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Sierpiński-Zygmund functions that are Darboux, almost continuous, or have a perfect road

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Abstract. In this paper we show that if the real line $\mathbb{R}$ is not a union of less than continuum many of its meager subsets then there exists an almost continuous Sierpiński–Zygmund function having a perfect road at each point. We also prove that it is consistent with ZFC that every Darboux function $f:\mathbb{R}\to\mathbb{R}$ is continuous on some set of cardinality continuum. In particular, both these results imply that the existence of a Sierpiński–Zygmund function which is either Darboux or almost continuous is independent of ZFC axioms. This gives a complete solution of a problem of Darji [4]. The paper contains also a construction (in ZFC) of an additive Sierpiński–Zygmund function with a perfect road at each point.

1 Introduction

Our terminology is standard. In particular, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the sets of all: positive integers, integers, rationals and reals, respectively. We shall consider only real-valued functions of one real variable. No distinction is made between a function and its graph. The family of all functions from a set X into Y will be denoted by Y^X. The symbol $\mathrm{card}\,(X)$ will stand for the cardinality of a set X. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. If A is a planar set, we denote its x-projection by $\mathrm{dom}\,(A)$. For $f,g\in\mathbb{R}^{\mathbb{R}}$ the notation $[f=g]$ means the set $\{x\in\mathbb{R}: f(x)=g(x)\}$.

If $\mathscr{J}$ is an ideal of subsets of $\mathbb{R}$, then

$$\begin{aligned}\mathrm{cov}\,(\mathscr{J}) &= \min\{\mathrm{card}\,(\mathscr{F}): \mathscr{F}\subset\mathscr{J}\ \&\ \bigcup\mathscr{F}=\mathbb{R}\}\\ \mathrm{non}\,(\mathscr{J}) &= \min\{\mathrm{card}\,(A): A\subset\mathbb{R}\ \&\ A\notin\mathscr{J}\}.\end{aligned}$$

(See [5].) The ideal of all meager subsets of $\mathbb{R}$ is denoted by $\mathscr{K}$. Recall also the following definitions.

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- $f\colon\mathbb{R}\to\mathbb{R}$ is of *Sierpiński-Zygmund type* (shortly, $f\in SZ$, or f is of S-Z type) if its restriction $f|M$ is discontinuous for each set $M\subset\mathbb{R}$ with $\operatorname{card}(M)=\mathfrak{c}$ [17].
- $f\colon\mathbb{R}\to\mathbb{R}$ *has a perfect road* at $x\in\mathbb{R}$ if there exists a perfect set C such that x is a bilateral limit point of C and $f|C$ is continuous at x. We say that f is of *perfect road type* (shortly, $f\in PR$, or f is of PR type) if f has a perfect road at each point [13].
- $f\colon\mathbb{R}\to\mathbb{R}$ is said to be *almost continuous* (in the sense of Stallings) if each open subset of the plane containing f contains also a continuous function $g\colon\mathbb{R}\to\mathbb{R}$ [18].
- $F\colon\mathbb{R}\times[0,1]\to\mathbb{R}$ is a *connectivity function* if the graph of its restriction $F|X$ is connected (in $\mathbb{R}^3$) for every connected $X\subset\mathbb{R}\times[0,1]$.
- $f\colon\mathbb{R}\to\mathbb{R}$ is *extendable* if there is a connectivity function $F\colon\mathbb{R}\times[0,1]\to\mathbb{R}$ such that $F(x,0)=f(x)$ for every $x\in\mathbb{R}$.

Recall also that if $f\colon\mathbb{R}\to\mathbb{R}$ intersects every *blocking set*, i.e., a closed set $K\subset\mathbb{R}^2$ whose domain is a non-degenerate interval, then f is almost continuous [9]. It is also well-known that each almost continuous function $f\colon\mathbb{R}\to\mathbb{R}$ is connected [18] and therefore that it has the Darboux property.

