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ORDINARY AND STRONG DENSITY CONTINUITY OF COMPLEX ANALYTIC FUNCTIONS

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Abstract

In the paper we prove that the complex analytic functions are (ordinarily) density continuous. This stays in contrast with the fact that even such a simple function as $G: \mathbb{R}^2 \to \mathbb{R}^2$, $G(x, y) = (x, y^3)$, is not density continuous [1]. We will also characterize those analytic functions which are strongly density continuous at the given point $a \in \mathbb{C}$. From this we conclude that a complex analytic function f is strongly density continuous if and only if $f(z) = a + bz$, where $a, b \in \mathbb{C}$ and b is either real or imaginary.

1. Preliminaries

The notation used throughout this paper is standard. In particular, the complex plane $\mathbb{C}$ will be identified with $\mathbb{R}^2$. All sets considered in the paper will be Lebesgue measurable. The two-dimensional Lebesgue measure of a set $A \subset \mathbb{C}$ will be denoted by $\lambda(A)$. Recall that 0 is a strong dispersion point of $A \subset \mathbb{C}$ if

$$(1) \qquad \lim_{a \to 0+,\, b \to 0+} \frac{\lambda\big(A \cap [(-a, a) \times (-b, b)]\big)}{\lambda\big((-a, a) \times (-b, b)\big)} = 0$$

and it is an (ordinary) dispersion point of A if in the above limit we replace rectangles with squares, i.e., we take $a = b$. It is also well known that the squares can be replaced with the balls $B(a) = \{z \in \mathbb{C}: |z| < a\}$, i.e., that 0 is a dispersion point of $A \subset \mathbb{C}$ if

$$(2) \qquad \lim_{r \to 0+} \frac{\lambda\big(A \cap B(r)\big)}{\lambda\big(B(r)\big)} = 0.$$

A point $z \in \mathbb{C}$ is a dispersion (strong dispersion) point of $A \subset \mathbb{C}$ if it is a dispersion (strong dispersion) point of $A - z$, and $z \in \mathbb{C}$ is a density (strong density) point of

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A if it is a dispersion (strong dispersion) point of the complement of A. (Compare Saks [5], pages 106, 128.) The strong density topology $\mathcal{T}_S$ on $\mathbb{C}$ is defined as the family of all measurable subsets A of $\mathbb{C}$ such that every $z \in A$ is a strong density point of A [2]. Similarly we define the density topology $\mathcal{T}_\mathcal{N}$ on $\mathbb{C}$ using the notion of ordinary density point on $\mathbb{C}$. (Compare [2] and [3].) Notice that the topologies $\mathcal{T}_\mathcal{N}$ and $\mathcal{T}_S$ are invariant under translations and under multiplications by positive real numbers.

A function $f: \mathbb{C} \to \mathbb{C}$ is density continuous (strongly density continuous) at $z \in \mathbb{C}$ if it is continuous with the topology $\mathcal{T}_\mathcal{N}$ ($\mathcal{T}_S$) used in the domain and the range. In particular, it is easy to see that $f: \mathbb{C} \to \mathbb{C}$ is density continuous (strongly density continuous) at $z \in \mathbb{C}$ if and only if for every $A \subset \mathbb{C} \setminus \{z\}$ if z is not a dispersion (strong dispersion) point of A then $f(z)$ is not a dispersion (strong dispersion) point of $f(A)$.

In what follows we will use also the following easy fact. It will be left without proof.

LEMMA 1.1. *Let $A \subset B(1)$ and $R_k \subset B(1)$, $k \in \mathbb{N}$, be such that*

$$\frac{\lambda(A \cap R_k)}{\lambda(R_k)} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

If the sets $K^j \subset B(1)$ for $j < n$ are disjoint and such that $\lambda\left(\bigcup_{j<n} K^j\right) = \lambda\left(B(1)\right)$ then there is a $j < n$ and an increasing sequence $\{k_p\}$ such that

$$\frac{\lambda(A \cap K^j \cap R_{k_p})}{\lambda(K^j \cap R_{k_p})} > \delta \quad \text{for all} \quad p \in \mathbb{N}.$$

We will also use the following version of the change of variables formula.

