^{-1}\bigl(g_\beta(x_\alpha)\bigr)\Bigr)$$

is not empty.

Obviously, $h \in SZ$. It is enough to verify that $f \circ h \in SZ$. So, fix $\alpha < \mathbf{c}$. Then

$$\{x\colon\ f \circ h(x) = g_\alpha(x)\} = \{x\colon\ h(x) \in f^{-1}\bigl(g_\alpha(x)\bigr)\} \subset \{x_\beta\colon\ \beta < \alpha\},$$

and so $\mathrm{card}((f \circ h) \cap g_\alpha) < \mathbf{c}$. $\square$

Note that we did not use the regularity assumption in implications (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii). In particular, if

$$\mathcal{R}_2^\star = \{f \in \mathbb{R}^{\mathbb{R}}\colon\ \mathrm{card}\bigl(\mathrm{rng}(f)\bigr) = \mathbf{c}\}$$

then

Corollary 4.18. $\mathcal{R}_2^\star \subset \mathcal{R}_2$.

We have also

Corollary 4.19. *If* $\mathbf{c}$ *is a regular cardinal then* $\mathcal{R}_1 \subset \mathcal{R}_2 = \mathcal{R}_2^\star$.

Example 4.20. There exist functions $f_0, f_1 \in \mathcal{R}_2^\star$ such that for every $h:\mathbb{R} \to \mathbb{R}$ either $f_0 \circ h \notin SZ$ or $f_1 \circ h \notin SZ$.

Proof. Indeed, decompose the real line onto two sets A_0 and A_1 such that $\mathrm{card}(A_i) = \mathbf{c}$ for $i < 2$, and define a function f_i such that $f_i(A_i) = 0$ and $f_i \restriction A_{1-i}$ is one-to-one. Fix an $h:\mathbb{R} \to \mathbb{R}$. Since $\mathbb{R} = h^{-1}(\mathbb{R}) = h^{-1}(A_0) \cup h^{-1}(A_1)$ there exists $i < 2$ such that $\mathrm{card}(h^{-1}(A_i)) = \mathbf{c}$. Then $\mathrm{card}((f_i \circ h)^{-1}(0)) = \mathrm{card}(h^{-1}(A_i)) = \mathbf{c}$, so $f_i \circ h \notin SZ$. □

Corollary 4.21. $c_{\mathrm{in}}(SZ) = 2$.

4.3. Coding functions by SZ-functions

In the previous sections we examined when for a given function $f \in \mathbb{R}^\mathbb{R}$ there exist two SZ-functions $g, h \in \mathbb{R}^\mathbb{R}$ such that $f \circ h = g$ or $h \circ f = g$. In this section we will ask for which $f \in \mathbb{R}^\mathbb{R}$ there exist SZ-functions $g, h \in \mathbb{R}^\mathbb{R}$ such that $f = g \circ h$ or $f = h \circ g$, i.e., that f is coded by two SZ-functions. Note that even when for some f the first set of questions have a positive answer with h being one-to-one, this does not imply the positive answer for the second set of questions, since the inverse of an SZ-function does not have to be SZ. In fact, it is consistent with ZFC that no SZ-function $h:\mathbb{R} \to \mathbb{R}$ has an SZ inverse. This happens in the iterated perfect set model, where there is no SZ-function from $\mathbb{R}$ onto $\mathbb{R}$ [1]. (If h^{-1} is SZ then it is onto $\mathbb{R}$ and any of its SZ-extension is an SZ-function from $\mathbb{R}$ onto $\mathbb{R}$.) The same example also shows, that the set of questions we consider in this section cannot have a positive answer in ZFC for any function from $\mathbb{R}$ onto $\mathbb{R}$, even for the identity function. Thus, we will work here with the additional set theoretical assumptions.

We will start with the following lemmas.

Lemma 4.22. *Assume that* $\mathbf{c}$ *is a regular cardinal. Then the class* $\mathcal{R}_1$ *is closed under the compositions of functions.*

Proof. Suppose that $f = f_2 \circ f_1$, $f_1, f_2 \in \mathcal{R}_1$ and $\mathrm{card}(f^{-1}(y_0)) = \mathbf{c}$ for $y_0 \in \mathbb{R}$. Then f is constant on the set $X = f^{-1}(y_0) = \bigcup\{(f_1)^{-1}(t)\colon t \in (f_2)^{-1}(y_0)\}$, so either f_1 or f_2 is constant on a set of cardinality $\mathbf{c}$, a contradiction. □

Note that if $\mathbf{c}$ is a singular cardinal then the conclusion of Lemma 4.22 is false.