In [4], Darji constructed (in ZFC) an example of an S-Z function of perfect road type and asked whether there exists an almost continuous (or just Darboux) S-Z function. Examples of such functions under additional set theoretical assumptions are known. For example, Ceder [2] showed that under the assumption of the Continuum Hypothesis CH there exists a connectivity (hence Darboux) S-Z function, and Kellum [10] noticed that Ceder's function is in fact almost continuous. In Section 2 we will generalize both constructions (Ceder's and Darji's) by showing that under the assumption that $\operatorname{cov}(\mathscr{K})=\mathfrak{c}$ (which is somewhat weaker than CH or Martin's Axiom MA [16, 5]) there exists an almost continuous S-Z function of PR type. On the other hand, in Section 5 we will show that there is a model of ZFC in which there is no Darboux S-Z function. Thus, some additional set theoretical assumptions are necessary in all of the examples mentioned above.

Sections 3 and 4 contain the constructions related to that from Section 2. In particular, Section 3 deals with the functions $f\colon\mathbb{R}\to\mathbb{R}$ continuous with respect to the qualitative topology on the domain and the natural topology on the range. In Section 4 we give a ZFC example of an additive S-Z function of PR type, generalizing the result of Darji from [4].

2 An almost continuous S-Z function of PR type

In our construction we will use the following easy and well known lemma.

Lemma 1. [8] *Suppose $U\subset\mathbb{R}$ and $f\colon U\to\mathbb{R}$ is continuous. Then there exists a G_δ set M containing U and a continuous function $g\colon M\to\mathbb{R}$ such that $g|U=f$.* □

The next lemma is a modification of [4, Lemma 3].

Lemma 2. *There exists a sequence* $\langle\langle H_\alpha, p_\alpha\rangle\colon \alpha < \mathfrak{c}\rangle$ *such that*

(1) $H_\alpha \cup \{p_\alpha\} \subset \mathbb{R}$ *is a compact perfect set and* p_α *is a bilaterally limit point of* H_α*;*
(2) $H = \bigcup_{\alpha<\mathfrak{c}} H_\alpha$ *is linearly independent over* $\mathbb{Q}$*;*
(3) $H_\alpha \cap H_\beta = \emptyset$ *for every* $\alpha < \beta < \mathfrak{c}$*;*
(4) *for every* $x \in \mathbb{R}$ *there exists continuum many* $\gamma < \mathfrak{c}$ *such that* $x = p_\gamma$*.*

Proof. Let K be a linearly independent perfect set. (See [7] or [11, p. 270].) Pick a proper perfect subset P of K, and let $\{s_{\alpha,n}\colon \alpha < \mathfrak{c}\ \&\ n \in \mathbb{Z}\setminus\{0\}\}$ be a one-to-one enumeration of $K \setminus P$. Moreover, let $\{p_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of $\mathbb{R}$ such that for every $x \in \mathbb{R}$ there exists continuum many $\gamma < \mathfrak{c}$ with $p_\gamma = x$. By induction on $\alpha < \mathfrak{c}$ choose sequences $\langle q_{\alpha,n}\colon \alpha < \mathfrak{c}\ \&\ n \in \mathbb{Z}\setminus\{0\}\rangle$ of non-zero rationals and $\langle C_{\alpha,n}\colon \alpha < \mathfrak{c}\ \&\ n \in \mathbb{Z}\setminus\{0\}\rangle$ of perfect sets such that for every $\alpha < \mathfrak{c}$ and $n \in \mathbb{Z}\setminus\{0\}$

$$C_{\alpha,n} \subset \left(p_\alpha, p_\alpha + \frac{1}{n}\right) \cap (q_{\alpha,n}s_{\alpha,n} + P),$$

where for $b < a$ we will understand (a,b) as the interval (b,a). Next, for each $\alpha < \mathfrak{c}$ define $H_\alpha = \bigcup\{C_{\alpha,n}\colon n \in \mathbb{Z}\setminus\{0\}\}$. It is easy to see that the family $\{H_\alpha \subset \mathbb{R}\colon \alpha < \mathfrak{c}\}$ has the desired properties. □

Theorem 1. *Assuming* $\operatorname{cov}(\mathscr{K}) = \mathfrak{c}$, *there exists an almost continuous S-Z function* $f\colon \mathbb{R} \to \mathbb{R}$ *which has a perfect road at each point.*

Proof. For $A \subset \mathbb{R}$ we denote $L(A) = A \times \mathbb{R}$. Let $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ be a one-to-one enumeration of $\mathbb{R}$ and $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ an enumeration of all continuous functions defined on G_δ subsets of $\mathbb{R}$.

