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Density continuous transformations on $\mathbb{R}^2$

Abstract

In the paper we study transformations from $\mathbb{R}^2$ into $\mathbb{R}$ and from $\mathbb{R}^2$ into $\mathbb{R}^2$ continuous with respect to different density topologies on the domain and range. In the former case the range will always be equipped with the one-dimensional density topology and the domain with either ordinary density or strong density topology. In the later case the domain and the range will be equipped simultaneously with the ordinary density or strong density topology. We will investigate the relations between these classes and with the class of ordinary continuous transformations. We will also examine relation between the (strong) density continuity of $f: \mathbb{R}^2 \to \mathbb{R}$ and of $f(\cdot, y)$, $f(x, \cdot)$. Similar question will be considered for transformations $F = (f, g): \mathbb{R}^2 \to \mathbb{R}^2$ and their coordinate functions f and g.

1 Preliminaries

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ or $\mathbb{R}^1$ stands for the set of real numbers and $\mathbb{R}^2$ for the plane. For $A \subset \mathbb{R}^2$ its outer two-dimensional Lebesgue measure is denoted by $\mathrm{m}_2(A)$. Similarly, $\mathrm{m}_1(A)$ stands for the outer one-dimensional Lebesgue measure of $A \subset \mathbb{R}$.

Recall that $x \in \mathbb{R}$ is a density point of $A \subset \mathbb{R}$ if its density

$$d_1(A, x) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{m}_1(A \cap (x-\varepsilon, x+\varepsilon))}{\mathrm{m}_1((x-\varepsilon, x+\varepsilon))} = 1;$$

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$(x,y) \in \mathbb{R}^2$ is an (ordinary) density point of $A \subset \mathbb{R}^2$ if

$$d_2(A,(x,y)) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{m}_2\left(A \cap [(x-\varepsilon,x+\varepsilon)\times(y-\varepsilon,y+\varepsilon)]\right)}{\mathrm{m}_2\left((x-\varepsilon,x+\varepsilon)\times(y-\varepsilon,y+\varepsilon)\right)} = 1;$$

and $(x,y) \in \mathbb{R}^2$ is a strong density point of $A \subset \mathbb{R}^2$ if

$$d_s(A,(x,y)) = \lim_{a\to 0^+,\, b\to 0^+} \frac{\mathrm{m}_2\left(A \cap [(x-a,x+a)\times(y-b,y+b)]\right)}{\mathrm{m}_2\left((x-a,x+a)\times(y-b,y+b)\right)} = 1.$$

(Compare Saks [11, pages 106, 128].) Recall also that a point p is a (strong) dispersion point of a set B if it is a (strong) density point of the complement of B. The strong density topology $\mathcal{T}_\mathcal{S}^2$ on $\mathbb{R}^2$ is defined as the family of all measurable subsets A of $\mathbb{R}^2$ such that every $a \in A$ is a strong density point of A [8]. Similarly we define density topologies $\mathcal{T}_\mathcal{N}^1$ and $\mathcal{T}_\mathcal{N}^2$ on $\mathbb{R}$ and $\mathbb{R}^2$, respectively, using the notions of density point on $\mathbb{R}$ and ordinary density point on $\mathbb{R}^2$. (Compare [8] and [10].) We will drop the superscript in this notation and write $\mathcal{T}_\mathcal{N}$ in all cases when the space is fixed. Notice also that $\mathcal{T}_\mathcal{S}^2 \subset \mathcal{T}_\mathcal{N}^2$ and recall that the sets open in these topologies are measurable. The symbol $\mathcal{T}_\mathcal{O}$ stands for the ordinary topology on $\mathbb{R}$ or on $\mathbb{R}^2$.

The class of all functions from $\mathbb{R}^2$ to $\mathbb{R}$ with the density topology $\mathcal{T}_\mathcal{N}^1$ on the range and either ordinary density topology $\mathcal{T}_\mathcal{N}^2$ or strong density topology $\mathcal{T}_\mathcal{S}^2$ on the domain will be denoted by $\mathcal{C}(\mathcal{T}_\mathcal{N}^2,\mathcal{T}_\mathcal{N}^1)$ and $\mathcal{C}(\mathcal{T}_\mathcal{S}^2,\mathcal{T}_\mathcal{N}^1)$, respectively. They will be termed density and strongly density continuous transformations, respectively. Notice that $\mathcal{C}(\mathcal{T}_\mathcal{S}^2,\mathcal{T}_\mathcal{N}^1) \subset \mathcal{C}(\mathcal{T}_\mathcal{N}^2,\mathcal{T}_\mathcal{N}^1)$.

The class of all functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ which are continuous with respect to $\mathcal{T}_\mathcal{N}$ ($\mathcal{T}_\mathcal{S}^2$, respectively) on the domain and the range will be denoted by $\mathcal{C}(\mathcal{T}_\mathcal{N}^2,\mathcal{T}_\mathcal{N}^2)$ ($\mathcal{C}(\mathcal{T}_\mathcal{S}^2,\mathcal{T}_\mathcal{S}^2)$, respectively). Functions belonging to $\mathcal{C}(\mathcal{T}_\mathcal{N}^2,\mathcal{T}_\mathcal{N}^2)$ and $\mathcal{C}(\mathcal{T}_\mathcal{S}^2,\mathcal{T}_\mathcal{S}^2)$ are called density continuous and strongly density continuous transformations, respectively.

The class of all ordinarily continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ will be denoted by $\mathcal{C}(\mathcal{T}_\mathcal{O},\mathcal{T}_\mathcal{O})$.

Let us notice that the topologies $\mathcal{T}_\mathcal{N}^2$ and $\mathcal{T}_\mathcal{S}^2$ are invariant under translations and exchange of coordinates. Thus, $C(x,y) = (y,x)$ and the translations $T_{(s,t)}(x,y) = (x+s,y+t)$ are both density and strongly density continuous.

Functions from $\mathbb{R}$ to $\mathbb{R}$ continuous with respect to the density topology on the domain and the range are called density continuous. The next proposition lists some useful properties of these functions.

Proposition 1.1 *If $f\colon \mathbb{R}\to\mathbb{R}$ is density continuous, then*

(a) *f is Baire 1 function; in particular, f is measurable;*

(b) *f has the Darboux property;*

(c) *if f is not constant, then there exists $p \in f[\mathbb{R}]$ such that* $\mathrm{m}_1\left(f^{-1}(p)\right) = 0$.

PROOF. (a) and (b) can be found in [1], if we recall that density continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are approximately continuous [9].