Proposition 4.23. *If* $\mathbf{c}$ *is a singular cardinal then every function from* $\mathbb{R}$ *into* $\mathbb{R}$ *is a composition of two functions from the class* $\mathcal{R}_1$.

Proof. Suppose that $\mathbb{R} = \{x_\alpha\colon \alpha < \mathbf{c}\}$, $\kappa = \mathrm{cf}(\mathbf{c}) < \mathbf{c}$ and $\langle \lambda_\alpha\colon \alpha < \kappa\rangle$ is an increasing sequence of cardinals such that $\mathbf{c} = \bigcup_{\alpha<\kappa} \lambda_\alpha$. Fix $f \in \mathbb{R}^\mathbb{R}$. For every $\alpha < \mathbf{c}$ let $X_\alpha = f^{-1}(x_\alpha)$ and let $X_\alpha = \bigcup_{\beta<\kappa} X_{\alpha,\beta}$ be a partition such that $\mathrm{card}(X_{\alpha,\beta}) \leqslant \lambda_\beta$

for every $\beta < \kappa$. Choose a sequence $\langle Y_\alpha\colon \alpha < \mathbf{c}\rangle$ of pairwise disjoint sets of reals, each of cardinality equal to κ; $Y_\alpha = \{y_{\alpha,\beta}\colon \beta < \kappa\}$ and define $f_1(x) = y_{\alpha,\beta}$ for $x \in X_{\alpha,\beta}$ and $\hat{f}_2(y_{\alpha,\beta}) = x_\alpha$ for $\alpha < \mathbf{c}$, $\beta < \kappa$. Let $f_2 \in \mathcal{R}_1$ be any extension of $\hat{f}_2$. Then $f = f_2 \circ f_1$. $\square$

Lemma 4.24. *Assume $f \in \mathcal{R}_1$. Then $f \in SZ$ if and only if* $\mathrm{card}(f \cap g) < \mathbf{c}$ *for each continuous nowhere constant function g defined on a G_δ-set.*

Proof. The implication "$\Rightarrow$" is obvious. To prove "$\Leftarrow$" assume that g is a continuous function defined on a G_δ-set G. Let $\langle G_n\rangle_{n<\omega}$ be a sequence of all maximal intervals in G (i.e., nonempty sets of the form $G \cap (a,b)$, for $a < b$) on which g is constant. Then $H = G \setminus \bigcup_{n<\omega} G_n$ is a G_δ set and $g \restriction H$ is nowhere constant. Moreover,

$$g = (g \restriction H) \cup \bigcup_{n<\omega} (g \restriction G_n)$$

and for each $n < \omega$, $g \restriction G_n$ is constant, so $\mathrm{card}((g \restriction G_n) \cap f) < \mathbf{c}$. Hence

$$g \cap f = \big((g \restriction H) \cap f\big) \cup \bigcup_{n<\omega} \big((g \restriction G_n) \cap f\big)$$

and $\mathrm{card}(g \cap f) < \mathbf{c}$ since $\mathrm{cf}(\mathbf{c}) > \omega$. $\square$

The next theorem tells us that for every sequence $\langle f_\alpha\colon \alpha < \mathbf{c}\rangle$ of $\mathcal{R}_1$ functions there exists a sequence $\langle f_\alpha^\triangleright\colon \alpha < \mathbf{c}\rangle$ of their SZ codes and an $\circ$-decoder function $h \in SZ$ such that every f_α can be "right $\circ$-decoded" by h from $f_\alpha^\triangleright$.

Theorem 4.25. *Assume that the real line is not a union of less than* $\mathbf{c}$ *many meager sets. Then for every family $\{f_\alpha\colon \alpha < \mathbf{c}\} \subset \mathcal{R}_1$ there is a family $\{f_\alpha^\triangleright\colon \alpha < \mathbf{c}\}$ of SZ-functions and a "decoding" function $h \in SZ$ and such that $f_\alpha^\triangleright \circ h = f_\alpha$ for each $\alpha < \mathbf{c}$.*

Proof. Let $\mathcal{C}_n = \{g_\alpha\colon \alpha < \mathbf{c}\}$ be an enumeration of all nowhere constant $g \in \mathcal{C}_{G_\delta}$ and let $\{x_\alpha\colon \alpha < \mathbf{c}\} = \mathbb{R}$. For every $\alpha < \mathbf{c}$ choose

$$h(x_\alpha) \in \mathbb{R} \setminus \Big(\{g_\beta(x_\alpha)\colon \beta \leqslant \alpha\} \cup \{h(x_\beta)\colon \beta < \alpha\}$$
$$\cup \bigcup \{g_\beta^{-1}(f_\nu(x_\alpha))\colon \beta, \nu \leqslant \alpha\}\Big).$$

Note that the choice can be made since every set $g_\beta^{-1}(f_\nu(x_\alpha))$ is meager and $\mathbb{R}$ is not a union of less than $\mathbf{c}$ many meager sets.