LEMMA 1.2. *Let $F: \mathbb{C} \to \mathbb{C}$, $U \subset \mathbb{C}$ be an open region and let $h: U \to \mathbb{C}$ be analytic with analytic inverse. Then*

$$\int_{h(U)} F \, d\lambda = \int_U (F \circ h) \cdot |h'(z)|^2 \, d\lambda.$$

PROOF. This immediately follows from the standard change of variables formula ([4], Thm. 7.26) if we notice that the Jacobian of transformation h is equal to $|\det h'(x, y)| = |h'(z)|^2$ which, in turn, follows immediately from Cauchy-Riemann equations. (Compare [4], p. 250, Exercise 6.) ■

In what follows we will also use the following notation for $\alpha \geq 0$ and $\varepsilon, r_0 > 0$:

$$K(\alpha, \varepsilon, r_0) = \{z = re^{i\varphi} \in \mathbb{C}: 0 \leq r < r_0 \ \& \ \alpha - \varepsilon < \varphi < \alpha + \varepsilon\}.$$

2. Functions bz^n

We will start with the following equivalent form of the property that 0 is a strong dispersion point of $A \subset \mathbb{C}$.

LEMMA 2.1. *The point 0 is a strong dispersion point of $A \subset \mathbb{C}$ if and only if for every $\alpha = m\pi/2$, $m \in \mathbb{N}$, and every parameter $\varepsilon \in (0, \pi/4)$ (that might depend of r)*

$$\lim_{r \to 0+} \frac{\lambda(A \cap K(\alpha, \varepsilon, r))}{\lambda(K(\alpha, \varepsilon, r))} = 0. \tag{3}$$

PROOF. Fix $A \subset \mathbb{C}$.

By way of contradiction assume first that (3) is false for some $\alpha = m\frac{\pi}{2}$, $m \in \mathbb{N}$. Then, there exists $\delta > 0$ and sequences $\varepsilon_k \in (0, \pi/4)$ and $r_k > 0$ such that r_k converges to 0 and

$$\frac{\lambda(A \cap K(\alpha, \varepsilon_k, r_k))}{\lambda(K(\alpha, \varepsilon_k, r_k))} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

For convenience we will assume $\alpha = 0$, the other cases being similar.

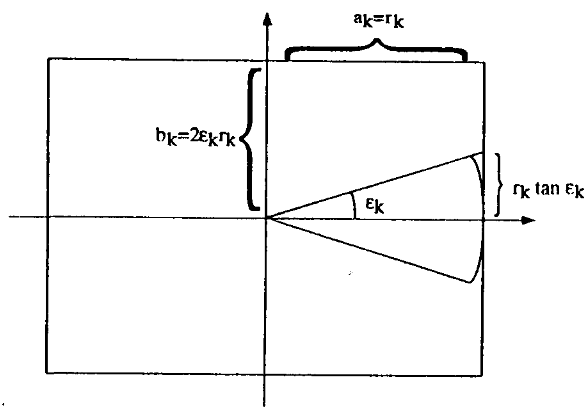

Figure 1.

Let $a_k = r_k$ and $b_k = 2\varepsilon_k r_k$. Then, $b_k \geq r_k \tan \varepsilon_k$, since $2 > \frac{\tan(\pi/4)}{\pi/4} > \frac{\tan \varepsilon_k}{\varepsilon_k}$. In particular, $K(\alpha, \varepsilon_k, r_k) \subset (-a_k, a_k) \times (-b_k, b_k)$. (See Figure 1.) Moreover, $\lambda((-a_k, a_k) \times (-b_k, b_k)) = 8r_k^2\varepsilon_k = 8\,\lambda(K(\alpha, \varepsilon_k, r_k))$. Hence

$$\frac{\lambda(A \cap [(-a_k, a_k) \times (-b_k, b_k)])}{\lambda((-a_k, a_k) \times (-b_k, b_k))} \geq \frac{\lambda(A \cap K(\alpha, \varepsilon_k, r_k))}{8\,\lambda(K(\alpha, \varepsilon_k, r_k))} > \frac{\delta}{8}$$

for all $k \in \mathbb{N}$, contradicting (1).