(c) follows from (a) and (b) since $\{f^{-1}(p)\colon p \in f[\mathbb{R}]\}$ is a partition of $\mathbb{R}$ into uncountable many measurable sets. □

In what follows $|v|$ stands for the length of a vector $v \in \mathbb{R}^n$. For an ordinary open subset U of $\mathbb{R}^n$ we say that $F\colon U \to \mathbb{R}^n$ is *bi-Lipschitz with constant L* ($L \geq 1$), if for every $q, r \in U$

$$L^{-1}|q-r| \leq |F(q) - F(r)| \leq L|q-r|.$$

Transformation $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is *locally bi-Lipschitz* if for every point $p \in \mathbb{R}^n$ there is $U \in \mathcal{T}_O$ containing p such that the restricted transformation $F|_U$ is bi-Lipschitz. We say that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is *locally bi-Lipschitz* if for every point $p \in \mathbb{R}^2$ there is an open rectangle $U = (a,b)\times(c,d)$ containing p and a constant $L \geq 1$ such that for every $x_0 \in (a,b)$ and $y_0 \in (c,d)$ the coordinate functions: $g_{y_0}\colon (a,b) \to \mathbb{R}$, $g_{y_0}(x) = f(x, y_0)$, and $h_{x_0}\colon (c,d) \to \mathbb{R}$, $h_{x_0}(y) = f(x_0, y)$, are bi-Lipschitz with constant L.

Recall also the following facts.

Proposition 1.2 *Locally convex functions from $\mathbb{R}$ to $\mathbb{R}$ are density continuous. In particular, analytic and piecewise linear functions from $\mathbb{R}$ to $\mathbb{R}$ are density continuous.*

PROOF. See [4]. □

Proposition 1.3 *Locally bi-Lipschitz transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ are density continuous.*

PROOF. See [2]. □

We will finish with the following easy proposition.

Proposition 1.4 *If $f, g\colon \mathbb{R}^n \to \mathbb{R}$, $n = 1, 2, \ldots$, are density (strongly density) continuous, then* $\max\{f,g\}$ *and* $\min\{f,g\}$ *are density (strongly density) continuous.*

PROOF. The proof of this proposition is precisely the same as the one for the density continuous functions from $\mathbb{R}$ to $\mathbb{R}$. (See [5, Theorem 2] or [3, Theorem 1(iii)].) □

2 Transformations from $\mathbb{R}^2$ into $\mathbb{R}$.

In this section we investigate the relation between (strong) density continuity of transformations $f\colon \mathbb{R}^2 \to \mathbb{R}$ and density continuity of their sections $f(x_0,\cdot)$ and $f(\cdot,y_0)$. We will also study the relations between $\mathcal{C}(\mathcal{T}^2_{\mathcal{N}},\mathcal{T}^1_{\mathcal{N}})$, $\mathcal{C}(\mathcal{T}^2_{\mathcal{S}},\mathcal{T}^1_{\mathcal{N}})$ and $\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})$. We start with the following easy fact.

Proposition 2.1 *Let $h\colon \mathbb{R}\to\mathbb{R}$ and define $f\colon\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=h(y)$. The following conditions are equivalent.*

(a) *h is density continuous.*

(b) *f is density continuous.*

(c) *f is strongly density continuous.*

PROOF. Let $A\in\mathcal{T}^1_{\mathcal{N}}$. Then $f^{-1}(A)=\mathbb{R}\times h^{-1}(A)$. It is easy to see that $h^{-1}(A)\in\mathcal{T}^1_{\mathcal{N}}$ if and only if $\mathbb{R}\times h^{-1}(A)\in\mathcal{T}^2_{\mathcal{N}}$ if and only if $\mathbb{R}\times h^{-1}(A)\in\mathcal{T}^2_{\mathcal{S}}$.
□

The next theorem will be fundamental in constructing most of the examples.

Theorem 2.2 *If $f\colon\mathbb{R}^2\to\mathbb{R}$ is locally bi-Lipschitz, then f is strongly density continuous.*

PROOF. Let $(x_0,y_0)\in\mathbb{R}^2$. We will show that f is strongly density continuous at (x_0,y_0). Without loss of generality we may assume that $(x_0,y_0)=(0,0)$ and that $f(0,0)=0$. Moreover, assume that $L\ge 1$ is such that for every $x_0,y_0\in(-1,1)$ the coordinate functions: $g_{y_0}\colon(-1,1)\to\mathbb{R}$, $g_{y_0}(x)=f(x,y_0)$, and $h_{x_0}\colon(-1,1)\to\mathbb{R}$, $h_{x_0}(y)=f(x_0,y)$ are bi-Lipschitz with constant L.

Let $A\subset\mathbb{R}$ be measurable such that $0\notin A$ and 0 is a dispersion point of A. It is enough to prove that $(0,0)$ is a strong dispersion point of $f^{-1}(A)$. Let $\varepsilon>0$. Since, by Proposition 1.3, functions g_0 and h_0 are density continuous, 0 is a dispersion point of $g_0^{-1}(A)$ and $h_0^{-1}(A)$. In particular, there exists $\delta_0>0$ such that for every $\delta\in(0,\delta_0)$

$$\mathrm{m}_1\left(g_0^{-1}(A)\cap(-\delta,\delta)\right)<2\delta\,\frac{\varepsilon}{8L^4}\quad\&\quad \mathrm{m}_1\left(h_0^{-1}(A)\cap(-\delta,\delta)\right)<2\delta\,\frac{\varepsilon}{8L^4}. \tag{1}$$

We will show that for every rectangle $R=(-a,a)\times(-b,b)$ with $0<a,b<\delta_0/(2L^2)$ we have

$$\mathrm{m}_2\left(f^{-1}(A)\cap R\right)\le\varepsilon\,\mathrm{m}_2\left(R\right).$$

This will finish the proof.

