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UNIFORMLY APPROACHABLE MAPS

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1. Preliminaries

Throughout the paper we will use the standard definitions and notation ([ABR], [E]).

Let X and Y be metric spaces. The goal is to study some intermediate classes of functions between the class $C_{uc}(X,Y)$ of all uniformly continuous mappings (briefly, UC) from X into Y and the class $C(X,Y)$ of all continuous functions $f\colon X\to Y$. These classes, defined below, have been intensively studied in [BD] and [BDP] mainly in the case when $Y=\boldsymbol{R}$. (Compare also [B,DP].) In this paper we will study them for general Y. In particular, we will consider the case when $X=Y$ is the complex plane $\boldsymbol{C}$.

Let X and Y be metric spaces and let $f\colon X\to Y$.

(1) For $K,M\subseteq X$ we say that $g\colon X\to Y$ is a (K,M)-*approximation* of f if g is a UC map such that $g[M]\subseteq f[M]$ and $g(x)=f(x)$ for each $x\in K$.
(2) The function f is *uniformly approachable* (briefly, UA) if f has a (K,M)-approximation for every compact $K\subseteq X$ and every $M\subseteq X$.
(3) The function f is *weakly uniformly approachable* (briefly, WUA) if f has a $(\{x\},M)$-approximation for every $x\in X$ and every $M\subseteq X$.

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The class of all uniformly approachable (weakly uniformly approachable, respectively) functions $f\colon X\to Y$ will be denoted by $C_{ua}(X,Y)$ ($C_{wua}(X,Y)$, respectively.) Notice that

$$C_{uc}(X,Y)\subseteq C_{ua}(X,Y)\subseteq C_{wua}(X,Y)\subseteq C(X,Y). \tag{1}$$

(See [DP]. Compare also [BD, Section 12].)

Indeed, every UC function f is UA, since $g=f$ is a (K,M)-approximation of f for every $M,K\subseteq X$. Every UA function is WUA, since $\{x\}$ is compact for every $x\in X$. To see that every WUA map f is continuous, it is enough to show that $f[\overline{M}]\subseteq\overline{f[M]}$ for every $M\subseteq X$, where $\overline{M}$ stands for the closure of M. So, take $x\in\overline{M}$ and an $(\{x\},M)$-approximation g of f. Then, $f(x)=g(x)\in g[\overline{M}]\subseteq\overline{g[M]}\subseteq\overline{f[M]}$.

We will discuss possible equations in (1) for different spaces X and Y in Section 3.

For a topological space X we say that a function $g\in C(X,\boldsymbol{R})$ is a *truncation* of $f\in C(X,\boldsymbol{R})$ if g is constant on every connected component of $\{x\in X\colon f(x)\neq g(x)\}$. In what follows we will use also the following fact which can be proved by straightforward transfinite induction diagonal argument.

Proposition 1.1. [BD, Thm 8.1] *Let X be a separable topological space. Then there is a set $M\subseteq X$ such that for every $f,g\in C(X,\boldsymbol{R})$, if $g[M]\subseteq f[M]$ and $f^{-1}(y)$ is at most countable for every $y\in\boldsymbol{R}$ then g is a truncation of f.*

The set M from Proposition 1.1 will be called a *magic set* (for X). The following fact is an easy but useful corollary of Proposition 1.1. It is a version of [BD, Cor 8.3].

Corollary 1.2. *Let X be a separable metric space. If $f\in C(X,\boldsymbol{R})$ is a one-to-one function without non-constant UC truncation then f is not WUA.*

2. Basic facts

In this section we collect some basic properties of the classes of UA- and WUA-maps.

Theorem 2.1. *Composition of* UA (WUA, respectively) *maps is a* UA (WUA, respectively) *map.*

Proof: Let $f\colon X \to Y$ and $f'\colon Y \to Z$ be UA maps. To show that $f' \circ f\colon X \to Z$ is UA take a compact subset K of X and a set $M \subseteq X$. Then $f[K]$ is a compact subset of Y. Choose a (K, M)-approximation $g\colon X \to Y$ of f and a $(f[K], f[M])$-approximation $g'\colon Y \to Z$ of f'. It is enough to show that $g' \circ g$ is a (K, M)-approximation of $f' \circ f$. Obviously $g' \circ g$ coincides with $f' \circ f$ on K. Also $g[M] \subseteq f[M]$ and $g'[f[M]] \subseteq f'[f[M]]$ implies that $(g' \circ g)[M] \subseteq (f' \circ f)[M]$. The proof is finished when we notice that $g' \circ g$ is UC as a composition of two UC functions.