Conversely, assume that (1) is false, i.e., that there exists $\delta > 0$ and sequences a_k, b_k converging to 0 such that

$$\frac{\lambda(A \cap [(-a_k, a_k) \times (-b_k, b_k)])}{\lambda((-a_k, a_k) \times (-b_k, b_k))} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, by Lemma 1.1 used with sets $K^j = \{re^{i\varphi} \in B(1) \colon j\frac{\pi}{2} < \varphi < (j+1)\frac{\pi}{2}\}$ for $j \in \{0, 1, 2, 3\}$, we can assume that the similar property holds when in the above limit the sequence of rectangles $\{(-a_k, a_k) \times (-b_k, b_k)\}$ is replaced by one of the following four sequences: $\{(0, a_k) \times (0, b_k)\}$, $\{(0, a_k) \times (-b_k, 0)\}$, $\{(-a_k, 0) \times (0, b_k)\}$, or $\{(-a_k, 0) \times (-b_k, 0)\}$. For convenience we will assume that this is the case for the first of these sequences, i.e., that

$$\frac{\lambda\big(A \cap [(0, a_k) \times (0, b_k)]\big)}{a_k b_k} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

Furthermore, choosing subsequence, if necessary, we can assume that either $a_k \leq b_k$ for all k or $b_k \leq a_k$ for all k. We will assume that

$$b_k \leq a_k \quad \text{for all} \quad k \in \mathbb{N}.$$

Let $r_k = \sqrt{a_k^2 + b_k^2}$ and $\varepsilon_k = \arctan \frac{b_k}{a_k \delta/2}$. Then $r_k \leq 2\, a_k$ and

$$(a_k \delta/2, a_k) \times (0, b_k) \subset K(0, \varepsilon_k, r_k) \subset (0, 2\, a_k) \times (-\frac{4}{\delta} b_k, \frac{4}{\delta} b_k).$$

(See Figure 2.)

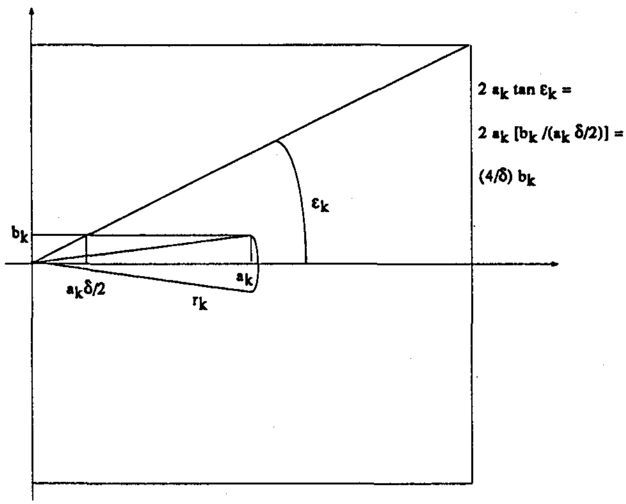

Figure 2.

In particular, $\lambda\big(K(0, \varepsilon_k, r_k)\big) \leq 2\, a_k\, 2\, \frac{4}{\delta} b_k = \frac{16}{\delta} a_k b_k$ and

$$\begin{aligned} \frac{\lambda\big(A \cap K(0, \varepsilon_k, r_k)\big)}{\lambda\big(K(0, \varepsilon_k, r_k)\big)} &\geq \frac{\lambda\big(A \cap [(a_k \delta/2, a_k) \times (0, b_k)]\big)}{\frac{16}{\delta} a_k b_k} \\ &\geq \frac{\delta}{16} \frac{\lambda\big(A \cap [(0, a_k) \times (0, b_k)]\big) - \lambda\big((0, a_k \delta/2] \times (0, b_k)\big)}{a_k b_k} \\ &\frac{\delta}{16}\left(\delta - \frac{\delta}{2}\right) = \frac{\delta^2}{32}, \end{aligned}$$

for all $k \in \mathbb{N}$, which contradicts (3). ■

In what follows we will need also the following inequality.