So choose a rectangle $R = (-a,a) \times (-b,b)$ with $0 < a,b < \delta_0/(2L^2)$. We assume that $a \le b$, the other case being similar. In the calculation that follows we will use the Fubini Theorem and the fact that for every bi-Lipschitz function $\psi\colon \mathbb{R} \to \mathbb{R}$ with constant L and for every measurable set $B \subset \mathbb{R}$ we have $\mathrm{m}_1\left(\psi^{-1}(B)\right) \le L\,\mathrm{m}_1(B)$.

$$
\begin{aligned}
\mathrm{m}_2\left(f^{-1}(A) \cap R\right) &= \int_{-a}^{a} \mathrm{m}_1\left(h_x^{-1}(A) \cap (-b,b)\right) dx \\
&\le \int_{-a}^{a} \mathrm{m}_1\left(h_x^{-1}(A \cap (h_x(0) - Lb, h_x(0) + Lb))\right) dx \\
&\le \int_{-a}^{a} L\,\mathrm{m}_1\left(A \cap (g_0(x) - Lb, g_0(x) + Lb)\right) dx \\
&= \int_{-a}^{a} L\,\mathrm{m}_1\left(g_0 \circ g_0^{-1}(A \cap (g_0(x) - Lb, g_0(x) + Lb))\right) dx \\
&\le \int_{-a}^{a} L^2\,\mathrm{m}_1\left(g_0^{-1}(A) \cap (x - L^2 b, x + L^2 b)\right) dx \\
&\le 2a\,L^2\,\mathrm{m}_1\left(g_0^{-1}(A) \cap (-a - L^2 b, a + L^2 b)\right) \quad (\text{as } a \le b) \\
&\le 2a\,L^2\,\mathrm{m}_1\left(g_0^{-1}(A) \cap (-2L^2 b, 2L^2 b)\right) \quad (\text{by } (1)) \\
&\le 2a\,L^2\,4L^2 b\,\frac{\varepsilon}{8L^4} \\
&= \varepsilon\,\mathrm{m}_2(R).
\end{aligned}
$$

This finishes the proof of Theorem 2.2. □

Now we are ready to examine the relations between the density continuity of transformations $f\colon \mathbb{R}^2 \to \mathbb{R}$ and density continuity of their sections $f(x_0,\cdot)$ and $f(\cdot,y_0)$. We will start with the following example.

Example 2.3 *There exists ordinary continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with density continuous sections $f(x_0,\cdot)$ and $f(\cdot,y_0)$ for all $x_0, y_0 \in \mathbb{R}$ which is neither density nor strongly density continuous.*

Proof. Choose decreasing sequence $\{a_n\}_{n=0}^{\infty}$ of positive numbers converging to 0 such that 0 a dispersion point of $A = \bigcup_{n=0}^{\infty}[a_{2n+1}, a_{2n}]$. (For example put $a_{2n+1} = [(n+5)!]^{-1}$ and $a_{2n} = [(n+5)!]^{-1} + [(n+6)!]^{-1}$.) Define $g\colon \mathbb{R} \to \mathbb{R}$ by putting $g(0) = 0$, $g(a_n) = a_{n+1}$ for all $n \in \mathbb{N}$ and extend it to an increasing function linearly on each of the intervals $(-\infty, 0]$, $[a_0, \infty)$ and $[a_{n+1}, a_n]$ for all $n \in \mathbb{N}$. It is easy to see that g is ordinary continuous. However, it is not density continuous, since 0 is a density point of $\mathbb{R} \setminus A$, while $0 = g^{-1}(0)$ is not a density point of $g^{-1}(\mathbb{R} \setminus A)$, since $g^{-1}(\mathbb{R} \setminus A) \cap (0, a_0] \subset A$.

Now, for $y \le x$ define

$$f(x,y) = \begin{cases} 0 & y \le x/3 \\ \frac{6g(x)}{x} \cdot y - 2g(x) & 0 < x/3 \le y \le x/2 \\ g(x) & x/2 \le y \end{cases}$$

and define $f(x,y) = f(y,x)$ for $x \le y$. Notice that for fixed $x > 0$ the function $f(x,\cdot)$ is linear for $x/3 \le y \le x/2$.

It is easy to see that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous. Also, sections $f(x_0,\cdot)$ and $f(\cdot,y_0)$ of f are density continuous for all $x_0, y_0 \in \mathbb{R}$, since they are piecewise density continuous. (They are either constant or bi-Lipschitz; see Propositions 1.2 and 1.4.)

To finish the proof it is enough to show that f is not density continuous. To see this, notice that

$$f^{-1}(A) \supset \{(x,y)\colon 0 < x/3 \le y \le x/2\} \cap \left[\bigcup_{n=0}^{\infty} [a_{2n+2}, a_{2n+1}] \times \mathbb{R}\right] = V$$

and that (ordinary) density of V at $(0,0)$ is $1/48$, since the ordinary density of $\{(x,y)\colon 0 < x/3 \le y \le x/2\}$ at $(0,0)$ is $1/48$ and 0 is a right density point of $\bigcup_{n=0}^{\infty} [a_{2n+2}, a_{2n+1}]$. Thus, $(0,0)$ is not a dispersion point of $f^{-1}(A)$ while $0 = f(0,0)$ is a dispersion point of A. □

Example 2.3 shows that we cannot conclude density continuity of function $f\colon \mathbb{R}^2 \to \mathbb{R}$ from the density continuity of its sections. What about the other way around? Can we conclude density continuity of sections of $f\colon \mathbb{R}^2 \to \mathbb{R}$ if f is either density continuous or strongly density continuous? The next example shows that the answer for this question is NO, in case of density continuity of f. Then we will show that the answer is YES in case when f is strongly density continuous.

Example 2.4 *There exists continuous and density continuous transformation $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(\cdot,0)$ is not density continuous.*

PROOF. Define f by

$$f(x,y) = \begin{cases} 0 & x \le 0 \\ 0 & x > 0 \text{ and } |y| > x^2 \\ (1 - \frac{|y|}{x^2})g(x) & x > 0 \text{ and } |y| \le x^2, \end{cases}$$

where g is as in Example 2.3. It is easy to see that f is continuous. f is density continuous at points $\ne (0,0)$ by Theorem 2.2 and Proposition 1.4. It is density continuous at $(0,0)$, because f equals to zero on a set $D = \{(x,y)\colon x \le 0 \text{ or } x > 0 \text{ and } |y| > x^2\}$ and $(0,0)$ is (ordinary) density point of D. Finally, $f(\cdot, 0)$ is not density continuous, since it is equal to g on $[0, \infty)$ and g is not right density continuous at 0. □

Now we will show that density continuity of $f(x_0, \cdot)$ and $f(\cdot, y_0)$ follows from the strong density continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$. The argument is essentially the same as in the proof that the sections of strongly approximately continuous functions from $\mathbb{R}^2$ into $\mathbb{R}$ are approximately continuous [8, p. 502]. For the convenience of the reader we will extract here the main parts of that proof in the form of two lemmas. They will imply both of these results.