The argument for WUA functions is identical, if we replace K with $\{x\}$. □

Pproposition 2.2. *Let F be a subset of a metric space X. If $f\colon X \to \boldsymbol{R}$ is UA (WUA, respectively) then so is its restriction $f|_F\colon F \to \boldsymbol{R}$.*

Proof: Let $f \in C_{ua}$ and $M, K \subseteq F$, K being compact. So, there exists (K, M)-approximation $g\colon X \to \boldsymbol{R}$ of f. It is easy to see that $g|_F$ is an (K, M)-approximation of $f|_F$. □

In [BD] it was shown that every continuous function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is UA. In fact, one can show the following stronger property.

Theorem 2.3 *If X is a uniform space then every continuous function $f\colon \boldsymbol{R} \to X$ is UA.*

Proof: Let K be a compact subset of $\boldsymbol{R}$ and $M \subseteq \boldsymbol{R}$. Choose an interval $[a, b]$ containing K such that either $a \in M$ or $(-\infty, a] \cap M = \emptyset$ and that either $b \in M$ or $[b, \infty) \cap M = \emptyset$. Let $r\colon \boldsymbol{R} \to [a, b]$ be the retraction, i.e., $r(x) = x$ for $x \in [a, b]$, $r(x) = a$ for $x < a$ and $r(x) = b$ for $x > b$. Clearly r is UC. Define g by $g = f|_{[a,b]} \circ r$. Then g is UC as a composition of two UC functions, and g coincides with f on $K \subseteq [a, b]$. It is also easy to see that $g[M] \subseteq f[M]$. □

In what follows we will often use the following criteria which is similar to that of [BD, Thm. 6.3].

Proposition 2.4. *Let (X, ρ) be a separable metric space and let $f\colon X \to \boldsymbol{R}$. If there exists an uncountable $Y \subseteq \boldsymbol{R}$ such that $f^{-1}(y)$ is non-empty and connected for every $y \in Y$ and that $\rho(f^{-1}(x), f^{-1}(y)) = 0$ for every $x, y \in Y$ then f is not WUA.*

Proof: Replacing X with $f^{-1}(Y)$, if necessary, we can assume that $X = f^{-1}(Y)$. Now, let D be a countable dense subset of X and put $M = f^{-1}(f[D])$. Then M is dense in X, since $D \subseteq M$. Moreover, M is a proper subset of X, since $Y = f[X]$ is uncountable unlike $f[M] = f[f^{-1}(f[D])] = f[D]$. Pick some $x \in X \setminus M$. Then, $x \notin M = f^{-1}(f[D]) = f^{-1}(f[f^{-1}(f[D])]) = f^{-1}(f[M])$, i.e., $f(x) \notin f[M]$. Our aim is to show that f has no $(\{x\}, M)$-approximation. By way of contradiction assume that there exists an UC function $g\colon X \to \boldsymbol{R}$ such that $g[M] \subseteq f[M]$. Then $g[M] \subseteq f[M] = f[D]$ is countable, hence zero-dimensional. (See [E, Chapter 7, §2, Corollary].) Since $C_d = f^{-1}(f(d)) \subseteq M$ is connected for every $d \in D$ and $g[C_d] \subseteq g[M] \subseteq f[M] = f[D]$, we conclude that g is constant on each C_d. Since our hypothesis yields that $\rho(C_d, C_{d'}) = 0$ for every $d, d' \in D$, the uniform continuity of g implies that all these constants coincide. Therefore $g|_M$ is constantly equal to some $b \in f[M]$. But $f(x) \notin f[M]$. So, $g(x) = b \neq f(x)$. Therefore g is not an $(\{x\}, M)$-approximation of f. □

As an immediate corollary we obtain the following. (See also [BD, Lemma 5.4].)

Corollary 2.5. *The function $h\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ given by formula $h(x, y) = x^2 - y^2$ is not WUA.*

Proof: By Proposition 2.4 function $h|_{[0,\infty)\times[0,\infty)}$ is not WUA. So, h is not WUA by Proposition 2.2. □

One of the most interesting properties of the class of UA functions is the fact that every perfect map[1] from $\boldsymbol{R}^n$ into

[1]A map $f\colon X \to \boldsymbol{R}^n$ is perfect if $f^{-1}(r)$ is compact for every $r \in \boldsymbol{R}^n$.