It is easy to observe that the function h is one-to-one and so, $h \in \mathcal{R}_1$. Also, by our choice, $\mathrm{card}([h = g]) < \mathbf{c}$ for every $g \in \mathcal{C}_n$. So, by Lemma 4.24, $h \in SZ$.

Now for $\nu < \mathbf{c}$ define $f_\nu^\triangleright$. Put $f_\nu^\triangleright(h(x_\alpha)) = f_\nu(x_\alpha)$ for every $\alpha < \mathbf{c}$ and for $x \notin \mathrm{rng}(h)$ define $f_\nu^\triangleright(x) = h(x)$. Clearly $f_\nu = f_\nu^\triangleright \circ h$ for every $\nu < \mathbf{c}$. To see that $f_\nu^\triangleright \in SZ$ first notice that $f_\nu^\triangleright \in \mathcal{R}_1$, since for every $y \in \mathbb{R}$ the set

$$(f_\nu^\triangleright)^{-1}(y) = \{h(x)\colon f_\nu^\triangleright(h(x)) = y\} \cup \{z \in \mathbb{R} \setminus \mathrm{rng}(h)\colon f_\nu^\triangleright(z) = y\}$$
$$\subset h[f_\nu^{-1}(y)] \cup h^{-1}(y)$$

has cardinality less than $\mathbf{c}$ as $h, f_\nu \in \mathcal{R}_1$. Moreover, for every $\beta < \mathbf{c}$

$$\begin{aligned}
[f_\nu^\triangleright = g_\beta] &= \{h(x)\colon\ f_\nu^\triangleright(h(x)) = g_\beta(h(x))\} \cup \{z \in \mathbb{R} \setminus \operatorname{rng}(h)\colon\ f_\nu^\triangleright(z) = g_\beta(z)\} \\
&= h[\{x\colon\ f_\nu(x) = g_\beta(h(x))\}] \cup \{z \in \mathbb{R} \setminus \operatorname{rng}(h)\colon\ h(z) = g_\beta(z)\} \\
&= h[\{x\colon\ h(x) \in g_\beta^{-1}(f_\nu(x))\}] \cup ([h = g_\beta] \setminus \operatorname{rng}(h)) \\
&\subset h[\{x_\alpha\colon\ \alpha < \max\{\beta, \nu\}\}] \cup [h = g_\beta].
\end{aligned}$$

Thus, $\operatorname{card}([f_\nu^\triangleright = g]) < \mathbf{c}$ for every $g \in \mathcal{C}_n$ and, by Lemma 4.24, $f_\nu^\triangleright \in SZ$. $\square$

Lemma 4.22 together with Theorem 4.25 yield to the following result:

Corollary 4.26. *Assume that the real line is not a union of less than* $\mathbf{c}$ *many meager sets and that* $\mathbf{c}$ *is a regular cardinal. For every* $f:\mathbb{R} \to \mathbb{R}$ *the following conditions are equivalent*:

(i) *there exist* $h, f^\triangleright \in SZ$ *such that* $f = f^\triangleright \circ h$;

(ii) $f \in \mathcal{R}_1$.

Note that Theorem 4.25 cannot be proved in ZFC since, as mentioned above, there exists a model V of ZFC in which no real function onto $\mathbb{R}$ (including the identity function) is a composition of two SZ-functions. Nevertheless, we have the following example.

Example 4.27. There exists an SZ-function $h:\mathbb{R} \to \mathbb{R}$ such that its nth composition h^n is SZ for every $n > 0$.

Proof. Let $\{g_\alpha\colon\ \alpha < \mathbf{c}\} = \mathcal{C}_{G_\delta}$ and $\{x_\alpha\colon\ \alpha < \mathbf{c}\} = \mathbb{R}$. For every $\gamma < \mathbf{c}$ choose

$$h(x_\gamma) \in \mathbb{R} \setminus (\{g_\beta(x_\alpha)\colon\ \alpha, \beta \leqslant \gamma\} \cup \{x_\alpha\colon\ \alpha \leqslant \gamma\}).$$