Lemma 2.5 *Let (X, σ) be a topological space, $y_0 \in X$ and let τ be a topology on $X \times X$ such that the following condition holds:*

(CL) *for every $U \in \tau$ and $x_0 \in X$*

$$\text{if } \; x_0 \in \mathrm{cl}_\sigma \{x\colon (x, y_0) \in U\}, \; \text{ then } \; (x_0, y_0) \in \mathrm{cl}_\tau U.$$

Then for every regular topological space Y and continuous $f\colon (X \times X, \tau) \to Y$ function $g\colon (X, \sigma) \to Y$ defined by $g(x) = f(x, y_0)$ is continuous.

PROOF. Let $x_0 \in X$. We will show that g is continuous at x_0. So let $V \subset Y$ be open neighborhood of $z_0 = g(x_0) = f(x_0, y_0)$. It is enough to prove that x_0 is a σ-interior point of $g^{-1}(V)$.

By way of contradiction assume that it is not the case. Then $x_0 \in \mathrm{cl}_\sigma\, g^{-1}(Y \setminus V)$. By the regularity of Y we can find disjoint open sets $U, W \subset Y$ such that $z_0 \in W$ and $Y \setminus V \subset U$. In particular,

$$x_0 \in \mathrm{cl}_\sigma\, g^{-1}(U) = \mathrm{cl}_\sigma \{x\colon (x, y_0) \in f^{-1}(U)\}.$$

Hence using (CL) to $f^{-1}(U) \in \tau$, we obtain

$$(x_0, y_0) \in \mathrm{cl}_\tau\, f^{-1}(U) \subset \mathrm{cl}_\tau\, f^{-1}(Y \setminus W)$$

which contradicts the fact that (x_0, y_0) is a τ-interior point of $f^{-1}(W)$. □

Lemma 2.6 $(\mathbb{R}, \mathcal{T}_{\mathcal{N}}^1)$ *and* $(\mathbb{R}^2, \mathcal{T}_{\mathcal{S}}^2)$ *satisfy condition* (CL) *of Lemma* 2.5 *for every $y_0 \in \mathbb{R}$, i.e.,*

(CL) *for every $U \in \mathcal{T}_{\mathcal{S}}^2$ and $x_0, y_0 \in \mathbb{R}$*

$$\text{if } \; x_0 \in \mathrm{cl}_{\mathcal{T}_{\mathcal{N}}^1} \{x\colon (x, y_0) \in U\}, \; \text{ then } \; (x_0, y_0) \in \mathrm{cl}_{\mathcal{T}_{\mathcal{S}}^2} U.$$

PROOF. The proof of this Lemma is implicitly contained in the proof of [8, Theorem 4]. For completeness sake we will sketch it here.

So let $U \in \mathcal{T}_{\mathcal{S}}^2$ and $x_0, y_0 \in \mathbb{R}$ be such that $x_0 \in \mathrm{cl}_{\mathcal{T}_{\mathcal{N}}^1} V$, where $V = \{x \colon (x, y_0) \in U\}$. Then there exists $\varepsilon > 0$ and a sequence of intervals $I_n = (a_n, b_n)$ centered at x_0 such that $\lim_{n\to\infty}(b_n - a_n) = 0$ and

$$\frac{\mathrm{m}_1(V \cap I_n)}{\mathrm{m}_1(I_n)} > \varepsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{2}$$

Now, fix $n \in \mathbb{N}$ and notice that for every $p \in V$ point (p, y_0) is an interior point of U. Thus, for $\varepsilon_n = \mathrm{m}_1(I_n)$ and every $p \in V$ there exists $\delta_p > 0$, such that

$$\mathrm{m}_2(U \cap R) > (1 - \varepsilon_n)\,\mathrm{m}_2(R) \quad \text{for every} \quad R \in \mathcal{R}_p, \tag{3}$$

where $\mathcal{R}_p$ is a family of all rectangles $(p-\delta, p+\delta) \times (y_0 - \delta', y_0 + \delta')$ with $\delta, \delta' \in (0, \delta_p)$. Notice that for $\mathcal{V}_p = \{(p - \delta, p + \delta) \subset I_n \colon 0 < \delta < \delta_p\}$ the collection $\{\mathcal{V}_p \colon p \in I_n \cap V\}$ is a Vitali cover of $I_n \cap V$. In particular, there exists a finite collection $\{J_i\}_{i=1}^m$ of disjoint intervals such that $J_i = (p_i - \delta_i, p_i + \delta_i) \in \mathcal{V}_{p_i}$ and

$$\mathrm{m}_1(I_n \cap V) - m_1\left(\bigcup_{i=1}^m J_i\right) \le m_1\left((I_n \cap V) \setminus \bigcup_{i=1}^m J_i\right) < \varepsilon_n^2. \tag{4}$$

Now, let $\delta' = \min\{\delta_i \colon i = 1, 2, \ldots, m\}$ and define $R_n = I_n \times (y_0 - \delta', y_0 + \delta')$. Then the rectangles $J_i^\star = J_i \times (y_0 - \delta', y_0 + \delta')$ are disjoint subsets of R_n, and

$$\begin{aligned}
\mathrm{m}_2(U \cap R_n) &\ge \sum_{i=1}^m \mathrm{m}_2(U \cap J_i^\star) && \text{(by (3))}\\
&\ge \sum_{i=1}^m (1 - \varepsilon_n)\,\mathrm{m}_2(J_i^\star)\\
&= (1 - \varepsilon_n)\,2\delta'\, m_1\left(\bigcup_{i=1}^m J_i\right) && \text{(by (4))}\\
&\ge (1 - \varepsilon_n)\,2\delta'\,(\mathrm{m}_1(I_n \cap V) - \varepsilon_n^2) && \text{(by (2))}\\
&\ge (1 - \varepsilon_n)\,2\delta'\,(\varepsilon\,\mathrm{m}_1(I_n) - \varepsilon_n^2)\\
&= (1 - \varepsilon_n)(\varepsilon - \varepsilon_n)\,2\delta'\,\varepsilon_n\\
&= (1 - \varepsilon_n)(\varepsilon - \varepsilon_n)\,\mathrm{m}_2(R_n).
\end{aligned}$$

Thus,

$$\frac{\mathrm{m}_2(U \cap R_n)}{\mathrm{m}_2(R_n)} \ge \frac{\varepsilon}{2}$$

for sufficiently large n since ε_n approaches 0. So $(x_0, y_0) \in \mathrm{cl}_{\mathcal{T}_{\mathcal{S}}^2} U$. □

As a corollary we conclude the following theorem.