$\boldsymbol{R}$ is UA [BD, Thm 5.2]. However, this theorem does not generalize for all functions from $\boldsymbol{R}^2$ into $\boldsymbol{R}^2$, as shown by the next example. (Compare with Example 4.1.)

Example 2.6. *The functions $f, g\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ given by the formulas $f(x, y) = x^2$ and $g(x, y) = y^2$ are UA, while the perfect map $H = (f, g)\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is not WUA.*

Proof: The function f is UA by Theorem 2.1 since it is a composite of a UC function $f_1\colon \boldsymbol{R}^2 \to \boldsymbol{R}$, $f_1(x, y) = x$, and a UA function $f_2\colon \boldsymbol{R} \to \boldsymbol{R}$, $f_2(x) = x^2$. (See Theorem 2.3.) Similarly we show that g is UA. To see that H is is not WUA notice that the function $h\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ from Corollary 2.5 is a composite of H and a UC function $h\colon \boldsymbol{R}^2 \to \boldsymbol{R}$, $h(x, y) = x-y$. Thus, by Theorem 2.1, H cannot be WUA. □

Example 2.6 shows also that a map $H = (f, g)\colon X \to Y \times Z$ does not have to be WUA even if $f\colon X \to Y$ and $g\colon X \to Z$ are both UA.

Now, if X is a metric space then $C(X, \boldsymbol{R})$ is a linear topological space and $C_{uc}(X, \boldsymbol{R})$ is its linear subspace. However, the classes $C_{ua}(X, \boldsymbol{R})$ and $C_{wua}(X, \boldsymbol{R})$ are not closed under addition, as shown by Example 2.6 and Corollary 2.5. In fact, the next example shows that the sum of WUA function and UC function does not have to be WUA. To formulate it easier, we need the following notation and lemma.

We will use the symbol C to denote the unit circle on the complex plane:

$$C = \{z \in \boldsymbol{C}\colon |z| = 1\}$$

with the standard distance and we will write $\boldsymbol{R}_c$ to denote the real line with the metric of $C \setminus \{-1\}$, i.e., given by formula

$$\rho(x, y) = \left| e^{2\boldsymbol{i} \arctan x} - e^{2\boldsymbol{i} \arctan y} \right|.$$

Lemma 2.7. (i) *Let Y be a metric space. A function $f \in C(\boldsymbol{R}_c, Y)$ is UC if and only if both limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ exist and are equal.*

(ii) *A function* $f \in C(\boldsymbol{R}_c, \boldsymbol{R})$ *is WUA provided that* $int(f[(-\infty, x)] \cap f[(x, \infty)]) \neq \emptyset$ *for every* $x \in \boldsymbol{R}_c$.

Proof: Since the spaces $\boldsymbol{R}_c$ and $C \setminus \{-1\}$ are isometric, we can assume that $f\colon C \setminus \{-1\} \to Y$.

(i) Now, if both limits $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ exist and are equal to $b \in \boldsymbol{R}$, then we can extend f to a continuous function F on C by putting $F(-1) = b$. Clearly F is UC, as a continuous function on the compact set C. So, f is UC.

Conversely, if $f\colon C \setminus \{-1\} \to Y$ is UC, then it can be extended uniquely to UC function $F\colon C \to Y$ and then clearly both limits exist and are equal to $F(-1)$.

(ii) Choose $x \in \boldsymbol{R}_c$ and $M \subseteq \boldsymbol{R}_c$. Clearly we can assume that $M \not\subseteq \{x\}$.

If M is not dense in $\boldsymbol{R}_c$ then we can find $a < b$ in $\boldsymbol{R}_c$ and $m \in M$ such that $(a, b) \cap M = \emptyset$ and either $a < b < x < m$ or $m < x < a < b$. Assume that $a < b < x < m$. Define $g|_{[b,m]} = f|_{[b,m]}$, $g(x) = f(m)$ for $x \in (-\infty, a) \cup (m, \infty)$ and extend it to (a, b) in a continuous way. Then, g is UC by (i) and it is easy to see that g is an $(\{x\}, M)$-approximation of f. The case $m < x < a < b$ is handled in a similar way.