Observe that $h \in SZ$. We shall verify that $h^n \in SZ$ for $n > 1$. Suppose that $g_\beta(x_\alpha) = h^n(x_\alpha)$. Let $x_\gamma = h^{n-1}(x_\alpha)$. Note that $\gamma > \alpha$ and $g_\beta(x_\alpha) = h(x_\gamma)$, so $\gamma < \beta$. Therefore $\{x\colon\ h^n(x) = g_\beta(x)\} \subset \{x_\alpha\colon\ \alpha < \beta\}$, so $\operatorname{card}(h^n \cap g_\beta) < \mathbf{c}$. $\square$

Now, we consider the following cardinals. (See [4].)

$$\begin{aligned}
c_r(SZ) &= \min\{\operatorname{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathcal{R}_1\ \&\ \neg\exists h \in SZ\ \forall f \in \mathcal{F}\ \exists f^\triangleright \in SZ\ f = f^\triangleright \circ h\} \\
&= \min\{\operatorname{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathcal{R}_1\ \&\ \forall h \in SZ\ \exists f \in \mathcal{F}\ \forall f^\triangleright \in SZ\ f \neq f^\triangleright \circ h\}
\end{aligned}$$

and

$$\begin{aligned}
c_l(SZ) &= \min\{\operatorname{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathcal{R}_1\ \&\ \neg\exists h \in SZ\ \forall f \in \mathcal{F}\ \exists f^\triangleleft \in SZ\ f = h \circ f^\triangleleft\} \\
&= \min\{\operatorname{card}(\mathcal{F})\colon\ \mathcal{F} \subset \mathcal{R}_1\ \&\ \forall h \in SZ\ \exists f \in \mathcal{F}\ \forall f^\triangleleft \in SZ\ f \neq h \circ f^\triangleleft\}.
\end{aligned}$$

(We will assign the value $(2^{\mathbf{c}})^+$ in case when the minimum is run over the empty set.)

Note, that by the remark above in the iterated perfect set model the following corollary holds.

Corollary 4.28. *It is consistent with* ZFC *that* $\mathbf{c} = \omega_2$ *and* $c_r(SZ) = c_l(SZ) = 1$.

Theorem 4.29. *Assume that the real line is not a union of less than* $\mathbf{c}$ *many meager sets and that* $\mathbf{c}$ *is a regular cardinal. Then*

$$\mathbf{c} < c_r(SZ) \leqslant 2^{\mathbf{c}}.$$

Proof. The inequality $\mathbf{c} < c_r(SZ)$ follows from Theorem 4.25. To prove the inequality $c_r(SZ) \leqslant 2^{\mathbf{c}}$ it is enough to show that for every $h \in SZ$ there exists $f \in SZ$ such that $g \circ h = f$ for no $g \in SZ$. Fix $h \in SZ$ and recall that $\mathrm{card}(\mathrm{rng}(h)) = \mathbf{c}$.

Set $f = h$ and suppose that $g \circ h = h$ for some $g \in \mathbb{R}^{\mathbb{R}}$. Then $g(h(x)) = h(x)$, so $\mathrm{rng}(h) \subset [g = \mathrm{id}]$ and consequently, $\mathrm{card}(g \cap \mathrm{id}) = \mathbf{c}$, hence $g \notin SZ$. $\square$

To determine how big can be the cardinal $c_r(SZ)$ we shall use the following poset:

$$\mathbb{P}^{\triangleright} = \{\langle p, E, G\rangle\colon\ p \in \mathbb{P}\ \&\ G \subseteq \mathcal{C}_n\ \&\ E \subseteq \mathbb{R}^{\mathbb{R}}\ \&\ \mathrm{card}(E) + \mathrm{card}(G) < \mathbf{c}\}$$

ordered by

$$\begin{aligned}&\langle p, E, G\rangle \leqslant \langle q, F, H\rangle\\ &\quad \text{iff}\quad p \supseteq q \text{ and } E \supseteq F \text{ and } G \supseteq H\\ &\qquad\quad \text{and } \forall x \in \mathrm{dom}(p)\setminus\mathrm{dom}(q)\ \forall f \in F\ \forall g \in H\ p(x) \notin g^{-1}(f(x)),\end{aligned}$$

where $\mathcal{C}_n$ is formed by nowhere constant $\mathcal{C}_{G_\delta}$ functions.

The following theorem can be proved analogously to [3, Theorem 3.4].

Theorem 4.30. *Let* $\lambda \geqslant \kappa \geqslant \omega_2$ *be cardinals such that* $\mathrm{cf}(\lambda) > \omega_1$ *and* κ *is regular. Then it is relatively consistent with* ZFC + CH *that* $2^{\mathbf{c}} = \lambda$ *and* $\mathrm{Lus}_\kappa(\mathbb{P}^{\triangleright})$ *holds.*

We will prove the following theorem.