Theorem 2.7 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $x_0, y_0 \in \mathbb{R}$ and $g, h\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = f(x, y_0)$ and $h(y) = f(x_0, y)$ for $x, y \in \mathbb{R}$.*

(i) *If f is strongly density continuous, then g and h are density continuous.*

(ii) *If f is strongly approximately continuous, then g and h are approximately continuous.*

Proof. It follows immediately from Lemmas 2.5 and 2.6 if we notice that the ordinary topology and the density topology on $\mathbb{R}$ are regular. □

We will finish this section with the following comparison of classes of density continuous $\mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1)$, strongly density continuous $\mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1)$ and ordinary continuous $\mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O})$ functions from $\mathbb{R}^2$ into $\mathbb{R}$.

Theorem 2.8

$$\begin{array}{ccccc} \mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1) & \subset & \mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1) & & \\ \cup & & \cup & & \\ \mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1) \cap \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) & \subset & \mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1) \cap \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) & \subset & \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) \end{array}$$

All the containments are proper.

Proof. The inclusions follow immediately from $\mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1) \subset \mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1)$.

To see that they are proper it is enough to show $\mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1) \not\subset \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O})$ for vertical inclusions and $\mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) \not\subset \mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1)$ and $\mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1) \cap \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) \not\subset \mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1)$ for horizontal inclusions.

$\mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1) \not\subset \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O})$ follows from Proposition 2.1 for an arbitrary function $f(x, y) = h(y)$, where $h\colon \mathbb{R} \to \mathbb{R}$ is density continuous and not continuous. For example, we can define

$$h(x) = \begin{cases} 0 & x \notin A \\ \frac{\min\{a_{2n} - x, x - a_{2n+1}\}}{a_{2n} - a_{2n+1}} & x \in (a_{2n+1}, a_{2n}), \end{cases}$$

where $A = \bigcup_{n=0}^{\infty} [a_{2n+1}, a_{2n}]$ is from Example 2.3.

$\mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) \not\subset \mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1)$ follows from Proposition 2.1 for any function $f(x, y) = g(y)$, where $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and not density continuous. For example, function g from Example 2.3 works.

$\mathcal{C}(\mathcal{T}_\mathcal{N}^2, \mathcal{T}_\mathcal{N}^1) \cap \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O}) \not\subset \mathcal{C}(\mathcal{T}_\mathcal{S}^2, \mathcal{T}_\mathcal{N}^1)$ is justified by function f from Example 2.4 since, by Theorem 2.7(i), f cannot be strongly density continuous. □

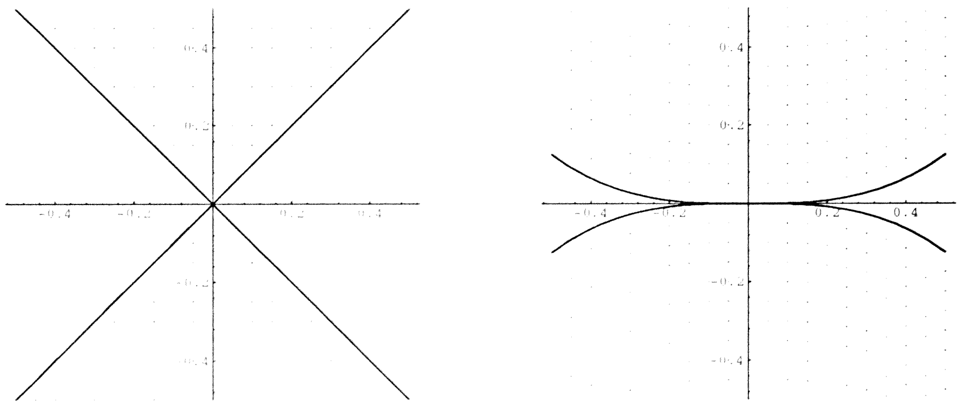

Figure 1: Sets $F^{-1}(A)$ (left) and A (right) from Example 3.2

3 Transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$.

In this section we consider the interrelations between (strongly) density continuous transformations $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ and their coordinate functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$.

If f and g are the coordinate functions of transformation $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, then $f, g\colon \mathbb{R}^2 \to \mathbb{R}$. It might happen, however, that either f or g depends of only one variable, e.g., $f(x,y) = h(y)$ for some $h\colon \mathbb{R} \to \mathbb{R}$. Then, according to Proposition 2.1, we can replace f with h when examining the (strong) density continuity of f. We will use this convention throughout the rest of the paper.

The next easy fact forms a base for the discussion of this section.

Theorem 3.1 *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation with coordinate functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$, i.e., such that $F(x,y) = (f(x,y), g(x,y))$.*

(a) *If F is strongly density continuous, then f and g are also strongly density continuous.*

(b) *If F is density continuous, then f and g are density continuous.*

PROOF. (a). Assume that F is strongly density continuous and let $A \in \mathcal{T}^1_{\mathcal{N}}$. We have to show that $f^{-1}(A), g^{-1}(A) \in \mathcal{T}^2_{\mathcal{S}}$. But $A \times \mathbb{R} \in \mathcal{T}^2_{\mathcal{S}}$ and so, $f^{-1}(A) = F^{-1}(A \times \mathbb{R}) \in \mathcal{T}^2_{\mathcal{S}}$. Similarly, $\mathbb{R} \times A \in \mathcal{T}^2_{\mathcal{S}}$ so that $g^{-1}(A) = F^{-1}(\mathbb{R} \times A) \in \mathcal{T}^2_{\mathcal{S}}$.

