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Topology and its Applications 63 (1995) 277–298

TOPOLOGY AND ITS APPLICATIONS

Topologies making a given ideal nowhere dense or meager

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Received 13 January 1994; revised 23 June 1994

Abstract

Let X be a set and let $\mathcal{J}$ be an ideal on X. In this paper we show how to find a topology τ on X such that τ-nowhere dense (or τ-meager) sets are exactly the sets in $\mathcal{J}$. We try to find the "best" possible topology with such property.

In Section 1 we discuss the ideals $\{\emptyset\}$ and $\mathcal{P}(X)$. We also show that for every ideal $\mathcal{J} \neq \mathcal{P}(X)$ there is a topology T_0 making it nowhere dense and that this topology is T_1 if $\bigcup \mathcal{J} = X$. Section 2 concerns principal ideals $\mathcal{P}(S)$ for $S \subset X$. It contains characterization of cardinal pairs $(\kappa, \lambda) = (|S|, |X \setminus S|)$ for which $\mathcal{P}(S)$ can be made nowhere dense or meager by compact Hausdorff, metric, and complete metric topologies. Section 3 deals with the ideals containing all singletons. We prove there that it is consistent with ZFC + CH that for every σ-ideal $\mathcal{J}$ on $\mathbb{R}$ containing all singletons and such that every element of $\mathcal{J}$ is either null or meager, there exists a Hausdorff zero-dimensional topology making $\mathcal{J}$ nowhere dense. Section 4 contains the discussion of the above theorem. In particular, it is noticed there that the theorem follows from CH for the ideals with the cofinality $\leqslant \omega_1$.

Keywords: Nowhere dense; Meager; Perfectly meager and universally null sets

AMS (MOS) Subj. Class.: Primary 54D80; secondary 54A35, 03E35

1. Preliminaries

Most notation used in this paper will follow [10]. If τ is a topology on X then $N(\tau, X)$ (respectively $M(\tau, X)$) is the family of all τ-nowhere dense (respectively τ-meager) sets. We also write $N(\tau)$ or $N(X)$ if the other parameter is clear from the context.

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SSDI 0166-8641(95)00074-3

For an ideal $\mathscr{I}$ on a nonempty set X we say that a topology τ on X *makes* $\mathscr{I}$ *meager* (*nowhere dense*, respectively) if $\mathscr{I}=M(\tau, X)$ ($\mathscr{I}=N(\tau, X)$, respectively).

We will start with some easy remarks.

Fact 1.1. *If τ makes $\mathscr{I}$ nowhere dense then:*
τ makes $\mathscr{I}$ meager if and only if $\mathscr{I}$ is a σ-ideal.

Since every ideal on a finite set is a σ-ideal we immediately conclude

Fact 1.2. *Let S be a finite subset of X and let $\mathscr{I}\subset\mathscr{P}(S)$ be an ideal. If τ is a topology on X then:*
τ makes $\mathscr{I}$ meager if and only if τ makes $\mathscr{I}$ nowhere dense.

Let us begin with considering two special cases of ideals: the trivial ideal $\{\emptyset\}$ and the improper ideal $\mathscr{P}(X)$. For the trivial ideal the following is true.

Fact 1.3. *If τ is a T_0 topology on X then the following conditions are equivalent:*
(1) *τ makes $\{\emptyset\}$ nowhere dense;*
(2) *τ makes $\{\emptyset\}$ meager;*
(3) *τ is a discrete topology on X.*

Proof. Equivalence of (1) and (2) and the implication "(3) $\Rightarrow$ (1)" are obvious.

To see that (1) implies (3) first notice that the set $\text{cl}(\{x\})$ is clopen for every $x\in X$, since otherwise a nowhere dense set $\text{cl}(\{x\})\setminus\text{int}(\text{cl}(\{x\}))$ would be nonempty.

Now, if $y\in\text{cl}(\{x\})$ then also $x\in\text{cl}(\{y\})$ since $\text{cl}(\{y\})$ is open and contains y. But X is T_0, so $y=x$. This means that $\{x\}=\text{cl}(\{x\})\in\tau$ for every $x\in X$. Fact 1.3 has been proved. □

Notice that in the above the assumption of X being T_0 is important since the indiscrete space $(X, \{\emptyset, X\})$ also makes the trivial ideal nowhere dense.

In the case of improper ideal $\mathscr{P}(X)$ on a set X the situation is a little bit more complicated, as described in the following fact.

Fact 1.4. *Let X be a nonempty set.*

(1) *There is no topology on X making $\mathscr{P}(X)$ nowhere dense.*

(2) *If X is finite then there is no topology on X making $\mathscr{P}(X)$ meager.*

(3) *There is neither a compact T_2 nor a complete metrizable topology on X making $\mathscr{P}(X)$ meager.*

(4) *If X is infinite then there is a metrizable topology τ on X making $\mathscr{P}(X)$ meager.*

Proof. (1) is obvious, since X is dense in itself.

(2) follows from (1), by Fact 1.2.

(3) follows immediately from the Baire Category Theorem.

To see (4) let Y be a set with the same cardinality as X and identify X with $Y \times \mathbb{Q}$, where $\mathbb{Q}$ stands for the set of rational numbers considered with the natural topology. If we equip Y with the discrete topology (or any other metrizable topology), then the product topology on $Y \times \mathbb{Q}$ is metrizable, and it makes $\mathscr{P}(X) = \mathscr{P}(Y \times \mathbb{Q})$ meager, since the sets $Y \times \{q\}$ are nowhere dense in $Y \times \mathbb{Q}$. □

Facts 1.3 and 1.4 fully describe the situation with the trivial and improper ideals. Thus, in what follows we will consider only proper and nontrivial ideals.

Now, let $\mathscr{I}$ be an ideal on X. Define $\tau(\mathscr{I}) = \{X \setminus A\colon A \in \mathscr{I}\} \cup \{\emptyset\}$. The following fact is easy to verify. (See [8]. Compare also Lemma 3.5.)

Theorem 1.5. *If $\mathscr{I}$ is a proper ideal on X, then $\tau(\mathscr{I})$ is a topology on X making $\mathscr{I}$ nowhere dense.*

It is easy to see that

Fact 1.6. *If $\bigcup \mathscr{I} = X$ then $\tau(\mathscr{I})$ is a T_1 but not a T_2 topology.*

However, in general, the elements of $X \setminus \bigcup \mathscr{I}$ are not separated by $\tau(\mathscr{I})$. Thus, to prove the next theorem, we need to modify topology $\tau(\mathscr{I})$.

Theorem 1.7. *For every proper ideal $\mathscr{I}$ on X there is a T_0 topology τ on X making $\mathscr{I}$ nowhere dense.*

Proof. Extend topology $\tau(\mathscr{I})$ to $\tau_0(\mathscr{I}) = \tau(\mathscr{I}) \cup \mathscr{P}(X \setminus \bigcup \mathscr{I})$. It is easy to see that $\tau_0(\mathscr{I})$ is a T_0 topology on X. It is also not difficult to see that all sets from $\mathscr{I}$ remain closed nowhere dense, while no new nowhere dense sets are added. □

Since Theorem 1.7 closes the problem of the existence of T_0 topological spaces making a given ideal nowhere dense (meager), for the rest of the paper we will study the spaces that are at least T_1. The paper is organized as follows. In Section 2 we consider the case where $\mathscr{I}$ is principal. This case seems to be well understood. However, two open problems are stated near the end of Section 2. Section 3 is devoted to the ideals containing all singletons. Here two main results are based on additional axioms whose role, in several instances, is not entirely clear.

2. Principal ideals

Recall that an ideal $\mathscr{I} \subseteq \mathscr{P}(X)$ is said to be *principal* if there exists a subset $S \subseteq X$ such that $\mathscr{I} = \mathscr{P}(S)$. This section is devoted to a problem of making such ideals nowhere dense or meager. Since such problems depend only on the cardinality of sets S and $X\setminus S$ the following definition will be useful. For cardinal numbers κ and λ we say that a topological space X (or its topology) is (κ, λ) *nowhere dense* (*meager*) provided there exists $S \in [X]^{\kappa}$ such that $|X\setminus S| = \lambda$ and $N(X) = \mathscr{P}(S)$ $(M(X) = \mathscr{P}(S))$.

Let us note the following obvious facts, that follow directly from Fact 1.1.

Fact 2.1. *If $k < \omega$ then topology τ is (k, λ) nowhere dense if and only if it is (k, λ) meager.*

Fact 2.2. *If a topology τ is (κ, λ) nowhere dense then it is (κ, λ) meager.*

Now, suppose that a topology τ on X makes $\mathscr{I} = \mathscr{P}(S)$ meager. For any $y \in X\setminus S$ the singleton $\{y\}$ must be dense in some nonempty open set U. Thus, $y \in V$ for all nonempty open subsets $V \subseteq U$. It follows that either $U = \{y\}$ or τ is not T_1. This observation can be stated as follows.

Fact 2.3. *If a T_1 topological space X makes $\mathscr{P}(S)$ meager then $\{y\}$ is open for every $y \in X\setminus S$.*

Lemma 2.4. *Let τ be a T_1 topology on X and let $S \subset X$. Topology τ makes $\mathscr{I} = \mathscr{P}(S)$ nowhere dense iff*

(1) *$\{y\}$ is open for every $y \in X\setminus S$; and*

(2) *$\operatorname{int}(S) = \emptyset$ (equivalently, $X\setminus S$ is dense in X).*

Proof. "$\Rightarrow$" Assume that $N(\tau) = \mathscr{I}$. (1) follows from Fact 2.3. Thus, S is closed. Now, if $U \subset S$ is open then it is empty since S is nowhere dense, which proves (2).