So, assume that M is a dense subset of $\boldsymbol{R}_c$. Let $m \in M \cap f^{-1}(\mathrm{int}(f[(-\infty, x)] \cap f[(x, \infty)]))$ and put $y = f(m) \in f[(-\infty, x)] \cap f[(x, \infty)]$. Choose $a \in (-\infty, x) \cap f^{-1}(y)$ and $b \in (x, \infty) \cap f^{-1}(y)$. Then, $a < x < b$ and $f(a) = f(b) = y = f(m)$. Define $g|_{[a,b]} = f|_{[a,b]}$ and $g(x) = f(m)$ for $x \in (-\infty, a) \cup (b, \infty)$. It is easy to see that g is a UC function which is an $(\{x\}, M)$-approximation of f. $\square$

Notice that the assumption in condition (ii) of Lemma 2.7 is not necessary for its conclusion as shown by the UC function $f\colon \boldsymbol{R}_c \to \boldsymbol{R}$, $f(x) = x/(1 + x^2)$. Notice also, that Lemma 2.7(i) implies that $C_{uc}(\boldsymbol{R}_c, \boldsymbol{R})$ is a subring of $C(\boldsymbol{R}_c, \boldsymbol{R})$, i.e., that $C_{uc}(\boldsymbol{R}_c, \boldsymbol{R})$ is closed under pointwise product of functions.

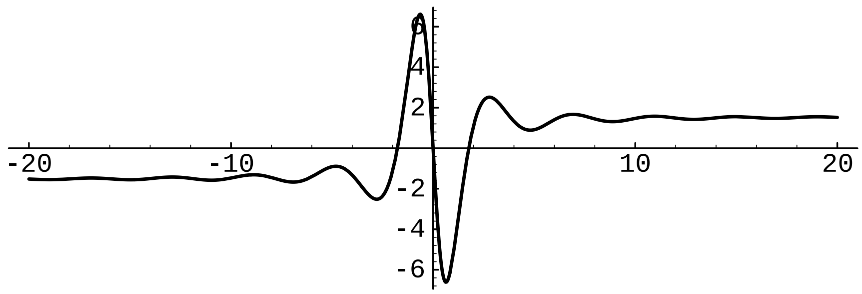

Figure 1: Graph of function f from Example 2.8.

Example 2.8. *Let $f, h\colon \boldsymbol{R}_c \to \boldsymbol{R}$ be given by formulas*

$$h(x) = \frac{12 \sin \frac{\pi}{2} x}{x^2 + 1}$$

and $f(x) = -h(x) + \arctan x$. (See Figure 1.) Then h is UC, f is WUA while $f + h$ is not WUA.

Proof: The function h is UC by Lemma 2.7(i). The increasing function $f + h = \arctan$ is not WUA by Corollary 1.2 and the other implication of Lemma 2.7(i). To finish the proof, it is enough to show that f satisfies the assumption of Lemma 2.7(ii). So, let $x \in \boldsymbol{R}_c$ and choose an integer number k such that $x \in [4k, 4(k+1)]$. Notice that

$$f(4k-1) = \arctan(4k-1) + \frac{12}{(4k-1)^2 + 1} > \arctan(4k-1) > -\frac{\pi}{2}$$

and

$$f(4k+5) = \arctan(4k+5) - \frac{12}{(4k+5)^2 + 1} < \arctan(4k+5) < \frac{\pi}{2}.$$

Thus, it is enough to show the inequality $f(4k-1) > f(4k+5)$, since then $(-\pi/2, f(4k-1)) \subseteq f[(-\infty, x)]$, $(f(4k+5), \pi/2) \subseteq f[(x, \infty)]$, and $(-\pi/2, f(4k-1)) \cap (f(4k+5), \pi/2) \neq \emptyset$. So, we can reduce our task to the proof of

(2)
$$\arctan(4k+5) - \arctan(4k-1) < \frac{12}{(4k+5)^2+1} + \frac{12}{(4k-1)^2+1}.$$

Now, if $4k+5 \leq 0$, then arctan is concave on the interval $[4k-1, 4k+5]$ and using differential approximation at the point $4k+5$ we obtain

$$\arctan(4k+5) - \arctan(4k-1) \leq 6 \arctan'(4k+5) = \frac{6}{(4k+5)^2+1}.$$

So, (2) holds. The case when $4k-1 \geq 0$ is handled by the fact that f is an odd function. Now, if $4k-1 < 0 < 4k+5$ then $k = 0$ or $k = -1$, i.e., $\{4k-1, 4k+5\} \cap \{-1, 1\} \neq \emptyset$, So, using the fact that the derivative of arctan is at most 1 we obtain

$$\begin{aligned} \arctan(4k+5) - \arctan(4k-1) &< 6 = \frac{12}{(\pm 1)^2+1} \\ &< \frac{12}{(4k+5)^2+1} + \frac{12}{(4k-1)^2+1}. \end{aligned}$$