Construct, by induction on $\alpha < \mathfrak{c}$, a sequence $\langle\langle C_\alpha, D_\alpha\rangle\colon \alpha < \mathfrak{c}\rangle$ such that for every $\alpha < \mathfrak{c}$

(1) $D_\alpha \subset \operatorname{dom}(g_\alpha)\setminus\bigcup_{\beta<\alpha}(C_\beta\cup D_\beta)$ is an at most countable set such that $g_\alpha|D_\alpha$ is a dense subset of $g_\alpha \setminus \bigcup_{\beta<\alpha}(g_\beta \cup L(C_\beta \cup D_\beta))$;
(2) C_α is equal to a set H_γ from Lemma 2 such that $x_\alpha = p_\gamma$ and C_α is disjoint from $\{x_\beta\colon \beta \le \alpha\} \cup \bigcup_{\beta\le\alpha} D_\beta \cup \bigcup_{\beta<\alpha} C_\beta$.

The choice as in (2) can be made, since the set $\{x_\beta\colon \beta \le \alpha\} \cup \bigcup_{\beta\le\alpha} D_\beta$ has cardinality less than continuum, and there are continuum many pairwise disjoint sets H_γ with $p_\gamma = x_\alpha$.

Now, define the values $f(x_\alpha)$ of the function f by induction on $\alpha < \mathfrak{c}$ as follows.

(a) $f(x_\alpha) = g_\beta(x_\alpha)$ provided $x_\alpha \in D_\beta$ for some $\beta < \mathfrak{c}$.
(b) $f(x_\alpha) \in \{y \in \mathbb{R}\colon |y - f(x_\beta)| < |x_\alpha - x_\beta|\}\setminus\{g_\gamma(x_\alpha)\colon \gamma \le \alpha\}$ provided $x_\alpha \in C_\beta$ for some $\beta < \mathfrak{c}$. (Note that $f(x_\beta)$ is already defined since, by (2), $\beta < \alpha$.)
(c) $f(x_\alpha) \in \mathbb{R}\setminus\{g_\gamma(x_\alpha)\colon \gamma \le \alpha\}$ otherwise.

We will show that f has the desired properties.

First notice that, by (b), $f|(C_\beta \cup \{x_\beta\})$ is continuous at x_β for every $\beta < \mathfrak{c}$. Therefore, $f \in PR$.

To prove that $f \in SZ$, by Lemma 1 it is enough to show that $\operatorname{card}([f = g_\beta]) < \mathfrak{c}$ for each $\beta < \mathfrak{c}$. But $[f = g_\beta] \subset \bigcup_{\alpha \le \beta} D_\alpha \cup \{x_\alpha : \alpha < \beta\}$, so $\operatorname{card}([f = g_\beta]) < \mathfrak{c}$. Hence, $f \in SZ$.

To verify that f is almost continuous choose a blocking set $F \subset \mathbb{R}^2$. It is enough to show that $f \cap F \neq \emptyset$. To see this, note that there exist a non-degenerate interval $J \subset \operatorname{dom}(F)$ and an upper semicontinuous function $h : J \to \mathbb{R}$ such that $h \subset F$. (See [10, Lemma 1].) Thus there exists an $\alpha_0 < \mathfrak{c}$ such that $g_{\alpha_0} = h|C(h)$, where $C(h)$ denotes the set of all points at which h is continuous. Then $\operatorname{dom} g_{\alpha_0}$ is residual in J and $g_{\alpha_0} \subset F$. In particular, if S is the set of all $\alpha < \mathfrak{c}$ such that $\operatorname{dom}(g_\alpha \cap F)$ is residual in some non-degenerate interval I then $S \neq \emptyset$.