LEMMA 2.2. *Let $f(z) = e^{i\varphi} z^n$, $A \subset \mathbb{C}$, $\alpha, \beta \geq 0$ and $\varepsilon, r > 0$. If*

$$A \cup K(\alpha, \varepsilon, r) \subset K(\beta, \frac{\pi}{n}, 1) \tag{4}$$

then for every $d \in (0,1)$

$$\lambda\big(f(A) \cap K(n\alpha + \varphi, n\varepsilon, r^n)\big) \geq n^2 (d\,r)^{2n-2} \big[\lambda\big(A \cap K(\alpha, \varepsilon, r)\big) - d^2 \varepsilon r^2\big].$$

PROOF. By (4) we can restrict f to $K(\beta, \frac{\pi}{n}, 1)$. Then, f is one-to-one and has an analytic inverse f^{-1}. Hence, using Lemma 1.2, we obtain

$$\begin{aligned}
\lambda\big(f(A) \cap K(n\alpha + \varphi, n\varepsilon, r^n)\big) &= \lambda\big(f(A) \cap f(K(\alpha, \varepsilon, r))\big) \\
&= \int_{f(K(\alpha,\varepsilon,r))} \chi_{f(A)}\, d\lambda \\
&= \int_{f(K(\alpha,\varepsilon,r))} \chi_A \circ f^{-1}\, d\lambda \\
&= \int_{K(\alpha,\varepsilon,r)} \chi_A \circ f^{-1} \circ f(z) \cdot |f'(z)|^2\, d\lambda \\
&= \int_{K(\alpha,\varepsilon,r)} \chi_A(z) \cdot n^2 |z|^{2n-2}\, d\lambda \\
&\geq \int_{K(\alpha,\varepsilon,r)\setminus K(\alpha,\varepsilon,dr)} \chi_A(z) \cdot n^2 |z|^{2n-2}\, d\lambda \\
&\geq \int_{K(\alpha,\varepsilon,r)\setminus K(\alpha,\varepsilon,dr)} \chi_A(z) \cdot n^2 (d\,r)^{2n-2}\, d\lambda \\
&= n^2 (d\,r)^{2n-2} \left[\int_{K(\alpha,\varepsilon,r)} \chi_A\, d\lambda - \int_{K(\alpha,\varepsilon,dr)} \chi_A\, d\lambda\right] \\
&\geq n^2 (d\,r)^{2n-2} \big[\lambda\big(A \cap K(\alpha, \varepsilon, r)\big) - \lambda\big(K(\alpha, \varepsilon, dr)\big)\big] \\
&= n^2 (d\,r)^{2n-2} \big[\lambda\big(A \cap K(\alpha, \varepsilon, r)\big) - d^2 \varepsilon r^2\big].
\end{aligned}$$

■

Now, we are ready for the proof of the main lemma.

LEMMA 2.3. *Let $b \in \mathbb{C}$ and $n \in \mathbb{N}$, $n \geq 1$. Then the function $f(z) = bz^n$ is density continuous at 0. Moreover, it is strongly density continuous at 0 if and only if b is either real or imaginary number.*

PROOF. If $b = 0$ then the lemma is certainly true. So, assume that $b \neq 0$. The topologies $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{S}}$ are invariant under multiplications by positive real numbers. So, without loss of generality we can assume that $|b| = 1$, i.e., that $b = e^{i\varphi}$ for some $\varphi \geq 0$. To prove that f is density continuous at 0 let $A \subset B(1)$ be such that 0 is not a dispersion point of A. We will show that $0 = f(0)$ is not a dispersion point of $f(A)$. To this order first notice that, by (2), there exists a sequence $r_k \in (0,1)$ converging to 0 and $\delta \in (0,1)$ such that

$$\frac{\lambda\big(A \cap B(r_k)\big)}{\lambda\big(B(r_k)\big)} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, by Lemma 1.1 used with sets $K^j = K(\frac{j\pi}{2n}, \frac{\pi}{4n}, 1)$, $j < 4n$, we can assume that for some $\alpha = \frac{j\pi}{2n}$ we have $A \subset K(\alpha, \frac{\pi}{4n}, 1)$ and

$$\frac{\lambda\big(A \cap K(\alpha, \frac{\pi}{4n}, r_k)\big)}{\lambda\big(K(\alpha, \frac{\pi}{4n}, r_k)\big)} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, by Lemma 2.2 used with $d = \delta/2$, and the above we have