□

Problem 2.9. *If X is a metric space, $f \in C_{ua}(X, \boldsymbol{R})$ and $g \in C_{uc}(X, \boldsymbol{R})$ is $f+g$ UA?*

Problem 2.10 *If $f \in C_{wua}(\boldsymbol{R}^2, \boldsymbol{R})$ and $g \in C_{uc}(\boldsymbol{R}^2, \boldsymbol{R})$ is $f+g$ WUA?*

3. Equations between C_{uc}, C_{ua}, C_{wua} and C.

In this section we will discuss all the possible spectra of equations and sharp inclusions in the formula

(3) $$C_{uc}(X, Y) \subseteq C_{ua}(X, Y) \subseteq C_{wua}(X, Y) \subseteq C(X, Y).$$

An important contribution for this discussion comes from the following fact.

Proposition 3.1. [BD, Thm. 12.1] *For any metric space X the equation $C_{uc}(X,\boldsymbol{R}) = C_{ua}(X,\boldsymbol{R})$ implies that $C_{uc}(X,\boldsymbol{R}) = C(X,\boldsymbol{R})$.*

In particular, the space Y in the following cases (ii)-(iv) cannot be equal to $\boldsymbol{R}$. On the other hand, we have $Y = \boldsymbol{R}$ in all remaining cases.

(i): $C_{uc}(X,Y) = C_{ua}(X,Y) = C_{wua}(X,Y) = C(X,Y)$. Any compact space X will do the job. For example $X = [0,1]$, $Y = \boldsymbol{R}$.

(ii): $C_{uc}(X,Y) = C_{ua}(X,Y) = C_{wua}(X,Y) \neq C(X,Y)$. Let M be a magic set for $\boldsymbol{R}$ from Proposition 1.1. Then $X = Y = M$ considered with the usual metric have the above property. (See [B, Example 13].) Spaces X and Y can be also chosen to be complete metric spaces [B, Cor. 18]. (Notice that in [B] term UA is used for what we call here WUA.)

(iii): $C_{uc}(X,Y) = C_{ua}(X,Y) \neq C_{wua}(X,Y) = C(X,Y)$.

We do not know such an example. So, the following problem remains open.

Problem 3.2. *Do there exist metric spaces X and Y with the property that $C_{uc}(X,Y) = C_{ua}(X,Y) \neq C_{wua}(X,Y) = C(X,Y)$?*

(iv): $C_{uc}(X,Y) = C_{ua}(X,Y) \neq C_{wua}(X,Y) \neq C(X,Y)$. Now, let $f\colon \boldsymbol{R}_c \to \boldsymbol{R}$ be as in Example 2.8 and let $M \subseteq \boldsymbol{R}_c$ be a magic set. We claim, that the above chain of equations and inequalities holds for $X = M$ and $Y = f[M]$.

First notice that both M and the complement of M are dense in $\boldsymbol{R}_c$. Otherwise, if (a,b) is a non-empty interval contained in either M or $\boldsymbol{R}_c \setminus M$ and if $g \in C(\boldsymbol{R}_c,\boldsymbol{R})$ is such that $g|_{\boldsymbol{R}_c\setminus(a,b)} = \arctan|_{\boldsymbol{R}_c\setminus(a,b)}$, $g[(a,b)] =$

arctan$[(a,b)]$, and $g \neq$ arctan, then g is an $(\{a\}, M)$-approximation of arctan which is not a truncation of arctan.

Next we will show that

$$\text{(4)} \qquad \text{any } h \in C_{wua}(M, f[M]) \text{ can be uniquely extended to } C(\boldsymbol{R}_c, \boldsymbol{R}).$$

Notice that this immediately implies $C_{wua}(M, f[M]) \neq C(M, f[M])$ since there exists a function $h \in C(M, f[M])$ with $h[M]$ having precisely two elements and, by (4), such h is not WUA.