"$\Leftarrow$" Assume (1) and (2). For any $A \in N(\tau)$ we must have $A \subseteq S$ because of (1). On the other hand, if S was dense in an open set U then, by (1), $U \subseteq S$ and, by (2), U is empty. □

Lemma 2.4 shows that any T_1 topology making $\mathscr{I} = \mathscr{P}(S)$ nowhere dense is a union of $\mathscr{P}(X\setminus S)$ and a family $\mathscr{F}$ of sets meeting S. In general $\mathscr{F}$ does not need to be closed under supersets but in case of S being a singleton these topologies have a nice characterization.

Theorem 2.5. *Let X be a set and let $s \in X$. Topology τ is a T_1 topology making $\mathscr{P}(\{s\})$ nowhere dense if and only if it has a basis of the form*

$$\mathscr{B} = \{\{x\}: x \in X\setminus\{s\}\} \cup \mathscr{F}, \tag{1}$$

where $\mathscr{F}$ is some nonmaximal filter on X such that $\bigcap \mathscr{F} = \{s\}$.

Proof. "$\Rightarrow$" Assume that τ is a T_1 topology on X making $\mathscr{P}(\{s\})$ nowhere dense. Take $\mathscr{F} = \{U \in \tau: s \in U\}$. Evidently $\mathscr{F}$ is closed under finite intersections. It is closed for taking supersets since, by Lemma 2.4, $\{x\}$ is open for every $x \in X\setminus\{s\}$.

Thus, $\mathscr{F}$ is a filter. It is not maximal, since $\{s\} \notin \mathscr{F}$. Moreover, $\bigcap \mathscr{F} = \{s\}$, because τ is T_1. Clearly $\mathscr{B}$ as in the equation (1) is a basis for τ.

"$\Leftarrow$" Assume that $\mathscr{B}$ is as in (1) above. It is easy to see that $\tau = \{\bigcup \mathscr{B}': \mathscr{B}' \subseteq \mathscr{B}\} = \mathscr{P}(X \setminus S) \cup \mathscr{F}$ is a T_1 topology making $\mathscr{P}(\{s\})$ nowhere dense. □

Corollary 2.6. *Any T_1 topology making $\mathscr{P}(\{s\})$ nowhere dense is also a T_2 topology.*

Notice that Corollary 2.6 is false for the ideal $\mathscr{P}(S)$ even if S has only two elements. For example, if we take different $a,b \in \mathbb{R} \setminus \omega$ and define a topology τ on $X = \omega \cup \{a, b\}$ as generated by sets $\{c\} \cup \omega \setminus F$, where $c \in \{a, b\}$ and $F \in [\omega]^{<\omega}$ then τ is a topology making $\mathscr{P}(S) = \mathscr{P}(\{a, b\})$ nowhere dense which is T_1 but not T_2.

In general, for an arbitrary nonempty $S \subseteq X$, T_1 topologies on X making $\mathscr{I} = \mathscr{P}(S)$ nowhere dense do not have any simple characterization analogous to Theorem 2.5. However, for finite sets S we have the following characterization.

Theorem 2.7. *If $S = \{s_1, s_2, \ldots, s_n\}$ is a finite nonempty subset of X and τ is a T_1 topology on X, then τ makes $\mathscr{P}(S)$ nowhere dense if and only if $X = X_1 \cup X_2 \cup \cdots \cup X_n$, where*

(a) $X_k \cap S = \{s_k\}$,

(b) $X_k \in \tau$,

(c) $\tau_k = \tau|_{X_k}$ *makes $\mathscr{P}(\{s_k\})$ nowhere dense,*

for every $k = 1, 2, \ldots, n$.

Proof. "$\Leftarrow$" follows easily from Theorem 2.5.

"$\Rightarrow$" For $k \in \{1, 2, \ldots, n\}$ let $X_k = (X \setminus S) \cup \{s_k\}$. Clearly (a) and (b) are satisfied, since τ is T_1. The easy checking of (3) is left to the reader. □

It is worth mentioning that the analog of Theorem 2.7 does not need to be true for infinite sets S, because $\tau|_S$ does not need to be discrete. For example, consider $\omega + 1$ with the order topology and take $(\omega + 1)^2$ with the product topology. Clearly, it makes $\mathscr{P}([\{\omega\} \times (\omega + 1)] \cup [(\omega + 1) \times \{\omega\}])$ nowhere dense while $\{(\omega, \omega)\}$ is not open in $\{\omega\} \times (\omega + 1)$.

The following facts follow easily from Theorems 2.5 and 2.7 and Fact 1.1.

Corollary 2.8. *If X is finite and $S \subset X$ is nonempty then there is no T_1 topology on X making $\mathscr{P}(S)$ either nowhere dense or meager. In particular, for any $l < \omega$ and $0 < k < \omega$ there is no (k, l) meager T_1 topology.*

In what follows we will use also the following example.

Example 2.9. Let λ be a limit ordinal and let $\mathscr{B} = \mathscr{P}(\lambda) \cup \{(\lambda + 1 \setminus \alpha: \alpha < \lambda\}$. A T_1 topology generated by $\mathscr{B}$ will be denoted by $\tau_{\text{ord}}(\lambda)$. It is easy to see that $\tau_{\text{ord}}(\lambda)$ makes $\mathscr{P}(\{\lambda\})$ nowhere dense.

The next corollary follows immediately from Theorem 2.5.

Corollary 2.10. *$\tau_{ord}(\omega)$ is a separable metrizable topology making $\mathscr{P}(\{\omega\})$ nowhere dense. In particular, it is $(1, \omega)$ nowhere dense.*

In fact, $\tau_{\text{ord}}(\omega)$ is just an order topology on $\omega + 1$.

The following examples indicate that even in the simplest possible case, discussed in Theorem 2.5, there is no unique $(1, \omega)$ nowhere dense topology.

Example 2.11. The topology induced from $\mathbb{R}^2$ on a countable set

$$X = \{(1/n, 1/m): 0 < n, m < \omega\} \cup \{(0, 0)\}$$

makes $\mathscr{P}(\{(0, 0)\})$ nowhere dense. Space X is not homeomorphic to $\tau_{\text{ord}}(\omega)$. However, the diagonal $D = \{(1/n, 1/n): 0 < n < \omega\}$ is homeomorphic to $\tau_{\text{ord}}(\omega)$.

Example 2.12. Let $\mathscr{F}$ be a nonprincipal ultrafilter on ω. The topology τ on $\omega + 1$ generated by $\mathscr{P}(\omega) \cup \{F \cup \{\omega\}: F \in \mathscr{F}\}$ makes $\mathscr{P}(\{\omega\})$ nowhere dense. The space $(\omega + 1, \tau)$ is $(1, \omega)$ nowhere dense and does not have any subspace homeomorphic to $\tau_{\text{ord}}(\omega)$. This is because in $\tau_{\text{ord}}(\omega)$ only cofinite sets containing ω are open unlike in any infinite subspace of $(\omega + 1, \tau)$ containing ω.

Corollary 2.10 and Examples 2.11 and 2.12 show a variety of $(1, \omega)$ nowhere dense topologies. It is easy to show that the space from Example 2.12 is normal but not metrizable, since ω has no countable basis.

The next theorem gives a necessary and sufficient condition for the existence of a metrizable topology which is (κ, λ) nowhere dense.

Theorem 2.13. *Let $\kappa > 0$ and λ be cardinal numbers. The following conditions are equivalent.*

(i) *There exists a metrizable space X which is (κ, λ) nowhere dense.*

(ii) *$\lambda^{\omega} \geqslant \kappa$ and $\lambda \geqslant \omega$.*

Proof. "(i) $\Rightarrow$ (ii)" Let τ be a metrizable topology on X making $\mathscr{I} = \mathscr{P}(S)$ nowhere dense with $|S| = \kappa$ and $|X \setminus S| = \lambda$. Then, by Lemma 2.4 $X \setminus S$ is dense in X. Since $S \neq \emptyset$, as $\kappa > 0$, we conclude that λ is infinite. Moreover, for every $x \in S$ there exists a sequence $s(x) = \{d_n\} \in (X \setminus S)^{\omega}$ that converges to x. Thus, the function $s: S \to (X \setminus S)^{\omega}$ is one-to-one and so, $\kappa = |S| \leqslant |X \setminus S|^{\omega} = \lambda^{\omega}$.

"(ii) $\Rightarrow$ (i)" Define a metric d on λ^{ω} by putting for different $f, g \in \lambda^{\omega}$,

$$d(f, g) = \frac{1}{1 + \min\{n \in \omega: f(n) \neq g(n)\}}.$$

Let Z_0 be the family of all $f \in \lambda^{\omega}$ such that f is equal to zero for almost all $n < \omega$. Thus,

$$|Z_0| = |\lambda^{<\omega}| = \lambda$$

and $Z_1 = \lambda^{\omega} \setminus Z_0$ has cardinality λ^{ω}.

Define a function $G : \lambda^\omega \to \mathbb{R}$ by

$$G(f) = \begin{cases} [1 + \max\{n \in \omega : f(n) \neq 0\}]^{-1}, & f \in Z_0, \\ 0, & f \in Z_1. \end{cases}$$

Now, consider the graph of G as a subspace of $\lambda^\omega \times \mathbb{R}$ with the natural product topology. Notice that $G_0 = G \cap (Z_0 \times \mathbb{R})$ is dense in G and points of G_0 are isolated in G. Thus, $G_1 = G \cap (Z_1 \times \mathbb{R}) = Z_1 \times \{0\}$ is closed and nowhere dense in G. So, $N(G) = \mathscr{P}(Z_1)$.