Theorem 4.31. *If* $\mathbf{c} = \omega_1$ *and* $\kappa > \mathbf{c}$ *is a regular cardinal then* $\mathrm{Lus}_\kappa(\mathbb{P}^{\triangleright})$ *implies that* $c_r(SZ) = \kappa$.

This and Theorem 4.30 will immediately imply the following corollary.

Corollary 4.32. *Let* $\lambda \geqslant \kappa \geqslant \omega_2$ *be cardinals such that* $\mathrm{cf}(\lambda) > \omega_1$ *and* κ *is regular. Then it is relatively consistent with* ZFC + CH *that* $2^{\mathbf{c}} = \lambda$ *and* $c_r(SZ) = \kappa$.

The proof of Theorem 4.31 will be split into three lemmas.

Lemma 4.33.

(i) *Assume that a union of less than continuum many meager sets is meager again. Then* $\mathrm{Lus}_\kappa(\mathbb{P}^{\triangleright}) \Rightarrow \mathrm{Lus}_\kappa(\mathbb{P})$.

(ii) *For any regular* κ *we have* $\mathrm{Lus}_\kappa(\mathbb{P}^{\triangleright}) \Rightarrow \mathrm{MA}_\kappa(\mathbb{P}^{\triangleright})$.

Proof. The proof is similar to the proof of Lemma 2.7. The only modification is that in the proof of (i) we must replace the condition "card$(r^{-1}(y)) = \mathbf{c}$ for every $y \in \mathbb{R}$" by "for every $y \in \mathbb{R}$ the level set $r^{-1}(y)$ is not meager" and that we choose

$$s(x) \in r^{-1}\big(q(x)\big) \setminus \bigcup \big\{g^{-1}\big(f(x)\big): \ f \in E \ \& \ g \in G\big\}. \qquad \square$$

Lemma 4.34. *Assume that* $\mathbf{c}$ *and* κ *are regular cardinals and* $\kappa > \mathbf{c}$. *Then* $\mathrm{Lus}_\kappa(\mathbb{P})$ *implies that* $c_r(SZ) \leqslant \kappa$.

Proof. Let $\langle G_\alpha: \ \alpha < \kappa\rangle$ be a κ-Lusin sequence of $\mathbb{P}$-filters and define

$$g_\alpha = \bigcup G_\alpha.$$

Then similarly as in the proof of Lemma 2.8 we can assume that each g_α is a total function from $\mathbb{R}$ into $\mathbb{R}$. Let $\{x_\xi: \ \xi < \mathbf{c}\}$ be an enumeration of $\mathbb{R}$. For every $\alpha < \kappa$ put

$$X_\alpha = \big\{x_\xi: \ g_\alpha(x_\xi) \neq g_\alpha(x_\eta) \text{ for every } \eta < \xi\big\}$$

and let $f_\alpha \in \mathcal{R}_1$ be an extension of $g_\alpha \restriction X_\alpha$. We will show that for an arbitrary $h \in \mathbb{R}^\mathbb{R}$ there is an $\alpha < \kappa$ such that $f_\alpha = f_\alpha^\triangleright \circ h$ for no $f_\alpha^\triangleright \in SZ$.

If $h \notin \mathcal{R}_1$ then $f_\alpha^\triangleright \circ h \notin \mathcal{R}_1$ for each $f_\alpha^\triangleright \in \mathbb{R}^\mathbb{R}$ and, since $f_\alpha \in \mathcal{R}_1$, $f_\alpha \neq f_\alpha^\triangleright \circ h$. So, assume that $h \in \mathcal{R}_1$. Then card(rng$(h)) = \mathbf{c}$, because $\mathbf{c}$ is a regular cardinal. For $\xi < \mathbf{c}$ let D_ξ be the set of all $p \in \mathbb{P}$ such that

$$\exists \gamma \geqslant \xi \big[(\forall \alpha \leqslant \gamma)\big(x_\alpha \in \mathrm{dom}(p)\big) \ \& \ (\forall \alpha < \gamma)\big(p(x_\alpha) \neq p(x_\gamma)\big) \ \& \ p(x_\gamma) = h(x_\gamma)\big]$$

and observe that every D_ξ is dense in $\mathbb{P}$.

Indeed, for every $p \in \mathbb{P}$ there is $\gamma \geqslant \xi$ with $x_\gamma \notin \mathrm{dom}(p)$ and $h(x_\gamma) \notin \mathrm{rng}(p)$. Choose $y \neq h(x_\gamma)$ and set

$$q = p \cup \big\{\big(x_\gamma, h(x_\gamma)\big)\big\} \cup \big\{(x_\eta, y): \ \eta < \gamma \ \& \ x_\eta \notin \mathrm{dom}(p)\big\}.$$

Then $q \in D_\xi$ and $q \leqslant p$.

