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W. Schwarz

MSC 1991: 11N37

DELAUNAY TRIANGULATION, *Delone triangulation* – A very important geometric structure in **computational geometry**, named after B.N. Delaunay.

Let $S = \{p_1, \ldots, p_n\}$ be a generic set of n points in $\mathbf{R}^d$. The straight-line dual of the **Voronoĭ diagram** generated by S is a **triangulation** of S, called the *Delaunay triangulation* and usually denoted by $\mathrm{DT}(S)$. The Delaunay triangulation of S is triangulation of the convex hull of S in $\mathbf{R}^d$ and the set of vertices of $DT(S)$ is S.

One of the equivalent definitions for $DT(S)$ is as follows: $DT(S)$ is a triangulation of S satisfying the '*empty sphere property*', i.e. no d-simplex of the triangulation of its circumsphere has a point of S in its interior.

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O.R. Musin

MSC 1991: 68U05

DENSITY TOPOLOGY – The density topology $\mathcal{T}_d$ on $\mathbf{R}$ is the family of all $X \subset \mathbf{R}$ with the property that every $x \in X$ is a density point of X, i.e., such that

$$\lim_{h \to 0^+} \frac{\mathrm{m}_i(X \cap (x-h, x+h))}{2h} = 1,$$

where m_i stands for the Lebesgue inner measure (cf. **Lebesgue measure**).

The density topology was first defined in 1952 by O. Haupt and Ch. Pauc [7], although its study did not start until 1961, when it was rediscovered by C. Goffman and D. Waterman [6]. In both cases it was introduced to show that the class $\mathcal{A}$ of approximately continuous functions (cf. **Approximate continuity**) coincides with the class $\mathcal{C}(\mathcal{T}_d)$ of all real functions that are continuous with respect to the density topology on the domain and the natural topology on the range. Thus, in a way, the density topology has been present in real analysis since 1915, when A. Denjoy defined and studied the class $\mathcal{A}$ [4]. The equation $\mathcal{A} = \mathcal{C}(\mathcal{T}_d)$ shows the importance of the density topology in real analysis, since the class $\mathcal{A}$ is strongly tied to the theory of Lebesgue integration and differentiation. For example, a bounded function is approximately continuous if and only if it is a **derivative**.

The topological properties of the density topology on $\mathbf{R}$ are known quite well. Every $X \in \mathcal{T}_d$ is Lebesgue measurable. The topology is connected, completely regular but not normal (cf. also **Connected space**; **Completely-regular space**; **Normal space**). A set $S \subset \mathbf{R}$ is $\mathcal{T}_d$-nowhere dense if and only if it has Lebesgue measure zero (cf. **Nowhere-dense set**). Also, $\mathbf{R}$ considered with the bitopological structure [8] of the density and natural topologies is normal in the bitopological sense. (This is known as the *Luzin–Menshov theorem* [1].)

The density topology on $\mathbf{R}^n$ for $n \geq 2$ is also defined from the notion of a density point. However, in this case there are different notions of the density point, depending on different neighbourhood bases at the point. For example, all points $x \in X \subset \mathbf{R}^2$ satisfying the condition

$$\lim_{\mathrm{diam}(S) \to 0} \frac{\mathrm{m}_i(X \cap S)}{\mathrm{m}_i(S)} = 1,$$

where the sets S are chosen among the squares centred at x, are called *ordinary density points* of X. This leads to the *ordinary density topology* on $\mathbf{R}^2$ [9]. Similarly, by choosing the sets S from the family of all rectangles centred at x with sides parallel to the axes one obtains the *strong density points* [9] and *strong density topology*. The ordinary density topology is completely regular, unlike the strong density topology [5] (cf. also **Completely-regular space**). However, from the real analysis point of view, the strong density topology is usually more useful [3].