$$\begin{aligned}
\frac{\lambda\big(f(A) \cap B(r_k^n)\big)}{\lambda\big(B(r_k^n)\big)} &\geq \frac{\lambda\big(f(A) \cap K(n\alpha + \varphi, \frac{\pi}{4}, r_k^n)\big)}{\pi r_k^{2n}} \\
&\geq \frac{n^2(\frac{\delta}{2}\, r_k)^{2n-2}[\lambda\big(A \cap K(\alpha, \frac{\pi}{4n}, r_k)\big) - \frac{\delta^2}{4}\frac{\pi}{4n} r_k^2]}{4n\frac{\pi}{4n} r_k^{2n}} \\
&= \frac{n}{4}\left(\frac{\delta}{2}\right)^{2n-2}\left[\frac{\lambda\big(A \cap K(\alpha, \frac{\pi}{4n}, r_k)\big)}{\lambda\big(K(\alpha, \frac{\pi}{4n}, r_k)\big)} - \frac{\frac{\delta^2}{4}\frac{\pi}{4n} r_k^2}{\frac{\pi}{4n} r_k^2}\right] \\
&> \frac{n}{4}\left(\frac{\delta}{2}\right)^{2n-2}\left[\delta - \frac{\delta^2}{4}\right] > 0
\end{aligned}$$

for every $k \in \mathbb{N}$. Therefore, by (2), 0 is not a dispersion point of $f(A)$. We proved that f is density continuous at 0.

To prove the second part, assume first that $b = e^{i\varphi}$ is real or imaginary. Thus, $\varphi = m\frac{\pi}{2}$ for some $m \in \mathbb{N}$. Let $A \subset B(1)$ be such that 0 is not a strong dispersion point of A. We will show that $0 = f(0)$ is not a strong dispersion point of $f(A)$.

By Lemma 2.1 we can find $\delta \in (0,1)$, $\beta = p\frac{\pi}{2}$, where $p \in \mathbb{N}$, and sequences $\varepsilon_k \in (0, \pi/4)$ and $r_k \in (0,1)$ such that r_k converges to 0 and

$$\frac{\lambda\big(A \cap K(\beta, \varepsilon_k, r_k)\big)}{\lambda\big(K(\beta, \varepsilon_k, r_k)\big)} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

By Lemma 1.1 used with sets $K^j = K(\frac{j\pi}{2n}, \frac{\pi}{4n}, 1)$, $j < 4n$, we can assume that for some $\alpha = \frac{j\pi}{2n}$

$$\frac{\lambda\big(A \cap K(\beta, \varepsilon_k, r_k) \cap K(\alpha, \frac{\pi}{4n}, 1)\big)}{\lambda\big(K(\beta, \varepsilon_k, r_k) \cap K(\alpha, \frac{\pi}{4n}, 1)\big)} > \delta \quad \text{for all} \quad k \in \mathbb{N}.$$

We can also assume that either $\varepsilon_k > \frac{\pi}{4n}$ for all k or $\varepsilon_k \leq \frac{\pi}{4n}$ for all k. However, the first case implies that 0 is not an ordinary dispersion point of A, since in this case we would have

$$\frac{\lambda\big(A \cap B(r_k)\big)}{\lambda\big(B(r_k)\big)} \geq \frac{\lambda\big(A \cap K(\beta, \varepsilon_k, r_k)\big)}{4n\,\lambda\big(K(\beta, \varepsilon_k, r_k)\big)} > \frac{\delta}{4n}$$

for all $k \in \mathbb{N}$. Thus, by the first part of the Lemma, 0 is not a (strong) dispersion point of $f(A)$.

So, assume that $\varepsilon_k \leq \frac{\pi}{4n}$ for all k. But then, $\alpha = \beta$ since otherwise $K(\beta, \varepsilon_k, r_k) \cap K(\alpha, \frac{\pi}{4n}, 1) = \emptyset$. Hence, $K(\beta, \varepsilon_k, r_k) \cap K(\alpha, \frac{\pi}{4n}, 1) = K(\alpha, \varepsilon_k, r_k)$ and

$$\frac{\lambda\big(A \cap K(\alpha, \varepsilon_k, r_k)\big)}{\lambda\big(K(\alpha, \varepsilon_k, r_k)\big)} > \delta \text{ for all } k \in \mathbb{N}.$$