By the regularity of κ, there exists $\alpha < \kappa$ such that G_α intersects every set D_ξ with $\xi < \mathbf{c}$. Note that this implies that card$(X_\alpha) = \mathbf{c}$. Now, suppose that $f_\alpha = f_\alpha^\triangleright \circ h$. We will show that $f_\alpha^\triangleright \notin SZ$.

To see it note first that if $Y_\alpha = \{x \in X_\alpha: \ f_\alpha(x) = h(x)\}$, then card$(Y_\alpha) = \mathbf{c}$, since G_α intersects every set D_ξ. So, $h \in \mathcal{R}_1$ and the regularity of $\mathbf{c}$ imply that

$$\mathrm{card}\big(\mathrm{rng}(h \restriction Y_\alpha)\big) = \mathbf{c}.$$

Finally, observe that $f_\alpha^\triangleright(h(x)) = h(x)$ when $f_\alpha(x) = h(x)$, so rng$(h \restriction Y_\alpha) \subset [f_\alpha^\triangleright = \mathrm{id}]$. Therefore card$(f_\alpha^\triangleright \cap \mathrm{id}) = \mathbf{c}$ and consequently, $f_\alpha^\triangleright \notin SZ$. $\square$

Lemma 4.35. *If* $\kappa > \mathbf{c} = \omega_1$ *then* $\mathrm{MA}_\kappa(\mathbb{P}^\triangleright)$ *implies that* $c_r(SZ) \geqslant \kappa$.

Proof. Let $\mathcal{F} \subseteq \mathcal{R}_1$ be such that card$(\mathcal{F}) < \kappa$. We shall find $h \in SZ$ such that for every $f \in \mathcal{F}$ there exists $f^\triangleright \in SZ$ with $f = f^\triangleright \circ h$.

Observe that for any $x \in \mathbb{R}$ the set

$$D_x = \{\langle p, E, H\rangle \in \mathbb{P}^{\rhd}\colon\ x \in \mathrm{dom}(p)\}$$

is dense in $\mathbb{P}^{\rhd}$.

Indeed, for $\langle q, E, H\rangle \in \mathbb{P}^{\rhd} \setminus D_x$ choose

$$y \in \mathbb{R} \setminus \bigcup \{g^{-1}(f(x))\colon\ g \in H \ \&\ f \in E\}.$$

The choice is possible since, by CH, the set $\bigcup\{g^{-1}(f(x))\colon\ g \in H \ \&\ f \in E\}$ is meager as a countable union of nowhere dense sets. Put $p = q \cup \{\langle x, y\rangle\}$. Then $\langle p, E, H\rangle \leqslant \langle q, E, H\rangle$ and $\langle p, E, H\rangle \in D_x$.

Note also that for any $f \in \mathbb{R}^{\mathbb{R}}$ and $g \in \mathcal{C}_n$ the set

$$E_{f,g} = \{\langle p, E, H\rangle\colon\ f \in E \ \&\ g \in H\}$$

is dense in $\mathbb{P}^{\rhd}$, because $\langle p, E \cup \{f\}, H \cup \{g\}\rangle$ extends $\langle p, E, H\rangle$. Let

$$\mathcal{D} = \{D_x\colon\ x \in \mathbb{R}\} \cup \{E_{\bar{g},\mathrm{id}}\colon\ g \in \mathcal{C}_{G_\delta}\} \cup \{E_{f,k}\colon\ f \in \mathcal{F} \ \&\ k \in \mathcal{C}_n\},$$

where $\bar{g} \in \mathbb{R}^{\mathbb{R}}$ extends $g \in \mathcal{C}_{G_\delta}$ by associating 0 at all undefined places. Then $\mathcal{D}$ is a family of less than κ many dense subsets of $\mathbb{P}^{\rhd}$. Let G be a $\mathcal{D}$-generic filter in $\mathbb{P}^{\rhd}$ and let

$$\hat{h} = \bigcup \{p\colon\ \exists E \subset \mathbb{R}^{\mathbb{R}}\ \exists H \subset \mathcal{C}_n\ \langle p, E, H\rangle \in G\}.$$

Since $G \cap D_x \neq \emptyset$ for every $x \in \mathbb{R}$, $\hat{h}$ is a total function from $\mathbb{R}$ into $\mathbb{R}$.