To see (4) let $h\colon M \to \boldsymbol{R}$ be a function without continuous extension into $\boldsymbol{R}_c$. Then there exist $x \in \boldsymbol{R}_c \setminus M$, $\varepsilon > 0$ and two sequences $\{x_n\}_{n\in\boldsymbol{N}}$ and $\{y_n\}_{n\in\boldsymbol{N}}$ in M converging to x such that $|h(x_n) - h(y_n)| \geq \varepsilon$ for all $n \in \boldsymbol{N}$. Let $a < b$ be such that $x_n, y_n \in [a,b]$ for all $n \in \boldsymbol{N}$. Since $f|_{[a,b]}$ is UC, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for every $x, y \in [a,b]$ with $|x-y| < \delta$. It is easy to see that this implies essentially the same property for any truncation g of f, i.e., that

$$\text{(5)} \qquad |g(x) - g(y)| < \varepsilon$$

for every truncation g of f and any $x, y \in [a,b]$ with $|x - y| < \delta$.

Now, choose $k \in \boldsymbol{N}$ such that $|x_k - y_k| < \delta$ and put $M_0 = \{y_k\}$. We claim that if $g\colon M \to h[M]$ is an $(\{x_k\}, M_0)$-approximation of h then g is not UC. Indeed, if g is an $(\{x_k\}, M_0)$-approximation of h then $g(x_k) = h(x_k)$ and $\{g(y_k)\} = g[M] \subseteq h[M] = \{h(y_k)\}$. So

$$\text{(6)} \qquad g(x_k) - g(y_k)| = |h(x_k) - h(y_k)| \geq \varepsilon.$$

On the other hand, if g were UC, then it could be extended to a UC function $\hat{g} : \boldsymbol{R}_c \to \boldsymbol{R}$ which must be a truncation of f, since $\hat{g}[M] = g[M] \subseteq f[M]$. But this and (6) contradict (5). This finishes the proof of (4).

Next we will show that for any $h \in C_{wua}(M, f[M])$

$$(7) \qquad h \in C_{ua}(M, f[M]) \text{ if and only if } \hat{h} \text{ is a truncation of } f \text{ with } \lim_{x\to\infty} \hat{h}(x) = \lim_{x\to-\infty} \hat{h}(x),$$

where $\hat{h}\colon \boldsymbol{R}_c \to \boldsymbol{R}$ is the unique extension of h, existing by (4). Notice that by Lemma 2.7(i) and (4) this will immediately imply $C_{uc}(M, f[M]) = C_{ua}(M, f[M])$.
The implication "$\Leftarrow$" follows immediately from Lemma 2.7(ii). To prove the other implication take $h \in C_{ua}(M, f[M])$. Then $\hat{h}$ is a truncation of f by the definition of M. It is also easy to see that the limits under question exist. Put $s_0 = \lim_{x\to-\infty} \hat{h}(x)$ and $s_1 = \lim_{x\to\infty} \hat{h}(x)$, and by way of contradiction assume that $s_0 \neq s_1$. Let $\varepsilon = |s_0 - s_1|/6$ and choose $a_0, a_1 \in M$ such that $h(a_i) = \hat{h}(a_i) \in (s_i - \varepsilon, s_i + \varepsilon)$ for $i = 0, 1$, $f[(-\infty, a_0]] \subseteq (-\pi/2 - \varepsilon, -\pi/2 + \varepsilon)$ and $f[(a_1, \infty)] \subseteq (\pi/2 - \varepsilon, \pi/2 + \varepsilon)$. We claim that there is no $(\{a_0, a_1\}, M)$-approximation $g\colon M \to f[M]$ of h which is UC. To see it assume by way of contradiction that there exists such g. Then g is UC so that g has a UC extension $\hat{g}\colon \boldsymbol{R}_c \to \boldsymbol{R}$ which must be a truncation of f by the choice of M. Now

$$(8) \qquad |\hat{g}(a_0) - \hat{g}(a_1)| = |h(a_0) - h(a_1)| > 4\varepsilon.$$

Moreover, if $b_0 = \lim_{x\to-\infty} \hat{g}(x)$ then either $b_0 = h(a_0)$, if $h(a_0) \notin (-\pi/2 - \varepsilon, -\pi/2 + \varepsilon)$, or otherwise $b_0 \in (-\pi/2 - \varepsilon, -\pi/2 + \varepsilon)$. In any case, $|b_0 - h(a_0)| \le 2\varepsilon$. Similarly, we argue that $|b_1 - h(a_1)| \le 2\varepsilon$, where $b_1 = \lim_{x\to\infty} \hat{g}(x)$. Combining this with (8) we obtain that $b_0 \neq b_1$. So, by Lemma 2.7(i), $\hat{g}$ is not UC. This contradiction finishes the proof of (8).