The proof of part (b) is identical. □

At this moment one might be tempted to prove the converse of (a) and (b) of Theorem 3.1. Indeed, such a claim would be obvious if either the ordinary or the strong density topology on $\mathbb{R}^2$ was a product topology of the

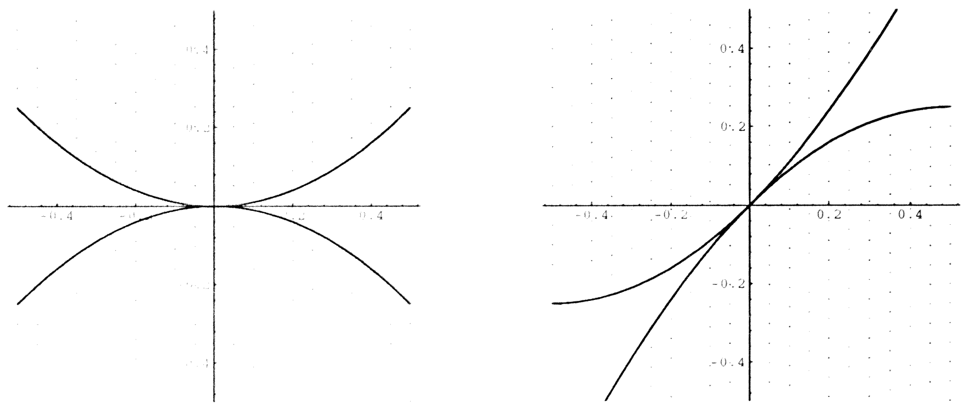

Figure 2: Sets $F^{-1}(B)$ (left) and B (right) from Example 3.3

one-dimensional density topology on $\mathbb{R}$. This, however, is not the case and the next two examples show that neither implication in (a) nor (b) of Theorem 3.1 can be reversed.

Example 3.2 *The transformation* $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (x, y^3)$, *is not density continuous, while its coordinate functions* $f(x) = x$ *and* $g(y) = y^3$ *are density continuous.*

PROOF. The functions f and g are density continuous, since all real analytic functions are density continuous. (See Proposition 1.2.) To see that F is not density continuous put $A = \{(u,v)\colon |v| > |u^3|\} \cup \{(0,0)\}$ and notice that $F^{-1}(A) = \{(u,v)\colon |v| > |u|\} \cup \{(0,0)\}$. (See Figure 1.) It is routine to check that $A \in \mathcal{T}_{\mathcal{N}}$, while $F^{-1}(A) \notin \mathcal{T}_{\mathcal{N}}$ since $d_2(F^{-1}(A),(0,0)) = 1/2 \neq 1$. □

Example 3.3 *The transformation* $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (x, x+y)$, *is not strongly density continuous, while its coordinate functions* $f(x,y) = x$ *and* $g(x,y) = x + y$ *are strongly density continuous.*

PROOF. f is strongly density continuous by Proposition 2.1. The function g is strongly density continuous by Theorem 2.2 since it is obviously bi-Lipschitz.

To see that F is not strongly density continuous put

$$\begin{aligned} B &= \{(u,v)\colon |v-u| > u^2\} \cup \{(0,0)\} \\ &= \{(u,v)\colon v > u + u^2 \ \text{ or } v < u - u^2\} \cup \{(0,0)\} \end{aligned}$$

and notice that $F^{-1}(B) = \{(u,v)\colon |v| > u^2\} \cup \{(0,0)\}$. (See Figure 2.) It is easy to check that $B \in \mathcal{T}_{\mathcal{S}}^2$, while $F^{-1}(B) \notin \mathcal{T}_{\mathcal{S}}^2$ because $(0,0)$ is not a strong density point of $F^{-1}(B)$. □

Notice that the transformation $F(x,y) = (x, x+y)$ is evidently bi-Lipschitz. In particular, the next corollary proves that Proposition 1.3 cannot be generalized to strongly density continuous transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$.

Corollary 3.4 *There exists a bi-Lipschitz transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$ which is not strongly density continuous.* □

Example 3.2 suggests that there is no real chance to reverse implication (b) of Theorem 3.1. Also, Example 3.3 does no leave much hope for reversing implication Theorem 3.1(b). We can still hope, however, that $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by $F(x,y) = (f(x), g(y))$ is strongly density continuous, provided $f, g\colon \mathbb{R} \to \mathbb{R}$ are density continuous. But, even this claim is too strong, as the precise condition in this direction is given by the next theorem.

Theorem 3.5 *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ and define the transformation $H\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $H(x,y) = (f(x), g(y))$. If H is not constant, then H is strongly density continuous if and only if functions f and g are density continuous and $\mathrm{m}_1\left(f^{-1}(p)\right) = \mathrm{m}_1\left(g^{-1}(p)\right) = 0$ for every $p \in \mathbb{R}$.*

PROOF. "$\Longrightarrow$" Assume first that H is strongly density continuous. Then f and g are density continuous by Theorem 3.1 and Proposition 2.1.

To prove the additional part first notice that neither f nor g is constant. To see this assume, to the contrary, that g is constant and equal to $b \in \mathbb{R}$. Then f is not constant, since we assumed that H is not constant. Thus, by Proposition 1.1(c), there exists $p \in f[\mathbb{R}]$ with $\mathrm{m}_1\left(f^{-1}(p)\right) = 0$. Now notice that $A = [\mathbb{R} \times (\mathbb{R} \setminus \{b\})] \cup \{(p,b)\} \in \mathcal{T}_S^2$ as a set of full measure, while $H^{-1}(A) = H^{-1}((p,b)) = f^{-1}(p) \times \mathbb{R}$ is non-empty and has two-dimensional Lebesgue measure zero. Thus, $H^{-1}(A) \notin \mathcal{T}_S^2$. This contradiction establishes that g is not constant. A similar argument shows that f is not constant.

Now, we will show that $\mathrm{m}_1\left(f^{-1}(p)\right) = 0$ for every $p \in \mathbb{R}$. This will imply that $\mathrm{m}_1\left(g^{-1}(p)\right) = 0$ for every $p \in \mathbb{R}$ by using the same argument for $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $G(x,y) = (g(x), f(y))$, which is also strongly density continuous.

So pick $p \in \mathbb{R}$ and, by way of contradiction, assume that $\mathrm{m}_1\left(f^{-1}(p)\right) > 0$. Then by the Lebesgue Density Theorem, there exists $a \in f^{-1}(p)$ with $d_1(f^{-1}(p), a) = 1$. Since g is not constant, by Proposition 1.1(c) we can find $q \in g[\mathbb{R}]$ such that $\mathrm{m}_1\left(g^{-1}(q)\right) = 0$. Put $A = [(\mathbb{R} \setminus \{p\}) \times \mathbb{R}] \cup \{(p,q)\}$. Then $A \in \mathcal{T}_S^2$ as a set of full measure. On the other hand,

$$H^{-1}(A) = [(\mathbb{R} \setminus f^{-1}(p)) \times \mathbb{R}] \cup [f^{-1}(p) \times g^{-1}(q)]$$

does not belong to $\mathcal{T}_S^2$. To see this pick $(a,b) \in f^{-1}(p) \times g^{-1}(q) \subset H^{-1}(A)$. Then

$$\begin{aligned} d_s(H^{-1}(A),(a,b)) &= d_s((\mathbb{R} \setminus f^{-1}(p)) \times \mathbb{R},(a,b)) \\ &= d_1(\mathbb{R} \setminus f^{-1}(p), a) \\ &= 1 - d_1(f^{-1}(p), a) = 1 - 1 \neq 1, \end{aligned}$$

since $\mathfrak{m}_2\left(f^{-1}(p) \times g^{-1}(q)\right) = 0$.