Observe that $\hat{h} \in SZ$. Indeed, fix $g \in \mathcal{C}_{G_\delta}$ and $\langle p, E, H\rangle \in E_{\bar{g},\mathrm{id}} \cap G$. Then

$$\{x\colon\ \hat{h}(x) = g(x)\} \subset \{x\colon\ \hat{h}(x) = \bar{g}(x)\} \subset \mathrm{dom}(p),$$

so $\mathrm{card}(\hat{h} \cap g) < \mathbf{c}$.

To define h note that by CH all level sets of $\hat{h}$ are countable. In particular, the set $\hat{h}^{-1}(\mathbb{Q})$ is also countable. For every $y \in \mathrm{rng}(h) \cap \mathcal{N}$ let $\hat{h}^{-1}(y) = \{x_{y,n}\colon\ n < \omega\}$. Choose a one-to-one sequence $\langle s_n\colon\ n < \omega\rangle$ of irrationals and define a function $h^*\colon \mathbb{R} \setminus h^{-1}(\mathbb{Q}) \to \mathcal{N}$ by $h^*(x_{y,n}) = \langle s_n, y\rangle$, where we identify $\mathcal{N} = \mathbb{R} \setminus \mathbb{Q}$ with $\mathcal{N} \times \mathcal{N}$ via natural homeomorphism. Note that h^* is one-to-one. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a one-to-one extension of h^*. Then $h \in SZ$. Indeed, suppose that $h \restriction X \in \mathcal{C}$ for some $X \in [\mathbb{R}]^{\mathbf{c}}$. Then $X_0 = X \setminus \hat{h}^{-1}(\mathbb{Q}) \in [\mathbb{R}]^{\mathbf{c}}$ and $h^* \restriction X_0 = h \restriction X_0 \in \mathcal{C}$, so

$$\hat{h} \restriction X_0 = \mathrm{pr}_y \circ h^* \restriction X_0 \in \mathcal{C},$$

contrary to $\hat{h} \in SZ$.

Now, for arbitrary $f \in \mathcal{F}$ define $\tilde{f}\colon \mathrm{rng}(h) \to \mathbb{R}$ by $\tilde{f}(y) = f(x)$ for $x = h^{-1}(y)$. We shall verify that $\tilde{f} \in SZ$.

First, note that $\tilde{f} \in \mathcal{R}_1$, because $f \in \mathcal{R}_1$. So, by Lemma 4.24, it is enough to verify that $\mathrm{card}(\tilde{f} \cap g) < \mathbf{c}$ for every $g \in \mathcal{C}_n$.

So, fix $g \in \mathcal{C}_n$ and suppose that $X = [\tilde{f} = g] \in [\mathrm{rng}(h)]^{\mathbf{c}}$. Then there exists $X_0 \in [\mathrm{rng}(h^*)]^{\mathbf{c}}$ such that $X_0 \subset [\tilde{f} = g]$. Therefore there are $n < \omega$ and $Z \in [\mathrm{rng}(\hat{h}) \cap \mathcal{N}]^{\mathbf{c}}$ such that $\{s_n\} \times Z \subset X_0$, so

$$Z \subset \{\hat{h}(x)\colon\ \langle s_n, \hat{h}(x)\rangle \in g^{-1}(f(x))\}.$$

Let $\varphi:\mathcal{N} \to \{s_n\} \times \mathcal{N}$ be a function defined by $\varphi(y) = \langle s_n, y\rangle$. Then φ is a homeomorphism, so $k = g \circ \varphi \restriction \varphi^{-1}(\mathrm{dom}(g)) \in \mathcal{C}_n$. Let $\langle p, E, H\rangle \in G \cap E_{f,k}$. Then

$$Y = \{x:\ \hat{h}(x) \in k^{-1}(f(x))\} \subset \mathrm{dom}(p),$$

so $\mathrm{card}(Y) < \mathbf{c}$. But $Z \subset \hat{h}(Y)$, contrary to $\mathrm{card}(Z) = \mathbf{c}$.

Finally, let $f^{\triangleright} : \mathbb{R} \to \mathbb{R}$ be an SZ-extension of $\tilde{f}$. Then $f = f^{\triangleright} \circ h$. □

Theorem 4.36. *Assume that the real line is not a union of less than* $\mathbf{c}$ *many meager sets and that* $\mathbf{c}$ *is a regular cardinal. Then*

$$c_l(SZ) > \mathbf{c}.$$

Proof. To see it take $\{f_\beta:\ \beta < \mathbf{c}\} \subset \mathcal{R}_1$. We will construct an $h \in SZ$ and a family $\{f_\beta^{\triangleleft}:\ \beta < \mathbf{c}\}$ of SZ-functions such that $f_\beta = h \circ f_\beta^{\triangleleft}$ for each $\beta < \mathbf{c}$.