Take $G_2 \in [G_1]^\kappa$ and define $X = G_0 \cup G_2$ with the subspace topology of G. It is easy to see that X has the desired properties.

This finishes the proof. □

Theorem 2.13 shows that there are no $(2^\mathfrak{c}, \omega)$ and $(2^\mathfrak{c}, \mathfrak{c})$ nowhere dense metrizable spaces. [1]

On the other hand, $\beta\mathbb{N}$ is a separable Hausdorff compact space which is $(2^\mathfrak{c}, \omega)$ nowhere dense. The existence of a $(2^\mathfrak{c}, \mathfrak{c})$ nowhere dense compact T_2 topology is stated in Corollary 2.16.

Let us also recall that the density of a topological space X is defined by

$$d(X) = \min\{|D| : D \text{ is dense in } X\} + \omega.$$

Theorem 2.14. *Let μ be the density of a compact Hausdorff topological space Y without isolated points. If $\mu \leqslant \lambda \leqslant |Y|$ then there exists a compact space X which is $(|Y|, \lambda)$ nowhere dense.*

Proof. The proof below is essentially a repetition of Alexandroff's duplicated sphere construction.

Let D be a dense subset of Y with cardinality λ. We define set X as $(D \times \{0\}) \cup (Y \times \{1\})$. The topology on X is defined by a local basis system. Points of $D \times \{0\}$ will be considered as isolated points. For $y \in Y$ a local base of $(y, 1)$ is defined as a family of all sets of the form

$$X \cap (U \times \{0, 1\}) \setminus \{(y, 0)\}$$

for every open set U in Y containing y. It is easy to see that X defined that way is Hausdorff and compact. Also, the set $D \times \{0\}$ is discrete and dense in X, since D was dense in Y and Y did not have isolated points. Hence, subspace $S = Y \times \{1\}$ is nowhere dense in X, while $X \setminus S = D \times \{0\}$ is an open discrete subspace of X of cardinality λ. Thus, X is $(|Y|, \lambda)$ nowhere dense.

The proof of the theorem has been completed. □

Corollary 2.15. *Let $\omega \leqslant \lambda \leqslant \kappa$. If $\kappa \geqslant \mathfrak{c}$ then the following conditions are equivalent:*

(i) *There exists a compact Hausdorff space which is (κ, λ) nowhere dense.*

[1] Letter $\mathfrak{c}$ stands for the cardinality of the continuum, i.e., $\mathfrak{c} = 2^\omega$.

(ii) *There exists a compactification* $\gamma\lambda$ *of* λ, *considered with the discrete topology, such that* $|\gamma\lambda \setminus \lambda| = \kappa$.

(iii) *There exists a compact Hausdorff space X with cardinality* κ *and density less than or equal to* λ.

Proof. "(i) ⇒ (ii)" follows directly from Lemma 2.4.

"(ii) ⇒ (iii)" is obvious.

"(iii) ⇒ (i)" follows from Theorem 2.14 used with $X' = X \times [0, 1]$ considered with the product topology, since then X' does not have isolated points. □

Since the density of $\beta\mathbb{N}$ is equal to ω and its cardinality is $2^{\mathfrak{c}}$ we conclude immediately that

Corollary 2.16. *There exists a compact Hausdorff topological space which is* $(2^{\mathfrak{c}}, \mathfrak{c})$ *nowhere dense.*

So, let us consider a general problem for which cardinal numbers κ and λ there exists a compact Hausdorff topological space which is (κ, λ) nowhere dense. If $\kappa = 0$ then the only T_0 space which is (κ, λ) nowhere dense is a discrete space of cardinality λ. Thus, λ must be finite. If $\kappa > 0$ then $\lambda \geqslant \omega$, since no finite set can have an accumulation point. So, assume that $\kappa > 0$ and $\lambda \geqslant \omega$. Now, if $\kappa \leqslant \lambda$ then we have the following example.

Example 2.17. If $\kappa > 0$ and $\lambda \geqslant \omega$ are cardinal numbers such that $\kappa \leqslant \lambda$ then there is a compact Hausdorff topological space X which is (κ, λ) nowhere dense.

Construction. To construct such a space, let Y be a free union of κ copies of the one-point compactification of a discrete space of cardinality λ. If κ is finite, take $X = Y$. If κ is infinite, define X as a one-point compactification of Y.

Notice also that if there is a (κ, λ) nowhere dense compact Hausdorff space then $\kappa \leqslant 2^{2^{\lambda}}$, since $|X| \leqslant 2^{2^{d(X)}}$ for every Hausdorff space X [7, Theorem 3.2]. Thus, the problem is reduced to $\lambda \geqslant \omega$ and $\lambda < \kappa \leqslant 2^{2^{\lambda}}$. Notice also, that for $\kappa = 2^{2^{\lambda}}$ and $\kappa = 2^{\lambda}$ there is a compact Hausdorff space which is (κ, λ) nowhere dense. This follows from Corollary 2.15 used with $X = \beta\lambda$ and $X = 2^{\lambda}$, respectively. This discussion can be summarized as follows.

Corollary 2.18 (GCH). *If* $\kappa > 0$ *and* λ *are cardinal numbers then the following conditions are equivalent.*

(i) *There exists a compact Hausdorff space which is* (κ, λ) *nowhere dense.*

(ii) *There exists a Hausdorff space which is* (κ, λ) *nowhere dense.*

(iii) $\lambda \geqslant \omega$ *and* $\kappa \leqslant 2^{2^{\lambda}}$.

It is unknown to the authors whether Corollary 2.18 remains true in ZFC. By Corollary 2.15 this question can be reformulated as the following problem.

Problem 2.19. Let λ be an infinite cardinal considered with the discrete topology. Can we prove in ZFC that for every cardinal $\lambda < \kappa < 2^{2^\lambda}$ there exists a compactification $\gamma\lambda$ of λ such that $|\gamma\lambda \setminus \lambda| = \kappa$?

Notice that the answer to Problem 2.19 is positive (in ZFC) for $\lambda = \omega$ and $\kappa = \omega_1$ [16, Corollary 6.16].

A more general problem can be stated that way.

Problem 2.20. Characterize completely regular spaces for which:

(a) for all cardinal numbers $\kappa \leqslant 2^{2^{|X|}}$ there is a compactification $\gamma\lambda$ of X such that $|\gamma X \setminus X| = \kappa$; or

(b) there is a compactification γX of X such that $|\gamma X \setminus X| = |X|$.

Notice that there are completely regular noncompact spaces for which (b) in the above fails. For example, it is easy to see that the only compactification of ω_1, considered with the order topology, is its one-point compactification $\omega_1 + 1$. (This is the case, since any continuous function from ω_1 into [0, 1] is eventually constant on ω_1.)

Some results of this section concerning (κ, λ) nowhere dense topological spaces are summarized in the next theorem.

Theorem 2.21. *In Table 1 we examine the existence of (κ, λ) nowhere dense topological spaces in the following classes: all topological spaces, T_0 spaces, T_1 spaces, T_2 spaces, compact spaces C, metrizable spaces M, complete metrizable spaces M^c and separable spaces S. Table 1 lists the best possible classes in which particular (κ, λ) nowhere dense spaces exist.*

Proof. Row $\lambda = 0$ follows from Fact 1.4(1). Column $\kappa = 0$ is discussed in Fact 1.3.

Examples from row $0 < \lambda < \omega$ and $\kappa > 0$ are as described in Theorem 1.7. They cannot be T_1 by Lemma 2.4, since no finite subset of a T_1 space can have an accumulation point.

Table 1
(κ, λ) nowhere dense spaces

	$\kappa = 0$	$0 < \kappa < \omega$	$\kappa = \omega$	$\kappa = \mathfrak{c}$	$\kappa = 2^{\mathfrak{c}}$
$\lambda = 0$	–	–	–	–	–
$0 < \lambda < \omega$	M^cC	T_0	T_0	T_0	T_0
$\lambda = \omega$	M^cS	CM	CM	CM	CT_2S
$\lambda = \mathfrak{c}$	M^c	M^c or CT_2	M^c or CT_2	M^c or CT_2	CT_2
$\lambda = 2^{\mathfrak{c}}$	M^c	M^c or CT_2	M^c or CT_2	M^c or CT_2	M^c or CT_2

Evidently, by Lemma 2.4, all the spaces from row $\lambda = \omega$ are separable (compact metric spaces are separable), while these from rows with $\lambda > \omega$ cannot be separable.

For $\lambda = \omega$ and $0 < \kappa \leqslant \omega$ the spaces from Example 2.17 are also metrizable, since they are countable.

For $\lambda = \omega$ and $\kappa = \mathfrak{c}$ notice that

$$X = ([0, 1] \times \{0\}) \cup \left\{\left(\frac{m}{n}, \frac{1}{n}\right) \in [0, 1]^2 \colon m \text{ and } n \text{ are relatively prime}\right\}$$

considered with the subspace topology of the plane is a compact metric $(\mathfrak{c}, \omega)$ nowhere dense space.

Since, by Theorem 2.13, there is no metrizable $(2^{\mathfrak{c}}, \omega)$ nowhere dense space, $\beta\mathbb{N}$ is the best possible $(2^{\mathfrak{c}}, \omega)$ nowhere dense space.

Similarly, by Theorem 2.13, there is no metrizable $(2^{\mathfrak{c}}, \mathfrak{c})$ nowhere dense space, so the example from Corollary 2.16 is the best possible.