To finish the proof it is enough to show that

(9) $f|_M \in C_{wua}(M, f[M]) \setminus C_{ua}(M, f[M])$.

But in Example 2.9 we proved that f is WUA, so Proposition 2.2 implies that $f|_M \in C_{wua}(M, f[M])$. The fact that $f|_M \notin C_{ua}(M, f[M])$ follows immediately from (8).

(v): $C_{uc}(X,Y) \neq C_{ua}(X,Y) = C_{wua}(X,Y) = C(X,Y)$ holds for $X = Y = \boldsymbol{R}$ by Theorem 2.3.

(vi): $C_{uc}(X,Y) \neq C_{ua}(X,Y) = C_{wua}(X,Y) \neq C(X,Y)$ holds for X being the Hedgehog with continuum many spikes [E, Example 4.1.3]E and $Y = \boldsymbol{R}$. For the proof see [BEP, Theorem 5.9]. (Actually it suffices to take just b spikes, where b is the smallest cardinality of an unbounded family in ω^ω.)

Problem 3.3. *Does there exist a separable metric space X with the property that $C_{uc}(X, \boldsymbol{R}) \neq C_{ua}(X, \boldsymbol{R}) = C_{wua}(X, \boldsymbol{R}) \neq C(X, \boldsymbol{R})$?*

(vii): $C_{uc}(X,Y) \neq C_{ua}(X,Y) \neq C_{wua}(X,Y) = C(X,Y)$ holds for $X = \boldsymbol{R} \setminus \{0\}$ and $Y = \boldsymbol{R}$. (See [Corollary 7.2 and Lemma 3.1].)

Problem 3.4. *Does there exist a connected metric space X with the property that $C_{uc}(X, \boldsymbol{R}) \neq C_{ua}(X, \boldsymbol{R}) \neq C_{wua}(X, \boldsymbol{R}) = C(X, \boldsymbol{R})$?*

(viii): $C_{uc}(X,Y) \neq C_{ua}(X,Y) \neq C_{wua}(X,Y) \neq C(X,Y)$ holds for $Y = \boldsymbol{R}$ and $X = \boldsymbol{R}_c$. The last inequality $C_{wua}(\boldsymbol{R}_c, \boldsymbol{R}) \neq C(\boldsymbol{R}_c, \boldsymbol{R})$ is witnessed by the identity function, which is not WUA by Corollary 1.2 and Lemma 2.7(i). This and Proposition 3.1 imply $C_{uc}(\boldsymbol{R}_c, \boldsymbol{R}) \neq C_{ua}(\boldsymbol{R}_c, \boldsymbol{R})$. The relation $C_{ua}(\boldsymbol{R}_c, \boldsymbol{R}) \neq C_{wua}(\boldsymbol{R}_c, \boldsymbol{R})$ is justified by the function f from Example 2.8. It is shown there that $f \in C_{wua}(\boldsymbol{R}_c, \boldsymbol{R})$. The fact that $f \notin C_{ua}(\boldsymbol{R}_c, \boldsymbol{R})$ follows from (9) and Proposition 2.2.

4. Complex polynomials and harmonic functions

In this section we will show that most complex analytic functions $f\colon \boldsymbol{C} \to \boldsymbol{C}$ are not WUA. We start with the following easy example.

Example 4.1. *The function $f\colon \boldsymbol{C} \to \boldsymbol{C}$ given by $f(z) = z^2$ is not WUA.*

Proof: If f were WUA than the real part of f, Re $f = \mathrm{pr}_x \circ f$, would be WUA, since $\mathrm{pr}_x\colon \boldsymbol{R}^2 \to \boldsymbol{R}$, $\mathrm{pr}_x(x, y) = x$, is UC. But $\mathrm{Re} f(x + iy) = x^2 - y^2$ is not WUA. (See Corollary 2.5.) □

Notice that $f(z) = z^2$ is perfect. So, even the simplest analytic perfect mappings do not have to be WUA.