A category analogue of the density topology, introduced by W. Wilczyński [10], is called the *$\mathcal{I}$-density topology*. It is Hausdorff, but not regular (cf. **Hausdorff space**; **Regular space**). The **weak topology** generated by the class of all $\mathcal{I}$-approximately continuous functions is known as the *deep $\mathcal{I}$-density topology*. It is completely regular, but not normal (cf. **Completely-regular space**; **Normal space**).

Most of the topological information concerning the topologies $\mathcal{T}_d$ and its category analogues can be found in [2]. This monograph contains an exhaustive study of sixteen different classes of continuous functions (from $\mathbf{R}$

to $\mathbf{R}$) that can be formed by putting the natural topology or either of these density topologies on the domain and the range.

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K. Ciesielski

MSC 1991: 26A15

DEPTH-FIRST SEARCH, *DFS* – A method of searching a finite directed or undirected **graph** $G = (V, E)$ along the edges of the graph and numbering its vertices. Depth-first search on undirected connected graphs works recursively, starting with an arbitrary vertex with DFS number 1 and, having numbered k vertices already, an unnumbered neighbour w of vertex v with number k will obtain number $k+1$ and defines a tree edge from w to v until all vertices are numbered. At the end the set of these tree edges represents a *spanning tree* of G. When depth-first search reaches a vertex v all of whose neighbours are already numbered and v has a DFS number larger than 1, then depth-first search backtracks one step to the predecessor w of v along the corresponding tree edge and tries to find a next unnumbered neighbour of w. If the graph is not connected (cf. **Graph, connectivity of a**), then the procedure starts again on every connected component of the graph. For directed graphs the procedure is more complicated but similar.

For the special case of *binary trees* this numbering is known as *pre-order*, a vertex ordering which plays an important role in computer science.

Depth-first search can be implemented in linear time $O(|V| + |E|)$ by using a representation of the graph as linked adjacency lists. Basing on depth-first search one can determine the 2-connected components of an undirected graph as well as the strongly connected components of a directed graph in linear time (cf. [1]). There are other, deep applications of depth-first search; for example, planarity of graphs (cf. **Graph, planar**) can be recognized in linear time using depth-first search [5].

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A. Brandstädt

MSC 1991: 05Cxx, 68R10

DESIGN WITH MUTUALLY ORTHOGONAL RESOLUTIONS – A combinatorial design D (cf. also **Block design**) with replication number r is said to be *μ-resolvable* if the blocks of D can be partitioned into classes (called *μ-resolution classes*) $R_1, \ldots, R_t$ ($t = r/\mu$) such that each element of D is contained in precisely μ blocks of each class. If $\mu = 1$, the design is called *resolvable*. Two μ-resolutions R and R' of D are called *orthogonal* if $|R_i \cap R'_j| \leq 1$ for all $R_i \in R$, $R'_j \in R'$. (It should be noted that the blocks of the design are considered as being labeled so that if a subset of the element set occurs as a block more than once, the blocks are treated as being distinct.) A set $Q = \{R^1, \ldots, R^d\}$ of d μ-resolutions of D is called a set of *mutually orthogonal μ-resolutions* if the μ-resolutions of Q are pairwise orthogonal. If $\mu = 1$, Q is a set of mutually orthogonal resolutions. If $d = 2$ and $\mu = 1$, the design is called *doubly resolvable*.

The existence of a **Room square** of side n is equivalent to the existence of a doubly resolvable $(n+1, 2, 1)$-BIBD (cf. also **Block design**). The rows form one resolution and the columns form an orthogonal resolution. A *Room d-cube* of side n is a d-dimensional array, each cell of which either is empty or contains an unordered pair of elements, such that each two-dimensional projection is a Room square of side n. The existence of a Room d-cube of side n is equivalent to the existence of a set of d mutually orthogonal resolutions of an $(n+1, 2, 1)$-BIBD. A pair of orthogonal μ-resolutions of a (v, k, λ)-BIBD can also be used to construct an array, a *Kirkman square* $\mathrm{KS}_k(v; \mu, \lambda)$.

An important open question in design theory is determining a good upper bound for d, where d is the size of the largest set of mutually orthogonal resolutions for