In the remaining seven cases the existence of compact T_2 spaces follows from Example 2.17. There is no compact metrizable example in these cases, since such an example would be separable. Moreover, in all this cases $\kappa \leqslant \lambda$. Thus, the complete metrizable spaces can be constructed as a free sum of κ many spaces $\omega + 1$ and of a discrete space of cardinality λ.

This finishes the proof of the theorem. □

For a similar theorem on (κ, λ) meager spaces we need also the following fact.

Fact 2.22. *If X is a compact T_2 or complete metrizable space then X is (κ, λ) nowhere dense if and only if X is (κ, λ) meager.*

Proof. Implication "$\Rightarrow$" follows from Fact 2.2.

For the other direction let X be a compact T_2 space which is (κ, λ) meager and let $S \subset X$ be such that $M(X) = \mathscr{P}(S)$. By Fact 2.3, $\{y\}$ is open for every $y \in X \setminus S$. So, S is closed in X. It is also meager, by our assumption. Hence, by the Baire Category Theorem, $\operatorname{int}(S) = \emptyset$. So, S is nowhere dense and $\mathscr{P}(S) = N(X)$. □

The results on (κ, λ) meager topological spaces similar to these of Theorem 2.21 are summarized in the next theorem.

Table 2
(κ, λ) meager spaces.

	$\kappa = 0$	$0 < \kappa < \omega$	$\kappa = \omega$	$\kappa = \mathfrak{c}$	$\kappa = 2^{\mathfrak{c}}$
$\lambda = 0$	–	–	MS	MS	M
$0 < \lambda < \omega$	$M^{\mathfrak{c}}C$	T_0	MS	MS	M
$\lambda = \omega$	$M^{\mathfrak{c}}S$	CM	CM	CM	M or CT_2S
$\lambda = \mathfrak{c}$	$M^{\mathfrak{c}}$	$M^{\mathfrak{c}}$ or CT_2	$M^{\mathfrak{c}}$ or CT_2	$M^{\mathfrak{c}}$ or CT_2	M or CT_2
$\lambda = 2^{\mathfrak{c}}$	$M^{\mathfrak{c}}$	$M^{\mathfrak{c}}$ or CT_2	$M^{\mathfrak{c}}$ or CT_2	$M^{\mathfrak{c}}$ or CT_2	$M^{\mathfrak{c}}$ or CT_2

Theorem 2.23. *In Table 2 we examine the existence of (κ, λ) meager topological spaces in the following classes: all topological spaces, T_0 spaces, T_1 spaces, T_2 spaces, compact spaces C, metrizable spaces M, complete metrizable spaces M^c and separable spaces S. Table 2 lists the best possible classes in which particular (κ, λ) meager spaces exist.*

Proof. The two first columns of Table 2 are identical to those of Table 1 by Fact 1.2.

For the remaining cases first notice that, by Fact 2.22, the compact T_2 entries and complete metrizable entries in Table 2 must be the same as those in Table 1.

The entries for $\lambda = 0$ and $\kappa \in \{\omega, 2^{\mathfrak{c}}\}$ are justified by Fact 1.4(4). For $\lambda = 0$ and $\kappa = \mathfrak{c}$ take $\mathbb{R} \times \mathbb{Q}$ with the subspace topology of the plane.

The noncompact metrizable (κ, λ) meager spaces for $\lambda > 0$ and $\kappa \geqslant \omega$ can be obtained by taking the free union of a $(\kappa, 0)$ meager metrizable example and a discrete space of cardinality λ. These spaces will be separable for $\lambda < \omega$ and $\kappa \in \{\omega, \mathfrak{c}\}$. They cannot be separable for $\kappa = 2^{\mathfrak{c}}$ since separable metric spaces have cardinality less then or equal to $\mathfrak{c}$. They cannot be separable for $\lambda > \omega$ by Fact 2.3.

This finishes the proof of the theorem. □

3. Ideals containing all singletons

Assume that X is an infinite set and $\mathscr{I} = [X]^{<\omega}$. By Fact 1.6 the family $\tau(\mathscr{I}) = \{X \setminus N: N \in \mathscr{I}\} \cup \{\emptyset\}$ is a T_1 topology with $N(\tau, X) = \mathscr{I}$. Taking the free union of such spaces we obtain the following.

Example 3.1. For any $k < \omega$ there exists a T_1 space X with k disjoint open sets and $N(\tau, X) = [X]^{<\omega}$.

This stands in contrast with the next lemma.

Lemma 3.2. *Let (τ, X) be a T_1 space with $[X]^{<\omega} \subseteq N(\tau, X)$. If there exists an infinite family $\{B_n: n < \omega\}$ of nonempty disjoint open sets, then there is an infinite set $Y \in N(\tau, X)$.*

Proof. Let $Y = \{y_n: n < \omega\}$ be a selector from $\{B_n: n < \omega\}$. If $V \in \tau$ is nonempty, then $V \cap B_n \cap Y \subseteq \{y_n\}$ for each $n < \omega$. Since $\{y_n\} \in N(\tau, X)$, Y cannot be dense in V. □

Lemma 3.3. *If X is an infinite Hausdorff space, then there exists an infinite family $\{U_n: n < \omega\}$ of nonempty pairwise disjoint open sets.*

Proof. A folklore diagonal argument is left to the reader. □

Theorem 3.4. *There is no Hausdorff topology on X making the ideal* $[X]^{<\omega}$ *nowhere dense.*

Proof. For finite X it is obvious. So, suppose that τ is a Hausdorff topology on an infinite set X such $[X]^{<\omega} \subseteq N(\tau, X)$. By Lemma 3.3 there exists an infinite family of open nonempty disjoint sets in X and, by Lemma 3.2, there exists an infinite nowhere dense subset of X. □

Notice also that $M(X)$ is always a σ-ideal. So we cannot have $M(X) = [X]^{<\omega}$. Hence,

Fact 3.5. *There is no topology making* $[X]^{<\omega}$ *meager.*

By Fact 1.6 for any uncountable set X there exists a (not second countable) T_1 topology on X making $[X]^{\leqslant\omega}$ nowhere dense. Theorem 3.6 proved below shows that the existence of such second countable topologies is independent of ZFC axioms. (This also follows from Kunen's theorem quoted below as Fact 3.7.)

Recall that an uncountable separable metric space is called a *Lusin space* if $M(X) = [X]^{\leqslant\omega}$. It is known that the Continuum Hypothesis CH implies the existence of Lusin spaces while under Martin's Axiom MA such spaces do not exist. (See [11, p. 205] or [9].) Thus, the existence of Lusin spaces is independent of ZFC axioms.

Theorem 3.6. *If there exists an uncountable, second countable* T_1 *space X such that* $M(\tau, X) \subseteq [X]^{\leqslant\omega}$ *then there exists a Lusin space.*

Proof. Let S be a union of boundaries from all sets of some countable open basis for X. Then, $X \setminus S$ is an uncountable T_1 zero-dimensional space. By the Urysohn Metrization Theorem [4, p. 256], $X \setminus S$ is metrizable, hence it is a Lusin space. □

Fact 3.7 (Kunen [9]). MA + $\neg$CH *implies that there is no uncountable Hausdorff space X with* $M(X) = [X]^{\leqslant\omega}$.

A topology τ on X is said to be *compatible with an ideal* $\mathscr{I}$ on X if for any $Y \subseteq X$ the following condition holds: if for every $x \in Y$ there exists an open neighborhood $x \in U$ such that $U \cap Y \in \mathscr{I}$, then $Y \in \mathscr{I}$.

The following facts are due to Nijåstad. (See [8,13].)

Lemma 3.8. *If a topology* τ *is compatible with an ideal* $\mathscr{I}$ *on X, then* $\tau^{\mathscr{I}} = \{U \setminus N: U \in \tau \text{ and } N \in \mathscr{I}\}$ *is a topology on X.*

Notice that any second countable topology on X is compatible with any σ-ideal on X. Thus, the family $\tau^{\mathcal{I}}$ in the next lemma is indeed a topology.

Lemma 3.9. *Let (X, τ) be a second countable topological space and let $\mathcal{I}$ be a σ-ideal on X such that $\mathcal{I} \cap \tau = \{\emptyset\}$. If $\tau^{\mathcal{I}}$ is a topology as in Lemma 3.8 then $N \in N(\tau^{\mathcal{I}}, X)$ if and only if $N = M \cup I$ for some $M \in N(\tau, X)$ and $I \in \mathcal{I}$.*

Proof. "$\Rightarrow$" Assume that $N \in N(\tau^{\mathcal{I}}, X)$. Let $\{U_n: n < \omega\}$ be a basis for τ. For each $n < \omega$ there exist a nonempty open $V_n \subseteq U_n$ and $I_n \in \mathcal{I}$ such that $N \cap (V_n \setminus I_n) = \emptyset$. Take $M = N \setminus \bigcup\{I_n: n < \omega\}$ and $I = N \cap \bigcup\{I_n: n < \omega\}$. We have $M \in N(\tau, X)$ and $I \in \mathcal{I}$. Clearly $N = M \cup I$.