The proof of implication "$\Longrightarrow$" is completed.

"$\Longleftarrow$" First notice that it is enough to show that the transformation $F(x,y) = (f(x),y)$ is strongly density continuous under our assumption, since $C(x,y) = (y,x)$ is strongly density continuous. Moreover, it is enough to show that $F(x,y) = (f(x),y)$ is strongly density continuous at $(0,0)$, since translations are strongly density continuous. We can also assume that $f(0) = 0$.

So let $A \subset \mathbb{R}^2$ be such that $(0,0)$ is not a strong dispersion point of A. We will finish the proof by showing that $F(0,0) = (0,0)$ is not a strong dispersion of $F[A]$. In the proof we will need the following easy lemma in which A^x stands for a vertical section of $A \subset \mathbb{R}^2$ given by x, i.e., $A^x = \{y\colon (x,y) \in A\}$.

Lemma 3.6 *Let $\varepsilon \in (0,1/2)$ and let $A \subset R = [-a,a] \times [-b,b]$ be measurable and such that*

$$\frac{\mathfrak{m}_2(A)}{\mathfrak{m}_2(R)} \geq \varepsilon.$$

If $B = \{x \in [-a,a]\colon \mathfrak{m}_1(A^x) \geq 2b\varepsilon^2\}$, then $\mathfrak{m}_1(B) \geq 2a\varepsilon^2$.

PROOF. Easy, by the Fubini Theorem. □

Now, we can come back to the proof of Theorem 3.5. Since $(0,0)$ is not a strong dispersion point of A, there exists $\varepsilon \in (0,.5)$ and a sequence of rectangles $R_n = [-a_n,a_n] \times [-b_n,b_n]$ such that $\max\{a_n,b_n\} \to 0$ and

$$\frac{\mathfrak{m}_2(A \cap R_n)}{\mathfrak{m}_2(R_n)} > 3\varepsilon \text{ for all } n \in \mathbb{N}. \tag{5}$$

Now, by induction on $i \in \mathbb{N}$, we define an increasing sequence $\{n_i\}_{i=0}^{\infty}$ of indices, sequences $\{c_i\}_{i=0}^{\infty}$, $\{d_i\}_{i=0}^{\infty}$ of positive numbers and a sequence $\{B_i\}_{i=0}^{\infty}$ of sets such that the following inductive conditions hold for every $i \in \mathbb{N}$

(i) $0 < c_i < d_i$ and $\frac{d_i}{c_{i-1}} < \frac{1}{i}$ for $i > 0$.

(ii) $B_i \subset [-a_{n_i},a_{n_i}] \cap f^{-1}((-d_i,d_i) \setminus (-c_i,c_i))$ is such that $\mathfrak{m}_1(B_i) \geq 2a_{n_i}\varepsilon^2$ and $\mathfrak{m}_1(A^x \cap [-b_{n_i},b_{n_i}]) \geq 2b_{n_i}\varepsilon^2$ for all $x \in B_i$.

To make the inductive step take $i \in \mathbb{N}$ such that the construction is already done for all natural numbers less than i. If $i = 0$ put $d_0 = 1$. Otherwise, choose $d_i \in (0, c_{i-1}/i)$. This guarantees satisfaction of (i) for step i.

Now, since 0 is a density point of $(-d_i, d_i)$ and f is density continuous, 0 is a density point of $f^{-1}((-d_i, d_i))$ and $(0,0)$ is a strong density point of $f^{-1}((-d_i, d_i)) \times \mathbb{R}$.

Choose $n_i \in \mathbb{N}$, $n_i > n_{i-1}$ for $i > 0$, such that

$$\frac{\mathrm{m}_2\left([f^{-1}((-d_i, d_i)) \times \mathbb{R}] \cap R_{n_i}\right)}{\mathrm{m}_2\left(R_{n_i}\right)} > 1 - \varepsilon.$$

Then, by (5),

$$\frac{\mathrm{m}_2\left(A \cap [f^{-1}((-d_i, d_i)) \times \mathbb{R}] \cap R_{n_i}\right)}{\mathrm{m}_2\left(R_{n_i}\right)} > 2\varepsilon. \tag{6}$$

Moreover, since

$$m_2\left(\left[\bigcap_{c>0} f^{-1}((-c,c)) \times \mathbb{R}\right] \cap R_{n_i}\right) = \mathrm{m}_2\left([f^{-1}(0) \times \mathbb{R}] \cap R_{n_i}\right) = 0$$

we can find $c_i \in (0, d_i)$ such that $\mathrm{m}_2\left([f^{-1}((-c_i, c_i)) \times \mathbb{R}] \cap R_{n_i}\right) < \varepsilon\, \mathrm{m}_2\left(R_{n_i}\right)$, i.e., that

$$\frac{\mathrm{m}_2\left([f^{-1}((-c_i, c_i)) \times \mathbb{R}] \cap R_{n_i}\right)}{\mathrm{m}_2\left(R_{n_i}\right)} < \varepsilon.$$

Hence, by (6),

$$\frac{\mathrm{m}_2\left(A \cap [f^{-1}((-d_i, d_i) \setminus (-c_i, c_i)) \times \mathbb{R}] \cap R_{n_i}\right)}{\mathrm{m}_2\left(R_{n_i}\right)} > \varepsilon.$$

Now, using Lemma 3.6 with $A \cap [f^{-1}((-d_i, d_i) \setminus (-c_i, c_i)) \times \mathbb{R}] \cap R_{n_i}$ and R_{n_i} we can find B_i satisfying (ii).

This completes the inductive construction.