Let $\mathcal{C}_n = \{g_\alpha:\ \alpha < \mathbf{c}\}$ be an enumeration of all nowhere constant $g \in \mathcal{C}_{G_\delta}$ and $\{z_\xi:\ \xi < \mathbf{c}\}$ be a one-to-one enumeration of $Z = \bigcup_{\xi<\mathbf{c}} \mathrm{rng}(f_\xi)$. Define inductively a sequence $\langle y_\xi:\ \xi < \mathbf{c}\rangle$ by choosing for every $\xi < \mathbf{c}$

$$y_\xi \in \mathbb{R} \setminus \Bigl(\{y_\zeta:\ \zeta < \xi\} \cup \bigcup \{g_\alpha^{-1}(z_\xi):\ \alpha \leqslant \xi\} \cup \bigcup \{g_\alpha[f_\beta^{-1}(z_\xi)]:\ \alpha, \beta \leqslant \xi\}\Bigr).$$

The choice can be made, since the exceptional set is a union of less than $\mathbf{c}$ many meager sets.

Now, let $Y = \{y_\xi:\ \xi < \mathbf{c}\}$, and define $h : Y \to \mathbb{R}$ by putting

$$h(y_\xi) = z_\xi$$

for every $\xi < \mathbf{c}$. Moreover, for every $\beta < \mathbf{c}$ define $f_\beta^{\triangleleft} : \mathbb{R} \to \mathbb{R}$ by a formula

$$f_\beta^{\triangleleft}(x) = y_\xi \quad \text{iff} \quad x \in f_\beta^{-1}(z_\xi).$$

Note that $f_\beta^{\triangleleft}$ is defined on $\mathbb{R}$ since $\mathrm{rng}(f_\beta) \subset Z$. Also, $f_\beta = h \circ f_\beta^{\triangleleft}$ for every $\beta < \mathbf{c}$, since for every $x \in \mathbb{R}$ there exists $\xi < \mathbf{c}$ such that $f_\beta(x) = z_\xi$, and $f_\beta(x) = z_\xi = h(y_\xi) = h(f_\beta^{\triangleleft}(x))$, as $x \in f_\beta^{-1}(z_\xi)$.

To see that $h \in SZ$ note first that h is one-to-one, so $h \in \mathcal{R}_1$. Thus, by Lemma 4.24, it is enough to show that $\mathrm{card}([h = g_\alpha]) < \mathbf{c}$ for every $\alpha < \mathbf{c}$. But if $g_\alpha(y_\xi) = h(y_\xi) = z_\xi$ then $y_\xi \in g_\alpha^{-1}(z_\xi)$ and, by the choice of y_ξ, $\alpha > \xi$. So, $[h = g_\alpha] \subseteq \{y_\xi:\ \xi < \alpha\}$ has cardinality less than $\mathbf{c}$, and $h \in SZ$.

Next fix $\beta < \mathbf{c}$ and notice that $f_\beta^{\triangleleft} \in \mathcal{R}_1$. To see that $f_\beta^{\triangleleft} \in SZ$ fix $\alpha < \mathbf{c}$. We will show that $\mathrm{card}([f_\beta^{\triangleleft} = g_\alpha]) < \mathbf{c}$. So, let $g_\alpha(x) = f_\beta^{\triangleleft}(x) = y_\xi$. Then $x \in f_\beta^{-1}(z_\xi)$ and

$$y_\xi = g_\alpha(x) \in g_\alpha[f_\beta^{-1}(z_\xi)].$$

So, by the choice of y_ξ, $\alpha > \xi$ or $\beta > \xi$. In particular, $[f_\beta^{\triangleleft} = g_\alpha] \subseteq \{y_\xi:\ \xi < \max(\alpha, \beta)\}$ has cardinality less than $\mathbf{c}$. □

Problem 4.37. Can it be proved in ZFC that $c_l(SZ) \leqslant 2^{\mathbf{c}}$? What about under CH?

5. Final remarks

Proofs of the following statements are left to the reader.

(1) Every function $f \in \mathbb{R}^{\mathbb{R}}$ is the uniform limit of a sequence of SZ-functions.

(2) Assuming $\mathrm{cf}(\mathbf{c}) = \omega_1$, every function $f \in \mathbb{R}^{\mathbb{R}}$ is the transfinite limit of a sequence of SZ-functions (cf. [14]).

(3) Assuming $\mathbf{c}$ is a regular cardinal, the discrete limits of sequences of SZ-functions are in the class SZ (cf. [5]).

(4) If $f, g \in SZ$, then $\max(f, g) \in SZ$ and $\min(f, g) \in SZ$ (hence the family SZ forms a lattice of functions).

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