"$\Leftarrow$" Now let $N = M \cup I$ where $M \in N(\tau, X)$ and $I \in \mathcal{I}$. For any nonempty open set $U \in \tau$ there exists a nonempty open set $V \subseteq U$ which is disjoint with M. Clearly, if $I' \in \mathcal{I}$ then $V' = V \setminus (I \cup I') \subseteq U \setminus I'$, $V' \in \tau^{\mathcal{I}}$, and $V' \cap N = \emptyset$. Also $V' \neq \emptyset$, since $\mathcal{I} \cap \tau = \{\emptyset\}$. Thus, $N \in N(\tau^{\mathcal{I}}, X)$. □

Lemma 3.10 (CH). *Let A be a set with $|A| = \mathfrak{c}$. If $\mathcal{I}$ is a σ-ideal on A containing all singletons, then there exists a Hausdorff topology τ' making $\mathcal{I}$ meager.*

Proof. Let τ_0 be a topology on A such that (A, τ_0) is a Lusin space. Then, $M(\tau_0, A) = [A]^{\leqslant\omega}$. Let $D \subset A$ be a countable dense subset and let $\mathcal{I}' = \mathcal{I}|_{A \setminus D} = \{J \cap (A \setminus D): J \in \mathcal{I}\}$. Clearly $\mathcal{I}'$ is a σ-ideal on A such that $\mathcal{I}' \cap \tau_0 = \{\emptyset\}$. Hence, by Lemma 3.8, $\tau' = \tau_0^{\mathcal{I}'}$ is a topology on A. Moreover, by Lemma 3.9, $M(\tau', A) = \{M \cup I: M \in M(\tau_0, A) \text{ and } I \in \mathcal{I}'\}$. It follows that

$$\begin{aligned} M(\tau', A) &= \{M \cup I: M \in M(\tau_0, A) \text{ and } I \in \mathcal{I}'\} \\ &= \{D' \cup I: D' \in [A]^{\leqslant\omega} \text{ and } I \in \mathcal{I}'\} \\ &= \mathcal{I}. \end{aligned}$$

Clearly τ' is Hausdorff, since τ_0 was. □

Theorem 3.11 (CH). *For any σ-ideal $\mathcal{I}$ on a set X of cardinality $\mathfrak{c}$ there exists a Hausdorff topology τ making $\mathcal{I}$ meager.*

Proof. If $A = \bigcup \mathcal{I}$ is uncountable let τ' be a topology on A as in Lemma 3.10. Define τ as a topology on X generated by $\tau' \cup \mathcal{P}(X \setminus A)$. It is easy to see that τ has the desired properties.

If $\bigcup \mathcal{I}$ is at most countable then a metrizable topology τ exists by Theorem 2.23. □

Notice that the topology τ from Theorem 3.11 does not need to be regular. To see this take $\mathcal{I} = [A]^{\leqslant\omega}$. Then, $\tau = \tau' = \tau_0^{\mathcal{I}'}$, as in Lemma 3.10. Let $a \in A$ be any condensation point for τ_0. There is a sequence $X = \{x_n: n < \omega\} \subset A \setminus (\{a\} \cup D)$ of τ_0-condensation points such that $\lim_{n\to\infty} x_n = a$, where D is a dense countable set from the proof of Lemma 3.10. Now take any τ_0-open neighborhood U of a and a set $C=(A\setminus U)\cup X$. The set C is closed in (A, τ') but it cannot be separated from a since if $C\subset W\in\tau'$ and $a\in V=V_0\setminus A\in\tau'$, where $V_0\in\tau_0$ and $I\in[A\setminus D]^{<\omega}$, then there exists $x_n\in V_0$ and $U\in\tau'$ such that $x_n\in U\subset T$ and $U\cap V\neq\emptyset$.

The following theorem implies that for a large number of natural σ-ideals $\mathcal{J}$ on $\mathbb{R}$ it is consistent with ZFC that there exists a zero-dimensional *regular* topological space X making $\mathcal{J}$ nowhere dense.

Theorem 3.12. *There exists a model of* ZFC + GCH *in which the following holds. There exists a one-to-one mapping* $e:\mathbb{R}\to 2^{\omega_2}$ *such that* $e[\mathbb{R}]$ *is a dense subspace of* 2^{ω_2} *having the following properties:*

(i) *every nowhere dense subset of* $e[\mathbb{R}]$ *is at most countable;*

(ii) *for every* $\alpha<\omega_2$ *and every* $r\in\mathbb{R}$ *there exists an ordinal* $\xi<\omega_1$ *such that* $e(r)(\alpha+\xi)=0$;

(iii) $e[A]$ *has an empty interior in* $e[\mathbb{R}]$ *for every meager and every null subset* A *of* $\mathbb{R}$.

Before we prove Theorem 3.12 we will show the following corollary.

Corollary 3.13. *There exists a model of* ZFC + GCH *in which the following holds. For every* σ*-ideal* $\mathcal{J}$ *of* $\mathbb{R}$ *containing all singletons and such that every element of* $\mathcal{J}$ *is either meager or null, there exists a Hausdorff zero-dimensional topology* $\tau_{\mathcal{J}}$ *on* $\mathbb{R}$ *making* $\mathcal{J}$ *nowhere dense.*

In particular, there exist Hausdorff zero-dimensional topologies on $\mathbb{R}$ *making the following ideals nowhere dense: perfectly meager sets, universally measure-zero sets, strong measure-zero sets, null sets, etc.*

Proof. Let τ be a topology on $\mathbb{R}$ generated by the mapping $e:\mathbb{R}\to 2^{\omega_2}$ from Theorem 3.12. Then, a base of (R, τ) induced by a standard product basis of 2^{ω_2} is given by sets

$$U_\varepsilon=\{r\in\mathbb{R}\colon \varepsilon\subset e(r)\}$$
$$=\{r\in\mathbb{R}\colon (\forall a\in\operatorname{dom}(\varepsilon))[e(r)(a)=\varepsilon(a)]\}$$

for all $\varepsilon\in H(\omega_2)$, where $H(A)$ is defined as a set of all functions from finite subsets of A into 2. Notice that sets U_ε are nonempty, since $e[\mathbb{R}]$ is dense in 2^{ω_2}.

Let $\mathcal{J}$ be a σ-ideal as stated in the assumptions. We will define $\tau_{\mathcal{J}}$ by modifying τ.

Let $\beta<\omega_2$ be such that the sets $\{U_\varepsilon\colon \varepsilon\in H(\beta)\}$ separate points of $\mathbb{R}$. Such β can be found, since $|\mathbb{R}|=\omega_1$. Let

$$\Lambda=\{\alpha<\omega_2\colon \text{cofinality of } \alpha \text{ is } \omega_1\}$$

and let $\{J_\alpha\colon \alpha\in\Lambda\}$ be an enumeration of $\mathcal{J}$ such that

$$J_\alpha=\emptyset \quad \text{for all } \alpha<\beta. \tag{2}$$

For $\alpha \in \Lambda$, $\xi < \omega_1$ and $i < 2$ we put

$$U^{\mathscr{J}}_{\langle \alpha+\xi,i\rangle} = \begin{cases} U_{\{\langle \alpha+\xi,0\rangle\}} \setminus J_\alpha, & i=0, \\ U_{\{\langle \alpha+\xi,1\rangle\}} \cup J_\alpha, & i=1, \end{cases}$$

and define a base of $\tau_{\mathscr{J}}$ as a family of all sets

$$U^{\mathscr{J}}_\varepsilon = \bigcap_{a \in \mathrm{dom}(\varepsilon)} U^{\mathscr{J}}_{\langle a,\varepsilon(a)\rangle}$$

for $\varepsilon \in H(\omega_2)$.

Notice that

$$U^{\mathscr{J}}_\varepsilon \Delta U_\varepsilon \in \mathscr{J} \quad \text{for every } \varepsilon \in H(\omega_2). \tag{3}$$

Clearly, $(\mathbb{R}, \tau_{\mathscr{J}})$ is zero-dimensional. It is Hausdorff, since, by (2), $U^{\mathscr{J}}_\varepsilon = U_\varepsilon$ for all $\varepsilon \in H(\beta)$ and $\{U_\varepsilon\colon \varepsilon \in H(\beta)\}$ separates points. Thus, it is enough to show that the nowhere dense sets of $(\mathbb{R}, \tau_{\mathscr{J}})$ are precisely the sets from $\mathscr{J}$.

It is easy to see that sets from $\mathscr{J}$ are closed, since if $J \in \mathscr{J}$ and $\alpha \in \Lambda$ is such that $J = J_\alpha$ then, by (ii) of Theorem 3.12,

$$\mathbb{R} \setminus \bigcup_{\xi<\omega_1} U^{\mathscr{J}}_{\{\langle \alpha+\xi,0\rangle\}} = \mathbb{R} \setminus \left(\bigcup_{\xi<\omega_1} U_{\{\langle \alpha+\xi,0\rangle\}} \setminus J_\alpha \right) = \mathbb{R} \setminus (\mathbb{R} \setminus J_\alpha) = J.$$

Also, the sets from $\mathscr{J}$ are nowhere dense, since, by (iii) of Theorem 3.12, $U_\varepsilon \notin \mathscr{J}$, while, by (2), $U^{\mathscr{J}}_\varepsilon \Delta U_\varepsilon \in \mathscr{J}$ for every $\varepsilon \in H(\omega_2)$.

So, we have proved that every set from $\mathscr{J}$ is nowhere dense in $(\mathbb{R}, \tau_{\mathscr{J}})$. To finish the proof choose a closed nowhere dense set F in $(\mathbb{R}, \tau_{\mathscr{J}})$. We will show that $F \in \mathscr{J}$.