From condition (ii) it follows immediately that 0 is not a dispersion point of $B = \bigcup_{i=0}^{\infty} B_i$. Hence, since $0 \notin B$ and f is density continuous, $0 = f(0)$ is not a dispersion point of $f[B]$. So, there exists $\delta > 0$ and a decreasing sequence $\{p_n\}_{n=0}^{\infty}$ converging to 0 such that

$$\frac{\mathrm{m}_1\left(f[B] \cap [-p_n, p_n]\right)}{2p_n} \geq 2\delta \quad \text{for every} \quad n \in \mathbb{N}.$$

For fixed $i \in \mathbb{N}$ let k_i be the smallest natural number such that $c_{k_i} \leq p_i$. Then $f[B] \cap [-p_i, p_i] = \bigcup_{j=k_i}^{\infty} f[B_j] \cap [-p_i, p_i]$ since $f[B_k] \subset (-d_k, d_k) \setminus$

$(-c_k, c_k)$ for every $k \in \mathbb{N}$. Hence,

$$\begin{aligned} & \frac{\mathrm{m}_1\left(f[B_{k_i}] \cap [-p_i, p_i]\right)}{2p_i} + \frac{2d_{k_i+1}}{2c_{k_i}} \\ \geq\ & \frac{\mathrm{m}_1\left(f[B_{k_i}] \cap [-p_i, p_i]\right)}{2p_i} + \frac{\mathrm{m}_1\left((-d_{k_i+1}, d_{k_i+1})\right)}{2p_i} \\ \geq\ & \frac{\mathrm{m}_1\left(f[B_{k_i}] \cap [-p_i, p_i]\right)}{2p_i} + \frac{m_1\left(\bigcup_{j=k_i+1}^{\infty} f[B_j] \cap [-p_i, p_i]\right)}{2p_i} \\ =\ & \frac{\mathrm{m}_1\left(f[B] \cap [-p_i, p_i]\right)}{2p_i} \\ \geq\ & 2\delta \end{aligned}$$

for every $i \in \mathbb{N}$. However, the sequence $\{\frac{d_{i+1}}{c_i}\}$ converges to 0. Thus, there exists $i_0 \in \mathbb{N}$ such that $\frac{d_{k_i+1}}{c_{k_i}} < \delta$ for $i \geq i_0$. In particular,

$$\frac{\mathrm{m}_1\left(f[B_{k_i}] \cap [-p_i, p_i]\right)}{2p_i} > \delta \quad \text{for} \quad i \geq i_0. \tag{7}$$

Now, notice that diameters of the rectangles $S_i = [-p_i, p_i] \times [-b_{n_{k_i}}, b_{n_{k_i}}]$ converge to 0. We will use these rectangles to show that $(0,0)$ is not a strong dispersion point of $F[A]$. So consider $F[A] \cap S_i$. For every $v \in f[B_{k_i}] \cap [-p_i, p_i]$ and each $x \in B_{k_i}$ such that $f(x) = v$ we have

$$(F[A] \cap S_i)^v = (F[A])^v \cap [-b_{n_{k_i}}, b_{n_{k_i}}] \supset A^x \cap [-b_{n_{k_i}}, b_{n_{k_i}}].$$

Hence, by (ii), $\mathrm{m}_1\left((F[A] \cap S_i)^v\right) \geq 2b_{n_{k_i}}\varepsilon^2$ for every $v \in f[B_{k_i}] \cap [-p_i, p_i]$. Therefore, by (7) and Fubini Theorem,

$$\mathrm{m}_2\left(F[A] \cap S_i\right) \geq 2b_{n_{k_i}}\varepsilon^2 \mathrm{m}_1\left(f[B_{k_i}] \cap [-p_i, p_i]\right) \geq 2b_{n_{k_i}}\varepsilon^2 \delta 2p_i = \mathrm{m}_2\left(S_i\right)\delta\varepsilon^2$$

for all $i \geq i_0$, i.e.,

$$\frac{\mathrm{m}_2\left(F[A] \cap S_i\right)}{\mathrm{m}_2\left(S_i\right)} \geq \delta\varepsilon^2 \quad \text{for every} \quad i \geq i_0.$$

Thus, $(0,0)$ is not a strong dispersion point of $F[A]$.

This finishes the proof of Theorem 3.5. □

Notice that the above proof can easily be modified to obtain the following local version of Theorem 3.5.

Theorem 3.7 *Let $f\colon \mathbb{R} \to \mathbb{R}$. Then $F(x,y) = (f(x), y)$ is strongly density continuous at $(a,b) \in \mathbb{R}^2$ if and only if f is density continuous at a and a is a dispersion point of $f^{-1}(\{f(p)\})$.* □

We will finish this section with the proof that there are no inclusion relations between $\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})$, $\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)$ and $\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)$ as stated in the theorem below.

Theorem 3.8 (a) $\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})\cap\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)\not\subset\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)$;

(b) $\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})\cap\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)\not\subset\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)$;

(c) $\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)\cap\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)\not\subset\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})$.

PROOF. (a) It is justified by $F\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by $F(x,y)=(x,x+y)$ from Example 3.3. It was proved there that F is not in $\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)$. F is also evidently bi-Lipschitz, so it is continuous and, by Proposition 1.3, it is in $\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)$.

(b) It is justified by $F\colon\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(x,y^3)$, since it is evidently in $\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})$. $F\notin\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)$ by Example 3.2 and $F\in\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)$ by Theorem 3.5, since $f(x)=x$ and $g(y)=y^3$ are density continuous homeomorphisms.

(c) It is justified by $F\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by $F(x,y)=(f(x),y)$, where $f(x)=x+h(x)$ and h is from Theorem 2.8.

Since h and f are clearly discontinuous at 0 we have $F\notin\mathcal{C}(\mathcal{T}_{\mathcal{O}},\mathcal{T}_{\mathcal{O}})$. To prove that $F\in\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)$ notice first that f is density continuous: at points $x\neq 0$, since it is there piecewise linear on $\mathbb{R}\setminus\{0\}$, and at 0, since f is the identity function on a set $\mathbb{R}\setminus A$, for which 0 is a density point. Now, $F\in\mathcal{C}(\mathcal{T}_{\mathcal{S}}^2,\mathcal{T}_{\mathcal{S}}^2)$ follows from Theorem 3.5 if we notice that none of the slopes of the linear pieces of h equals -1, i.e., $f^{-1}(p)$ is at most countable for every $p\in\mathbb{R}$.

To see that $F\in\mathcal{C}(\mathcal{T}_{\mathcal{N}}^2,\mathcal{T}_{\mathcal{N}}^2)$ we have to consider two kinds of points. F is density continuous at points of $\mathbb{R}^2\setminus(A\times\mathbb{R})\in\mathcal{T}_{\mathcal{N}}^2$ since it is the identity mapping on this set. F is continuous at points of $A\times\mathbb{R}$ since at every point of this set F is either locally bi-Lipschitz (Proposition 1.3) or is a maximum (minimum) of two functions with this property (see Proposition 1.4). □

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