Let $\alpha = \min S$ and I be a non-degenerate interval such that $\operatorname{dom}(g_\alpha \cap F)$ is residual in I. But F is closed and g_α is continuous. So, $g_\alpha|I \subset F$. Moreover, by the minimality of α, for each $\beta < \alpha$ the set $I \cap [g_\beta = g_\alpha] \subset \operatorname{dom}(g_\beta \cap F)$ is nowhere dense in I. Consequently,

$$\begin{aligned} I \cap \operatorname{dom}\Big[g_\alpha \setminus \bigcup_{\beta<\alpha} (g_\beta \cup L(C_\beta \cup D_\beta))\Big] \\ = (I \cap \operatorname{dom}(g_\alpha)) \setminus \bigcup_{\beta<\alpha} \Big(I \cap ([g_\beta = g_\alpha] \cup C_\beta \cup D_\beta)\Big) \neq \emptyset, \end{aligned}$$

since $\operatorname{cov}(\mathscr{K}) = \mathfrak{c}$. Thus, by (1), $I \cap D_\alpha \neq \emptyset$. Let $x \in I \cap D_\alpha$. Then, by (a), $\langle x, f(x)\rangle = \langle x, g_\alpha(x)\rangle \in f \cap F$. □

Remark. Note that an S-Z function of perfect road type is not extendable. (See [4].) So, Theorem 1 gives a new and easy example of an almost continuous function that has a perfect road at each point and is not an extendable function. The first example of such a function was constructed (in ZFC) in [15].

3 The qualitative case

Now we shall consider $\mathbb{R}$ with the fine topology q generated by the ideal $\mathscr{K}$. This topology is called the *qualitative* topology. Recall that a set G is open in the qualitative topology if it can be written in the form $U \setminus P$, where U is open in the Euclidean topology and P is of the first category. (Note that it is an example of a *-topology in the sense of Hashimoto [6] or $\mathscr{J}$-topology in the sense of Vaidyanathaswamy [19] with respect to the ideal $\mathscr{K}$ of meager sets.)

For a set $A \subset \mathbb{R}$ and a function $f : A \to \mathbb{R}$ we say that f is *q-continuous* at a point $x_0 \in A$ if f is continuous at x_0 as a real function defined on the subspace A of the space $\langle \mathbb{R}, q\rangle$.

Lemma 3. *For every set $A \subset \mathbb{R}$ and a function $f : A \to \mathbb{R}$,*

(1) if $A \in \mathscr{K}$, then f is q-continuous;
(2) if f is continuous, then it is q-continuous;

(3) if A is q-dense in itself and f is q-continuous, then f is continuous.

Proof. Statements (1) and (2) are evident. For (3), see [12, Th. 4] or [3, Cor. 1.1.8]. □

Proposition 1. *If* $\operatorname{cov}(\mathscr{K}) = \operatorname{non}(\mathscr{K}) = \mathfrak{c}$ *then there exists an almost continuous function* $f\colon\mathbb{R}\to\mathbb{R}$ *of perfect road type such that* $f|M$ *is not q-continuous for every* $M \notin \mathscr{K}$.

Proof. Assume $M \notin \mathscr{K}$. Then M is not q-nowhere dense. So, there exists an interval I such that $M \cap I$ is q-dense in itself. In particular, $M \cap I \notin \mathscr{K}$, so $\operatorname{card}(M \cap I) = \mathfrak{c}$. Let f be the function constructed in Theorem 1. Then $f|(M \cap I)$ is discontinuous so, by Lemma 3, $f|(M \cap I)$ is not q-continuous. Therefore, $f|M$ also is not q-continuous. □

4 An additive S-Z function of PR type

Theorem 2. *There exists an additive Sierpiński–Zygmund function* $f\colon\mathbb{R}\to\mathbb{R}$ *of perfect road type.*

Proof. Let $\widehat{H} = \{h_\alpha\colon \alpha < \mathfrak{c}\}$ be a Hamel basis which contains the set H constructed in Lemma 2 and let $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ be a well-ordering of all continuous functions defined on G_δ subsets of $\mathbb{R}$. For each $\alpha < \mathfrak{c}$ choose a set $\widehat{H}_\alpha$ and an $\hat{f}(h_\alpha)$ such that

(a) $\widehat{H}_\alpha$ is equal to a set H_γ from Lemma 2 such that $h_\alpha = p_\gamma$ and $\widehat{H}_\alpha$ is disjoint from $\{h_\beta\colon \beta \le \alpha\}$;
(b) $\hat{f}(h_\alpha) \ne q g_\beta(x) - f_\alpha(t)$ for all $\beta \le \alpha$, $q \in \mathbb{Q}$, $x \in \operatorname{lin}(\{h_\beta\colon \beta \le \alpha\})$ and $t \in \operatorname{lin}(\{h_\beta\colon \beta < \alpha\})$, where $\operatorname{lin}(A)$ denotes the linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ generated by A, and f_α is the additive extension of $\hat{f}|\{h_\beta\colon \beta < \alpha\}$.