Let $\{\varepsilon_s \in H(\omega_2)\colon s \in S\}$ be such that $F = \mathbb{R} \setminus \bigcup_{s\in S} U^{\mathscr{J}}_{\varepsilon_s}$. Since F is nowhere dense, for every $\varepsilon \in H(\omega_2)$ there is an $s \in S$ such that $U^{\mathscr{J}}_\varepsilon \cap U^{\mathscr{J}}_{\varepsilon_s} \neq \emptyset$. This implies that for every $\varepsilon \in H(\omega_2)$ there exists an $s \in S$ such that

$$U_\varepsilon \cap U_{\varepsilon_s} \neq \emptyset.$$

By a simple closure operation we can find a countable infinite set $T \subset \omega_2$ such that for every $\varepsilon \in H(T)$ there is an $s \in S$ such that

$$U_\varepsilon \cap U_{\varepsilon_s} \neq \emptyset \quad \text{and} \quad \varepsilon_s \in H(T).$$

Let $S_0 = \{s \in S\colon \varepsilon_s \in H(T)\}$. Thus, S_0 is at most countable and for every $\varepsilon \in H(\omega_2)$ there is an $s \in S_0$ such that

$$U_\varepsilon \cap U_{\varepsilon_s} \neq \emptyset,$$

i.e., the set $F'_0 = \mathbb{R} \setminus \bigcup_{s\in S_0} U_{\varepsilon_s}$ is nowhere dense in $(\mathbb{R}, \tau)$. So, by (i) of Theorem 3.12, F'_0 is at most countable and, in particular, $F'_0 \in \mathscr{J}$. But if $F_0 = \mathbb{R} \setminus \bigcup_{s\in S_0} U^{\mathscr{J}}_{\varepsilon_s}$ then, by (3), $F_0 \Delta F'_0 \in \mathscr{J}$. Hence, $F_0 \in \mathscr{J}$. But $F \subset F_0$. Thus, $F \in \mathscr{J}$.

This finishes the proof of the corollary. □

Proof of Theorem 3.12. Let V be a model of ZFC + GCH. We will find a generic extension of V in which the theorem holds.

The forcing we will use is ω-closed and satisfies ω_2-cc. Thus, it preserves cardinal numbers and the real numbers from V and from its extension are the same. We will denote this set of real numbers by $\mathbb{R}$.

Let

$$\mathscr{E}=\left\{\{\varepsilon_n\}_{n<\omega}\in[H(\omega_2)]^{\leqslant\omega}:\ (\forall\varepsilon\in H(\omega_2))(\exists n<\omega)[\varepsilon\cup\varepsilon_n\in H(\omega_2)]\right\} \tag{4}$$

and let

$$\mathbb{S}=\left\{\{f_r\in H_\omega(\omega_2):\ r\in A\}:\ A\in[\mathbb{R}]^{\leqslant\omega}\right\},$$

where $H_\omega(X)$ stands for the functions from at most countable subsets of X into 2. The forcing notion we will use is defined as $\mathbb{P}=\mathbb{S}\times[\mathscr{E}]^{\leqslant\omega}$. The partial order on $\mathbb{P}$ is defined by

$$\begin{aligned}&\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\leqslant\langle\langle f'_r\rangle_{r\in A'},\ \mathscr{D}'\rangle\\ &\quad\Leftrightarrow\mathscr{D}\supset\mathscr{D}'\text{ and }A\supset A'\text{ and }(\forall r\in A')[f_r\supset f'_r]\\ &\qquad\text{and }(\forall r\in A\setminus A')(\forall\{\varepsilon_n\}\in\mathscr{D}')(\exists n<\omega)[\varepsilon_n\subset f_r]\end{aligned} \tag{5}$$

for all $\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle,\ \langle\langle f'_r\rangle_{r\in A'},\ \mathscr{D}'\rangle\in\mathbb{P}$.

It is easy to see that $\mathbb{P}$ is ω-closed. To see that $\mathbb{P}$ satisfies ω_2-cc take a sequence $\{\langle\langle f_r^\xi\rangle_{r\in A^\xi},\ \mathscr{D}^\xi\rangle\in\mathbb{P}:\ \xi<\omega_2\}$ and notice that we can choose a subset $I\in[\omega_2]^{\omega_2}$ such that $A^\xi=A^\zeta$ for all $\xi,\ \zeta\in I$. Now, using CH and a standard Δ-system argument, we can find $\xi,\zeta\in I$ such that $f_r^\xi\cup f_r^\zeta\in H_\omega(\omega_2)$ for every $r\in A=A^\xi=A^\zeta$. Clearly, $\langle\langle f_r^\xi\cup f_r^\zeta\rangle_{r\in A},\ \mathscr{D}^\xi\cup\mathscr{D}^\zeta\rangle$ extends $\langle\langle f_r^\xi\rangle_{r\in A^\xi},\ \mathscr{D}^\xi\rangle$ and $\langle\langle f_r^\zeta\rangle_{r\in A^\zeta},\ \mathscr{D}^\zeta\rangle$.

Now, let G be a V-generic filter over $\mathbb{P}$ and define a mapping $e:\mathbb{R}\to2^{\omega_2}$ by putting for $s\in\mathbb{R}$,

$$e(s)=\bigcup\left\{g_s:\ (\exists\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\in G)[s\in A\text{ and }g_s=f_s]\right\}.$$

In order to show that this definition is correct it has to be argued that the following sets are dense in $\mathbb{P}$ for every $s\in\mathbb{R}$ and $\xi<\omega_1$,

$$D_s=\left\{\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\in\mathbb{P}:\ s\in A\right\}$$

and

$$D_s^\xi=\left\{\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\in\mathbb{P}:\ s\in A\text{ and }\xi\in\operatorname{dom}(f_s)\right\}.$$

To see that D_s is dense in $\mathbb{P}$ take $\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\in\mathbb{P}$ and assume that $s\notin A$. Let $\mathscr{D}=\{\{\varepsilon_n^k\}:\ k<\omega\}$ and define by induction on $m<\omega$ a sequence $\{n_m\}_{m<\omega}$ such that $\bigcup_{k<m}\varepsilon_{n_k}^k\in H(\omega_2)$ for every $m<\omega$. This can be done by the definition (4) of $\mathscr{E}$. Then, $f_s=\bigcup_{k<\omega}\varepsilon_{n_k}^k\in H_\omega(\omega_2)$ and $\langle\langle f_r\rangle_{r\in A\cup\{s\}},\ \mathscr{D}\rangle$ extends $\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle$.

To see that D_s^ξ is dense in $\mathbb{P}$ take $\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\in\mathbb{P}$. By the density of D_s we can assume that $s\in A$. If $\xi\notin\operatorname{dom}(f_s)$, extend f_s onto ξ arbitrarily and notice that such obtained condition extends $\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle$.

Thus, we proved that indeed $e:\mathbb{R}\to2^{\omega_2}$. To see that e is one-to-one it is enough to see that the set

$$\begin{aligned}D_{s,t}=\{\langle\langle f_r\rangle_{r\in A},\ \mathscr{D}\rangle\in\mathbb{P}:\ &s,\ t\in A\text{ and}\\ &(\exists\xi\in\operatorname{dom}(f_s)\cap\operatorname{dom}(f_t))[f_s(\xi)\neq f_t(\xi)]\}\end{aligned}$$

is dense in $\mathbb{P}$ for every distinct $s, t \in \mathbb{R}$. Similarly, the density of $e[\mathbb{R}]$ in 2^{ω_2} follows from the density of

$$D_\varepsilon = \{\langle\langle f_r\rangle_{r\in A}, \mathscr{D}\rangle \in \mathbb{P}\colon (\exists r \in A)[\varepsilon \subset f_r]\}$$

for every $\varepsilon \in H(\omega_2)$. The density of both these types of sets can be proved similarly as that of D_s^ξ and D_s, respectively. We will leave it as an exercise.

To argue for (iii) let $\varepsilon \in H(\omega_2)$ and let J be either meager or null. We will show that $U_\varepsilon \not\subset J$. We can assume that J is a Borel set, extending it, if necessary. Thus, J is already in V and the set

$$D_\varepsilon^J = \{\langle\langle f_r\rangle_{r\in A}, \mathscr{D}\rangle \in \mathbb{P}\colon (\exists r \in A\setminus J)[\varepsilon \subset f_r]\}$$

belongs to V. The density of this set is proved similarly as that of D_ε. This easily implies (iii).

To prove (ii) it is enough to notice that for every $\alpha < \omega_2$ and $s \in \mathbb{R}$ the set

$$E_s^\alpha = \{\langle\langle f_r\rangle_{r\in A}, \mathscr{D}\rangle \in \mathbb{P}\colon s \in A \text{ and } (\exists\xi < \omega_1)[\langle \alpha+\xi, 0\rangle \in f_s]\}$$

is dense in $\mathbb{P}$. The proof of this fact is identical to the proof of the density of D_s^ξ.

To finish the proof it is enough to show (i). So, let F be a closed nowhere dense subset of $e[\mathbb{R}]$. Choose a family $\{\varepsilon_s \in H(\omega_2)\colon s \in S\}$ such that $F = e[\mathbb{R}]\setminus \bigcup_{s\in S}[\varepsilon_s]$, where $[\varepsilon] = \{f \in 2^{\omega_2}\colon \varepsilon \subset f\}$. Since F is nowhere dense in $e[\mathbb{R}]$, for every $\varepsilon \in H(\omega_2)$ there is $s \in S$ such that $e[\mathbb{R}] \cap [\varepsilon] \cap [\varepsilon_s] \neq \emptyset$. Similarly as in Corollary 3.13 we can find a countable subset S_0 of S such that for every $\varepsilon \in H(\omega_2)$ there is $s \in S_0$ such that $e[\mathbb{R}] \cap [\varepsilon] \cap [\varepsilon_s] \neq \emptyset$. Notice that $F_0 = e[\mathbb{R}]\setminus \bigcup_{s\in S_0}[\varepsilon_s]$ contains F and that $D = \{\varepsilon_s\colon s \in S_0\} \in \mathscr{E}$. We will show that F_0 is at most countable.