We can also assume that $A \subset K(\alpha, \frac{\pi}{4n}, 1)$. Then, by Lemma 2.2 used with $d = \delta/2$, and the above we have

$$\begin{aligned}\frac{\lambda\big(f(A) \cap K(n\alpha + \varphi, n\varepsilon_k, r_k^n)\big)}{\lambda\big(K(n\alpha + \varphi, n\varepsilon_k, r_k^n)\big)} &\geq \frac{n^2(\frac{\delta}{2} r_k)^{2n-2}\big[\lambda\big(A \cap K(\alpha, \varepsilon_k, r_k)\big) - \frac{\delta^2}{4}\varepsilon_k r_k^2\big]}{n\varepsilon_k r_k^{2n}} \\ &= n\left(\frac{\delta}{2}\right)^{2n-2}\left[\frac{\lambda\big(A \cap K(\alpha, \varepsilon_k, r_k)\big)}{\lambda\big(K(\alpha, \varepsilon_k, r_k)\big)} - \frac{\frac{\delta^2}{4}\varepsilon_k r_k^2}{\varepsilon_k r_k^2}\right] \\ &> n\left(\frac{\delta}{2}\right)^{2n-2}\left[\delta - \frac{\delta^2}{4}\right] > 0\end{aligned}$$

for every $k \in \mathbb{N}$. But notice that $n\alpha + \varphi = (j + m)\frac{\pi}{2}$. Therefore, by Lemma 2.1, 0 is not a strong dispersion point of $f(A)$. We proved that f is strong density continuous at 0.

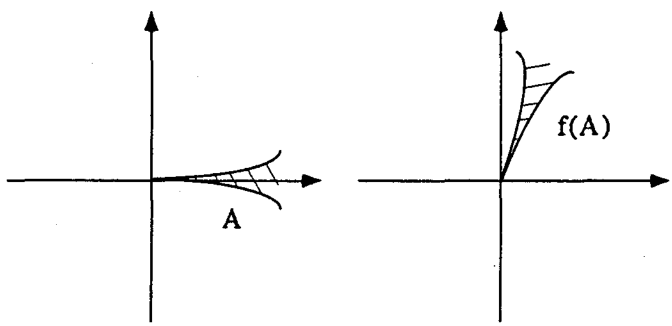

Figure 3.

To finish the proof let us assume that $b = e^{i\varphi}$ is neither real nor imaginary. Thus, $\varphi = p\frac{\pi}{2}$ for some $p > 0$, $p \notin \mathbb{N}$. Let $A = \{(x, y) \colon x > 0 \ \& \ -x^2 < y < x^2\}$. It is easy to see that 0 is not a strong dispersion point of A. On the other hand, $f(A)$ does not contain any axis. (See Figure 3.) Using this fact it is not difficult to argue that 0 is a strong dispersion point of $f(A)$. ■

3. General case

We will need the following fact.

LEMMA 3.1. *Let f be analytic on a neighborhood of 0 and assume that $f(z) = \sum_{k=n}^{\infty} a_k z^k$ where $n > 0$ and $a_n \neq 0$. If $g(z) = a_n z^n$ then there exists $r > 0$ such that for every $A \subset B(r)$*

$$\frac{1}{4n}\lambda\big(g(A)\big) \leq \lambda\big(f(A)\big) \leq 4\lambda\big(g(A)\big). \tag{5}$$

In particular, f is density (strongly density) continuous at 0 if and only if g is density (strongly density) continuous at 0.

PROOF. Notice that condition (5) imples that 0 is a dispersion (strong dispersion) point of $f(A)$ if and only if 0 is a dispersion (strong dispersion) point of $g(A)$. Thus, the additional part follows immediately from (5).

To prove (5) let us first choose $r_0 > 0$ such that f' does not have any zeros in $B(r_0) \setminus \{0\}$.

Notice that it is enough to prove that for every $j < n$ we can find $r > 0$ such that

$$\frac{1}{4}\lambda\big(g(A)\big) \leq \lambda\big(f(A)\big) \leq 4\lambda\big(g(A)\big).$$

holds for all $A \subset K^j = K(\alpha, \frac{\pi}{n}, r)$, where $\alpha = \frac{2\pi j}{n}$. So, choose $\alpha = \frac{2\pi j}{n}$ and consider functions f and g as restricted to K^j. Then, g is one-to-one and has an analytic inverse $g^{-1}\colon g(K^j) \to K^j$.