Moreover, if $h_\alpha \in \widehat{H}_\beta$ for some $\beta < \mathfrak{c}$ then, by (a), $\beta < \alpha$ and we will additionally require that

(c) $|\hat{f}(h_\alpha) - \hat{f}(h_\beta)| \le |h_\alpha - h_\beta|$.

Let $f\colon\mathbb{R}\to\mathbb{R}$ be the additive extension of $\hat{f}\colon\widehat{H}\to\mathbb{R}$.

To prove that f is a function of S-Z type it is enough to verify that $\operatorname{card}(f \cap g_\alpha) < \mathfrak{c}$ for every $\alpha < \mathfrak{c}$. So, fix $\alpha < \mathfrak{c}$ and assume that $f(x) = g_\alpha(x)$. Let γ be the first ordinal such that $x \in \operatorname{lin}(\{h_\beta\colon \beta \le \gamma\})$. Then $x = p h_\gamma + t_0$, where $p \in \mathbb{Q}\setminus\{0\}$ and $t_0 \in \operatorname{lin}(\{h_\beta\colon \beta < \gamma\})$. So $h_\gamma = qx - t$, where $q = p^{-1} \in \mathbb{Q}$ and $t = q t_0 \in \operatorname{lin}(\{h_\beta\colon \beta < \gamma\})$. Moreover, $\hat{f}(h_\gamma) = f(h_\gamma) = qf(x) - f(t) = q g_\alpha(x) - f(t)$, so $\gamma < \alpha$. Thus, by (b), $[f = g_\alpha] \subset \operatorname{lin}(\{h_\beta\colon \beta < \alpha\})$ and $\operatorname{card}(f \cap g_\alpha) < \mathfrak{c}$.

Now we shall verify that f has a perfect road at each $x \in \mathbb{R}$. For $x = h_\alpha \in \widehat{H}$ it is obvious by (c), since $f|(\widehat{H}_\alpha \cup \{h_\alpha\})$ is continuous at h_α. So, assume that $x = \sum_{i=1}^n q_i h_{\alpha_i}$, where all q_i are rationals. Then x is a bilaterally limit point of a perfect set $\widehat{H}_x = \sum_{i=1}^n q_i \widehat{H}_{\alpha_i} \cup \{x\}$ and $f|\widehat{H}_x$ is continuous at x. □

5 A model with no Darboux S-Z function

In this section we will show that in the iterated perfect set (Sacks) model there is no Darboux Sierpiński-Zygmund function. We will describe here only those properties of this model that are necessary to follow the argument. More details can be found in [14] and [1].

Let V be a model of ZFC+CH and let $V[G_{\omega_2}]$ be a model of ZFC+$\mathfrak{c} = \omega_2$ obtained as a generic extension of V over the forcing $\mathbb{P}$, which is a countable support iteration of the perfect set (Sacks) forcing. Then V and $V[G_{\omega_2}]$ have the same cardinals. Moreover, in $V[G_{\omega_2}]$ there exists an increasing sequence $\langle V[G_\alpha]: \alpha \leq \omega_2\rangle$ (of proper classes in $V[G_{\omega_2}]$, given by a formula) with the following properties. ($V[G_\alpha]$ is a generic extension of V obtained by extending V with the part G_α of G_{ω_2} which belongs to the α-iteration of Sacks forcing.)