Since the set

$$E_D = \{\langle\langle f_r\rangle_{r\in A}, \mathscr{D}\rangle \in \mathbb{P}\colon D \in \mathscr{D}\}$$

is dense in $\mathbb{P}$ we can find $\langle\langle f_r'\rangle_{r\in A'}, \mathscr{D}'\rangle \in G$ such that $D \in \mathscr{D}'$. We will show that $F_0 \subset A'$.

Let $t \in \mathbb{R}\setminus A'$. It is enough to show that $e[t] \in \bigcup_{s\in S_0}[\varepsilon_s]$. Choose $\langle\langle f_r\rangle_{r\in A}, \mathscr{D} > \in G$ extending $\langle\langle f_r'\rangle_{r\in A'}, \mathscr{D}'\rangle$ and such that $t \in A$. Then, by (5), there exists $s \in S_0$ such that $\varepsilon_s \subset f_t \subset e(t)$. Thus, $e(t) \in [\varepsilon_s]$.

This finishes the proof of the theorem. □

4. Remarks on Theorem 3.12

First notice that if we assume only the Continuum Hypothesis then the following weaker version of Theorem 3.12 can be proved.

Theorem 4.1 (CH). *There exists a one-to-one mapping $e\colon \mathbb{R} \to 2^{\omega_1\times\omega_1}$ such that $e[\mathbb{R}]$ is a dense subspace of $2^{\omega_1\times\omega_1}$ and:*

(i) *every nowhere dense subset of $e[\mathbb{R}]$ is at most countable;*

(ii) *for every $\alpha < \omega_1$ and every distinct $r, s \in \mathbb{R}$ there exist $\xi < \eta < \omega_1$ such that $e(r)(\langle\alpha, \xi\rangle) = 0$ and $e(s)(\langle\alpha, \eta\rangle) = 1$;*

(iii) $e[A]$ *has an empty interior in* $e[\mathbb{R}]$ *for every* $A \subset \mathbb{R}$ *which is either meager or null.*

Sketch of proof. Change the definition of forcing $\mathbb{P}$ in Theorem 3.12 by replacing ω_2 with $\omega_1 \times \omega_1$.

Let $\mathscr{F}$ be the family of all dense subsets of $\mathbb{P}$ considered in the proof of Theorem 3.12, i.e., sets D_s, D_s^ξ, $D_{s,t}$, D_ε^J, E_s^α and E_D, where indexes are chosen as in the proof. (Evidently in the definition of E_s^α we replace $\alpha+\xi$ with $\langle\alpha, \xi\rangle$.) Then $\mathscr{F}$ has cardinality ω_1 and we can easily construct, by transfinite induction of length ω_1, a filter G in $\mathbb{P}$ intersecting every set in $\mathscr{F}$. But this is all we need to conclude (i)–(iii). □

Although Theorem 4.1 is very similar to Theorem 3.12 we cannot, in general, deduce from it the conclusion of Corollary 3.13. This is the case, since to this order we would need to modify the topology of $e[\mathbb{R}] \subset 2^{\omega_1 \times \omega_1}$ by $|\mathscr{J}|$-many sets and we have only ω_1 coordinates to make this adjustment. However, if there exists $\{J_\alpha: \alpha < \omega_1\} \subset \mathscr{J}$ cofinal in $\mathscr{J}$ [2] (i.e., such that for every $J \in \mathscr{J}$ there is $\alpha < \omega_1$ with $J \subset J_\alpha$), then similarly as in Corollary 3.13 we can find a zero-dimensional Hausdorff topology on $\mathbb{R}$ making $\mathscr{J}$ meager. In particular, we can conclude the following:

Corollary 4.2 (CH). *If* $\mathscr{J}$ *is a* σ*-ideal on* $\mathbb{R}$ *containing all singletons and having cofinality* ω_1 *(i.e., with cofinal subfamily of cardinality* ω_1*) then there is a zero-dimensional Hausdorff topology on* $\mathbb{R}$ *making* $\mathscr{J}$ *nowhere dense.*

In particular, there exist zero-dimensional Hausdorff topologies making null sets and (ordinary) meager sets nowhere dense.

In the recent years two refinements of the natural topology on $\mathbb{R}$ have been intensively studied: the density topology and the $\mathscr{J}$-density topology. (For summary of topological properties of these topologies see [2].) They make, respectively, null sets and ordinary meager sets nowhere dense. However, both of these topologies are connected. Moreover, the $\mathscr{J}$-density topology is Hausdorff but not regular. The density topology is completely regular but not normal. In this context the following questions seems to be interesting.

Problem 4.3. Can we find in ZFC a zero-dimensional Hausdorff topology on $\mathbb{R}$ making meager sets nowhere dense?

It is easy to see that under Martin's Axiom such a topology exists.

[2] Under CH it is not the case for the ideals of perfectly meager sets and universally null sets. This follows from Lemma 4.10.

Problem 4.4. Can topologies from Corollaries 3.13 or 4.2 be normal? Compact? Metrizable? In particular, can we have such topologies for the ideals of meager sets or null sets?

Notice that neither of Theorem 3.12, Corollary 3.13 and Corollary 4.2 can be proved in full generality in ZFC. More precisely, Corollaries 3.13 and 4.2 fail under MA + $\neg$CH for the ideal $[\mathbb{R}]^{\leqslant\omega}$ by Fact 3.7. The corollaries also fail for the ideal *SMZ* of strong measure-zero sets under the Borel Conjecture, i.e., when $SMZ = [\mathbb{R}]^{\leqslant\omega}$. This is the case since every Lusin set has strong measure zero, i.e., the Borel Conjecture implies that there are no Lusin sets.

In what follows we will present few additional examples showing farther limitations on possible generalizations of Theorem 3.12 and Corollary 3.13 or 4.2.

The first fact shows that the assumptions on the ideal $\mathscr{I}$ in Corollary 3.13 cannot be completely disregarded if we like to make $\mathscr{I}$ nowhere dense.

Fact 4.5. *If X has at least two elements and $\mathscr{I}$ is a maximal ideal on X, then there is no Hausdorff topology making $\mathscr{I}$ nowhere dense.*

Proof. Suppose that τ is a Hausdorff topology on X making $\mathscr{I}$ nowhere dense. By the maximality of $\mathscr{I}$ out of every two disjoint open sets one must belong to $\mathscr{I}$. So, X cannot have two disjoint nonempty open sets. Thus, as a Hausdorff space, X must be a singleton. □

If we like to make $\mathscr{I}$ only meager in Corollary 3.13 the situation is not so clear. The next fact shows that we cannot make a measurable ideal meager by a Hausdorff topology.

Fact 4.6. *Assume that there exists a measurable cardinal κ, let $|X| \geqslant \kappa$ and let $\mathscr{I}$ be a maximal σ-ideal on X containing all singletons. Then there is no Hausdorff topology making $\mathscr{I}$ meager.*

Proof. By way of contradiction assume that there is a Hausdorff topology τ on X making $\mathscr{I}$ meager. The Hausdorff property implies that all points, with possible one exception, have open meager neighborhoods. Let $\mathscr{U}$ be a maximal family of pairwise disjoint open meager sets. Then, $\mathbb{R} \setminus \bigcup \mathscr{U}$ is closed and nowhere dense. To get a contradiction it is enough to show that $\bigcup \mathscr{U}$ is meager in X. But this follows immediately from the following fact.

$$\text{If } \mathscr{U} \text{ is a family of disjoint open and meager sets then } \bigcup \mathscr{U} \text{ is meager.} \qquad (6)$$

To argue for (6) let N_U^n be nowhere dense subsets of X such that $U = \bigcup_{n<\omega} N_U^n$ for every $U \in \mathscr{U}$. Then every set $N_n = \bigcup_{U \in \mathscr{U}} N_U^n$ is nowhere dense since sets from $\mathscr{U}$ separate sets $\{N_U^n\}_{U \in \mathscr{U}}$. So, $\bigcup \mathscr{U} = \bigcup_{n<\omega} N_n$ is meager.

The fact has been proved. □

Clearly in Fact 4.6 the cardinality of X is greater than $\mathfrak{c}$. It could be expected that the same result could be obtained if you replace the 0-1 universal measure with a finite universal measure, i.e., a measurable cardinal with a real measurable cardinal. However, this is not the case, as our next example shows.

Example 4.7. If μ is a universal σ-additive measure on $\mathbb{R}$ which extends the Lebesgue measure and $\mathscr{I}=\{X: \mu(X)=0\}$, then there is a Hausdorff topology on $\mathbb{R}$ making $\mathscr{I}$ nowhere dense.