Clearly all linear functions from $\boldsymbol{R}^2$ into $\boldsymbol{R}^2$ are UC. Hence all mappings $\boldsymbol{C} \to \boldsymbol{C}$, $z \mapsto az+b$ $(a, b \in \boldsymbol{C})$ are UC. In fact, it seems that these are the only complex analytic functions that are WUA. We do not have a proof of this general statement. However, the examples below should explain the reasons that stands behind this conjecture.

In all the examples that follow we will prove that the complex-valued function $f(z)$ in question is not WUA by showing that its imaginary part $h(x, y) = \mathrm{Im} f(x + iy)$ is not WUA. This is enough by Theorem 2.1. To prove that h is not WUA we will apply Proposition 2.2 to an appropriate restriction h_A of the function h to a subspace A of $\boldsymbol{R}^2$. The verification that h_A is not WUA will be done by applying Proposition 2.4

Example 4.2. (1) *The for $n > 1$ the n-th power function $\boldsymbol{C} \to \boldsymbol{C}$, $z \mapsto z^n$, is not WUA.*
(2) *The exponential function $\boldsymbol{C} \to \boldsymbol{C}$, $z \mapsto e^z$, is not WUA.*

Proof: To see (1) let $h(x, y) = (x^2 + y^2)^{n/2} \sin[n \arctan(y/x)]$. Take as A those points of the upper half-plain which have the argument in the interval $(0, \pi/n)$. Then for every $c > 0$ the level curve $L_c = h^{-1}(c)$ is connected and has two asymptotes: the rays $\mathrm{Arg}(z) = 0$ and $\mathrm{Arg}(z) = \pi/n$. Thus the distance between L_c and $L_{c'}$ is zero for $c, c' > 0$. The application of Proposition 2.4 finishes the proof.

In case (2) we have $h(x, y) = e^x \sin y$. Take $A = \{z \in \boldsymbol{C} : \mathrm{Re}z > 0 \ \& \ 0 < \mathrm{Im}z < \pi/2\}$. Then for $c > 1$ the level curve $L_c = h^{-1}(c) = \{(\ln[c/\sin y], y) \colon 0 < y < \pi/2\}$ is connected as a graph of continuous function $x(y) = \ln[c/\sin y]$. Moreover, all curves $\{L_c\}_{c>1}$ have as common asymptote the line $\mathrm{Im}z = 0$. Thus the distance between L_c and $L_{c'}$ is zero for $c, c' > 1$. Now Proposition 2.4 can be applied. □

In both cases above we used the fact that the function $h(x, y) = \mathrm{Im}f(x + iy)$ has many level curves whose connected components are of distance zero. (This was done by noticing that they have the same asymptote.) The same argument works also for other elementary entire functions like $\sin z$, $\cos z$ and complex polynomial functions of degree greater than 1. In fact, by Liouville's theorem for positive harmonic functions [ABR, Theorem 3.1], for every entire analytic function f the level curves of $h = \mathrm{Im}f$ are unbounded and it seems conceivable that for any non-linear f the function h always has many level curves whose connected components are of distance zero. This leads us to the following conjecture.

Conjecture 4.3. *If $f \colon \boldsymbol{C} \to \boldsymbol{C}$ is entire analytic function then f is WUA if and only if f is a linear.*

We will finish this discussion with one more example.

Example 4.4. *The inverse function $f \colon \boldsymbol{C} \setminus \{0\} \to \boldsymbol{C}$, $f(z) = 1/z$, is not WUA.*

Proof: Let A be the first quadrant $\{(x, y) \colon x > 0 \ \& \ y > 0\}$. Since $h(x, y) = \mathrm{Im}f(x + iy) = -y/\sqrt{x^2 + y^2}$ it is easy to see that for $-1 < c < 0$ the level curve $L_c = h^{-1}(c)$ is the ray in A of the line $y = -c(1 - c^2)^{-1/2}x$. The distance between L_c and $L_{c'}$ is clearly zero for $c, c' > 0$. So, Proposition 2.4 can be applied. □

It is not difficult to extend the argument of Example 4.4 to all rational functions with essential poles. This suggests that Conjecture 4.3 can be true also for all meromorphic functions.

We also do not know the answer for the following natural question. (Compare with Example 2.8.)

Question 4.5. If $f \in C_{ua}(\boldsymbol{C}, \boldsymbol{C})$ ($f \in C_{wua}(\boldsymbol{C}, \boldsymbol{C})$, respectively) and $g : \boldsymbol{C} \to \boldsymbol{C}$ is linear, is it always true that $f + g$ is UA (WUA, respectively)?

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