(A) CH holds in $V[G_\alpha]$ for every $\alpha < \omega_2$.
(B) For every $\alpha < \omega_2$ of uncountable cofinality and every $s \in 2^\omega \cap V[G_\alpha]$ there exists $\beta < \alpha$ such that $s \in V[G_\beta]$.
(C) For every $\alpha < \omega_2$ and $a, b \in \mathbb{R}$, $a < b$, there exists $s \in (a, b) \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$ (a Sacks number over $V[G_\alpha]$) such that for every $x \in \mathbb{R} \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$ there exists a continuous function $g \in \mathbb{R}^\mathbb{R} \cap V[G_{\omega_2}]$ coded in $V[G_\alpha]$ (i.e., such that $g|\mathbb{Q} \in V[G_\alpha]$) with the property that $g(x) = s$.

Property (A) follows immediately from the fact that CH holds in V and we iterate forcings of cardinality $\mathfrak{c}$. Properties (B) and (C) can be found in [1, Thm. 3.3(a)] and in [14, Sec. 4, p. 581], respectively.

Note also, that property (B) can be modified as follows.

(B′) For every $\alpha < \omega_2$ of uncountable cofinality and every $p \in (\mathbb{R}^\mathbb{Q} \cup \mathbb{R}) \cap V[G_\alpha]$ there exists $\beta < \alpha$ such that $p \in V[G_\beta]$.

The part concerning $p \in \mathbb{R}$ follows from the fact that a real number can be identified with its binary representation, i.e., a function $s: \omega \to 2$. This also implies the part for $p \in \mathbb{R}^\mathbb{Q}$, since any such p can be identified with $\hat{p}: \mathbb{Q} \times \omega \to 2$, $\hat{p}(q, n) = p(q)(n)$, and further, with a function from 2^ω by identifying $\mathbb{Q} \times \omega$ with ω via bijection from V.

Now, let $h \in \mathbb{R}^\mathbb{R} \cap V[G_{\omega_2}]$ be an SZ function and let $a = \inf h[\mathbb{R}]$, $b = \sup h[\mathbb{R}]$. Then $-\infty \leq a < b \leq \infty$. We will show that $(a, b) \not\subset h[\mathbb{R}]$.

To prove this let $C(\mathbb{R})$ stand for the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and define, for $\beta < \omega_2$,

$$S_\beta = h\left[\mathbb{R} \cap V[G_\beta]\right] \\ \cup \bigcup \left\{ \{x, y\}: (\exists g \in C(\mathbb{R}) \cap V[G_{\omega_2}])(g|\mathbb{Q} \in V[G_\beta] \ \& \ \langle x, y\rangle \in g \cap h) \right\}.$$

Note that, by (A), the set $(\mathbb{R} \cup \mathbb{R}^\mathbb{Q}) \cap V[G_\beta]$ has cardinality $\leq \omega_1$ and that $\operatorname{card}(h \cap g) \leq \omega_1$ for every $g \in C(\mathbb{R}) \cap V[G_{\omega_2}]$. So, $\operatorname{card}(S_\beta) \leq \omega_1$ for every $\beta < \omega_2$. Define $\Gamma: \omega_2 \to \omega_2$ by putting $\Gamma(\beta) = \sup\{\gamma(x): x \in S_\beta\}$, where

$\gamma(x) = \min\{\beta : x \in V[G_\beta]\}$, and let $\alpha < \omega_2$ be of uncountable cofinality such that $\Gamma(\beta) < \alpha$ for every $\beta < \alpha$. Then, by (B′),

(i) $h(x) \in V[G_\alpha]$ for every $x \in \mathbb{R} \cap V[G_\alpha]$;
(ii) $h \cap g \subset V[G_\alpha]$ for every $g \in C(\mathbb{R})$ coded in $V[G_\alpha]$.

Now, let $s \in (a, b) \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$ be a number from (C). It is enough to prove that $s \notin h[\mathbb{R}]$.

But $s \notin h\left[\mathbb{R} \cap V[G_\alpha]\right]$ by (i). So, let $x \in \mathbb{R} \cap (V[G_{\omega_2}] \setminus V[G_\alpha])$. It is enough to show that $h(x) \neq s$. But, by (C), there exists a continuous function $g: \mathbb{R} \to \mathbb{R}$ coded in $V[G_\alpha]$ such that $g(x) = s$. So, $h(x) \neq s$, since otherwise $\langle x, s\rangle \in h \cap g$ and, by (ii), $s \in V[G_\alpha]$. This contradiction finishes the proof. □

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