Proof. Let τ be the density topology on $\mathbb{R}$, i.e., τ is the family of all measurable subsets of $\mathbb{R}$ such that every point of A is its density point. (See [14, p. 90].) Then $\tau^{\mathscr{I}}=\{U\setminus I: U\in\tau \text{ and } I\in\mathscr{I}\}$ is a Hausdorff topology making $\mathscr{I}$ nowhere dense. To see this notice that $\tau\cap\mathscr{I}=\{\emptyset\}$ so, by the "$\Leftarrow$" part of Lemma 3.9 (the assumption of Lemma 3.9 that X is second countable was not used in the proof of part "$\Leftarrow$"), we have $\mathscr{I}\subset N(\tau, \mathbb{R})$. On the other hand let $Y\in N(\tau, \mathbb{R})$ and let $\mathscr{V}=\{V\in\tau: Y\cap V\in\mathscr{I}\}=\{V\in\tau: (\exists I\in\mathscr{I})[Y\cap V\setminus I=\emptyset]\}$. It is easy to see that $\bigcup\mathscr{V}$ is τ-dense in $\mathbb{R}$, since $\mathbb{R}\setminus\bigcup\mathscr{V}\subset Y$. Let $\mathscr{V}'$ be some maximal pairwise disjoint subfamily of $\mathscr{V}$. Then, $\mathscr{V}'$ is countable and $\bigcup\mathscr{V}'$ is also τ-dense in $\mathbb{R}$. Now, we have $Y\subset\bigcup_{V\in\mathscr{V}'}(V\cap Y)\cup(\mathbb{R}\setminus\bigcup\mathscr{V}')$. But $\mathbb{R}\setminus\bigcup\mathscr{V}'\in N(\tau, \mathbb{R})$ so it is Lebesgue null. It follows that $\mu(Y)=0$. □

It is worth noticing that the topology $\tau^{\mathscr{I}}$ from Example 4.7 is not normal. To see this recall that the density topology τ on $\mathbb{R}$ is not normal [14, p. 90]. So, let C_1 and C_2 be two disjoint τ-closed subsets of $\mathbb{R}$ which cannot be separated by τ. Suppose that C_1 and C_2 can be separated by $\tau^{\mathscr{I}}$. Then C_1 and C_2 are contained in disjoint sets $E_1\setminus I_1$ and $E_2\setminus I_2$, respectively, where E_1 and E_2 are in τ and $\mu(I_1)=\mu(I_2)=0$. We may assume that $E_1\cap C_2=E_2\cap C_1=\emptyset$. Since $E_1\cap E_2\subseteq I_1\cup I_2$ and μ is an extension of the Lebesgue measure $E_1\cap E_2$ is Lebesgue null. It follows that $E_1\setminus E_2$, $E_2\setminus E_1\in\tau$ separate C_1 and C_2, which is a contradiction.

Recall that the *weight* of a topological space is defined by

$$w(X)=\min\{|\mathscr{B}|: \mathscr{B} \text{ is an open basis for } \tau\}+\omega.$$

The *cellularity* of X is defined as

$$c(X)=\sup\{|\mathscr{C}|: \mathscr{C} \text{ is a pairwise disjoint family of open sets}\}+\omega.$$

If $\mathscr{I}$ is an ideal on a set X then $\mathscr{C}\subset\mathscr{I}$ is *cofinal* in $\mathscr{I}$ if for each $I\in\mathscr{I}$ there exists a set $C\in\mathscr{C}$ such that $I\subset C$. We define *confinality* of $\mathscr{I}$ by

$$\operatorname{cf}(\mathscr{I})=\min\{|\mathscr{C}|: \mathscr{C}\subset\mathscr{I} \text{ is cofinal in } \mathscr{I}\}.$$

Lemma 4.8. *Let τ be a topology on a set X. Then* $\operatorname{cf}(M(\tau, X))\leqslant w(X)^{c(X)}$.

Proof. Let $\mathscr{B}$ be a basis for τ with $|\mathscr{B}|\leqslant w(X)$. For every $A\in N(\tau, X)$ there exists a maximal pairwise disjoint family $\mathscr{C}_A\subset\mathscr{B}$ such that $A\cap\bigcup\mathscr{C}_A=\emptyset$. Clearly, $\bigcup\mathscr{C}_A$ is dense in X. It follows that

$$\mathscr{I}=\left\{\mathbb{R}\setminus\bigcup\mathscr{C}': \mathscr{C}'\subset\mathscr{B} \text{ is a family of pairwise disjoint sets}\right\}$$

is cofinal in $N(\tau, X)$. Hence, $\mathrm{cf}(N(\tau, X)) \leqslant |\mathscr{J}| \leqslant w(X)^{c(X)}$. So

$$\mathrm{cf}(M(\tau, X)) \leqslant [\mathrm{cf}(N(\tau, X))]^{\omega} \leqslant \left(w(X)^{c(X)}\right)^{\omega} = w(X)^{c(X)}. \qquad \square$$

Recall that an ideal $\mathscr{J}$ on X is *ω_1-saturated* if every family $\mathscr{F} \subset \mathscr{P}(X) \setminus \mathscr{J}$ of pairwise disjoint sets is at most countable. In particular, the ideal $\mathscr{J}$ from Example 4.7 is ω_1-saturated. It also contains $[\mathbb{R}]^{<\mathfrak{c}}$ if measure μ is continuum additive. Thus, the next fact tells us, in particular, that for a continuum additive measure μ in Example 4.7 the ideal $\mathscr{J}$ cannot be made meager by a metrizable topology.

Fact 4.9. *Let $|X| = \mathfrak{c}$ and let $\mathscr{J}$ be an ω_1-saturated σ-ideal on X such that $[X]^{<\mathfrak{c}} \subset \mathscr{J}$. If a Hausdorff topology τ on X is making $\mathscr{J}$ meager then $w(X, \tau) \leqslant \mathfrak{c}$. In particular, there is no metrizable topology making $\mathscr{J}$ meager.*

Proof. Let τ be a Hausdorff topology making $\mathscr{J}$ meager and by way of contradiction, assume that $w(X, \tau) \leqslant \mathfrak{c}$. Condition (6) in the proof of Fact 4.6 shows that the set of all points which have meager open neighborhoods is meager. Therefore, without loss of generality, we may assume that $\tau \cap \mathscr{J} = \{\emptyset\}$. Hence, $c(X) \leqslant \omega$, since $\mathscr{J}$ is ω_1-saturated. So, by Lemma 4.8, we obtain $\mathrm{cf}(M(\tau, X)) \leqslant \mathfrak{c}$. Let $\{M_\beta$: $\beta < \mathfrak{c}\}$ be a cofinal family in $M(\tau, X)$. For $\alpha < \omega_1$ and $\beta < \mathfrak{c}$, select

$$x_\alpha^\beta \in X \setminus \left[M_\beta \cup \{x_\delta^\gamma\colon \gamma = \beta \text{ and } \delta < \alpha \text{ or } \gamma < \beta\}\right].$$

Clearly, the sets $X_\alpha = \{x_\alpha^\beta\colon \beta < \mathfrak{c}\} \notin \mathscr{J}$ are pairwise disjoint subsets of $\mathbb{R}$. But this contradicts ω_1-saturation of $\mathscr{J}$. $\square$

We would like to conclude this discussion with a few remarks on universally null and perfectly meager sets. Recall that a set $X \subseteq \mathbb{R}$ is *perfectly meager* ($X \in PM$) if $P \cap X$ is meager in P for every perfect set $P \subseteq \mathbb{R}$. A set $Y \subseteq \mathbb{R}$ is *universally null* ($Y \in UN$) if $\mu(Y) = 0$ for every continuous Borel probability measure on $\mathbb{R}$. For more information on PM and UN see [11] or [1]. In general these two ideals have many similar properties. In particular, it is an open problem whether $UN \neq PM$ is provable in ZFC [3]. In this situation it would be particularly desirable to know more about topologies making UN or PM meager.

For an ideal $\mathscr{J}$ on a set X we define

$$\mathrm{non}(\mathscr{J}) = \min\{|Y|\colon Y \subseteq X \text{ and } Y \notin \mathscr{J}\}.$$

Lemma 4.10. $\mathrm{cf}(UN) > \mathrm{non}(UN)$ *and* $\mathrm{cf}(PM) > \mathrm{non}(PM)$.

Proof. Following an idea of Reclaw [15] we take two disjoint compact perfect subsets C and D of a linearly independent perfect set $Z \subseteq \mathbb{R}$. (See [12, Theorem 19, p. 206].) The mapping $h: C \times D \to C + D$, $h(c, d) = c + d$, is a homeomorphism.

To prove $\mathrm{cf}(UN) > \mathrm{non}(UN)$, choose a universally null set $X \subseteq D$ such that $|X| = \mathrm{non}(UN)$. We can find such a choice by a theorem of Grzegorek [5]. Notice that any selector Y from $\{C+x: x \in X\}$ is also universally null because $g=\pi_2 \circ h^{-1}$ (π_2 is projection onto the second coordinate) is an injective continuous mapping from Y onto X. Now, take any family $\mathscr{C}=\{C_x: x \in X\}$ of UN sets such that $|\mathscr{C}|=|X|=\mathrm{non}(UN)$. We will show that $\mathscr{C}$ does not cover UN, i.e., that $\mathrm{cf}(UN) > |\mathscr{C}|=\mathrm{non}(UN)$. To this end for every $x \in X$ select $y_x \in (C+x)\setminus C_x$. Then, $Y=\{y_x: x \in X\} \in UN$ but $Y \not\subseteq C_x$ for any $x \in X$. Hence $\mathscr{C}$ is not cofinal in UN.

The proof of $\mathrm{cf}(PM) > \mathrm{non}(PM)$ is similar. We just need to take X to be Grzegorek's set from Theorem 1 of [6]. □

Since under Martin's Axiom $\mathrm{non}(UN)=\mathrm{non}(PM)=\mathfrak{c}$, Lemma 4.8 yields the following.

Theorem 4.11 (MA). *There is no second countable topology on* $\mathbb{R}$ *making UN or PM meager.*

The following problem seems to be interesting.

Problem 4.12. How good topologies making *UN* or *PM* meager (or nowhere dense) can be found in ZFC?

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