Put $h = f \circ g^{-1}$. Then, h is analytic and

$$\begin{aligned} h'(z) &= f'(g^{-1}(z))(g^{-1})'(z) \\ &= \left(\sum_{k=n}^{\infty} k a_k \big(g^{-1}(z)\big)^{k-1}\right) \frac{1}{n a_n \big(g^{-1}(z)\big)^{n-1}} \\ &= \sum_{k=n}^{\infty} \frac{k a_k}{n a_n} \big(g^{-1}(z)\big)^{k-n}. \end{aligned}$$

Thus, we can pick $r \in (0, r_0)$ such that $|h'(z) - 1| = |h'(z) - h'(0)| < \frac{1}{2}$, i.e., that

$$\frac{1}{2} < |h'(z)| < \frac{3}{2} < 2 \tag{6}$$

for all $z \in g(K^j)$ with $|z| < r$. We will show that this choice of r implies (5).

Since f' does not have any zeros in $Z = K^j \cap B(r) \setminus \{0\}$, the set Z can be covered by open sets $S \subset Z$ such that f has an inverse $f^{-1}\colon f(S) \to S$. So, pick $S \subset Z$ with this property. Since Lebesgue measure is countable additive, we may assume that $A \subset S$. Then, function h restricted to $g(S)$ is one-to-one and, by

Lemma 1.2,

$$\begin{aligned}\lambda(f(A)) &= \int_{f(S)} \chi_{f(A)}\, d\lambda \\ &= \int_{h(g(S))} \chi_A \circ f^{-1}\, d\lambda \\ &= \int_{g(S)} (\chi_A \circ f^{-1} \circ f \circ g^{-1}(z)) \cdot |h'(z)|^2\, d\lambda \\ &= \int_{g(S)} \chi_{g(A)}(z) \cdot |h'(z)|^2\, d\lambda.\end{aligned}$$

But, by (6), $\frac{1}{4} < |h'(z)|^2 < 4$ for all $z \in g(S) \subset g(K^j)$. Thus

$$\lambda\big(f(A)\big) = \int_{g(S)} \chi_{g(A)}(z) \cdot |h'(z)|^2\, d\lambda \geq \int_{g(S)} \chi_{g(A)}(z) \cdot \frac{1}{4}\, d\lambda = \frac{1}{4}\lambda\big(g(A)\big)$$

and

$$\lambda\big(f(A)\big) = \int_{g(S)} \chi_{g(A)}(z) \cdot |h'(z)|^2\, d\lambda \leq \int_{g(S)} \chi_{g(A)}(z) \cdot 4\, d\lambda = 4\lambda\big(g(A)\big).$$

■

Now, we are ready for the main theorems.

THEOREM 3.2. *Every analytic function is density continuous.*

PROOF. It follows immediately from Lemmas 2.3, 3.1 and the fact that the topology $\mathcal{T}_{\mathcal{N}}$ is invariant under translations. ■

THEOREM 3.3. *A non-constant analytic function f is strongly density continuous at z if and only if $f^{(n)}(z)$ is either real or imaginary number, where $n = \min\{k > 0: f^{(k)}(z) \neq 0\}$.*

PROOF. Since topology $\mathcal{T}_S$ is invariant under translations we can assume that $z = 0$. Then, Theorem follows immediately from Lemmas 2.3 and 3.1. ■

THEOREM 3.4. *An analytic function f is strongly density continuous on its domain if and only if $f(z) = a + bz$ where $a, b \in \mathbb{C}$ and b is either real or imaginary number.*

PROOF. First assume that $f(z) = a + bz$ where $a, b \in \mathbb{C}$. If f is constant, then evidently f is strongly density continuous. So, assume f is not constant and pick an arbitrary $z \in \mathbb{C}$. Then $\min\{k > 0: f^{(k)}(z) \neq 0\} = 1$ and $f^{(1)}(z) = b$. So, by Theorem 3.3, f is strong density continuous at z if and only if b is either real or imaginary number.

So, assume that f is not linear. Then, f' is analytic and not constant. Therefore, there exists z in the domain of f such that $f'(z)$ is neither real nor imaginary number. Thus, by Theorem 3.3, f is not strongly density continuous at z. ■

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