Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 226–235

Krzysztof Ciesielski, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, (kcies@wvnvms.wvnet.edu)

Lee Larson, Department of Mathematics, University of Louisville, Louisville, KY 40292, (lmlars01@ulkyvx.louisville.edu)

UNIFORMLY ANTISYMMETRIC FUNCTIONS

Abstract

An example is given of a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the property that for each x there is an $\varepsilon(x) > 0$ such that $|f(x+h) - f(x-h)| \geq 1$ whenever $0 < h < \varepsilon(x)$. Further properties of such functions are also discussed.

1. An unbounded example

A function $f : \mathbb{R} \to \mathbb{R}$ is *weakly continuous* at x_0 if there are sequences $a_n \nearrow 0$ and $b_n \searrow 0$ such that

$$\lim_{n\to\infty} f(x_0 + a_n) = f(x_0) = \lim_{n\to\infty} f(x_0 + b_n).$$

A surprising theorem from cluster set theory [2, p. 82] states that an *arbitrary* function is weakly continuous everywhere on the complement of a countable set.

A function $f: \mathbb{R} \to \mathbb{R}$ is symmetrically continuous at $x_0 \in \mathbb{R}$ if

$$\lim_{h\to 0^+} f(x_0 + h) - f(x_0 - h) = 0;$$

i.e., if for every $\varepsilon > 0$

$$(1) \qquad (\exists d > 0)(\forall 0 < h < d)\ |f(x_0 + h) - f(x_0 - h)| < \varepsilon.$$

Key Words: symmetric continuity, Hamel basis

Mathematical Reviews subject classification: 26A15, 26A21

Received by the editors October 15, 1993

Motivated by the definitions given above, we define a function to be *weakly symmetrically continuous* at x_0 if there exists a sequence $h_n \searrow 0$ such that

$$\lim_{n\to\infty} f(x_0 + h_n) - f(x_0 - h_n) = 0.$$

It is easy to construct functions which are nowhere symmetrically continuous. For example, the characteristic function of any dense Hamel basis is such a function.[1] Any nowhere symmetrically continuous function f must satisfy the property that for every x there exists $\varepsilon > 0$ such that

$$(\forall d > 0)(\exists 0 < h < d)\ |f(x-h) - f(x+h)| \geq \varepsilon. \tag{2}$$

In asking how badly the symmetric continuity of f can break, several people[2] have asked whether a function can be nowhere weakly symmetrically continuous. This is the same as asking whether the quantifiers in (2) can be exchanged. That is, whether there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ there exists $\varepsilon > 0$ such that

$$(\exists d > 0)(\forall 0 < h < d)\ |f(x-h) - f(x+h)| \geq \varepsilon.$$

It is clear that this last question is equivalent to asking whether the following statement holds:

($\star$) There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ there is a $d > 0$ (depending on x) so that whenever $0 < h < d$,

$$|f(x-h) - f(x+h)| \geq d.$$

A function f satisfying ($\star$) will be called a *uniformly antisymmetric function.* In view of the theorem cited above for weak continuity, the existence of such a function is by no means obvious.

The purpose of this note is to show that uniformly antisymmetric functions exist. The existence of such functions can be inferred from the following theorem.

Theorem 1.1 *There exists a partition $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ of $\mathbb{R}$ such that for every $x \in \mathbb{R}$ the set*

$$S_x = \bigcup_{n\in\mathbb{N}} \{h > 0 : x - h, x + h \in P_n\} \tag{3}$$

is finite.

[1] In fact, a Hamel basis $\mathcal{B}$ can be chosen to be both first category and measure zero. For example, $\mathcal{B}$ can be constructed such that $\mathcal{B} \setminus C$ is dense and countable, where C is the Cantor ternary set. Thus $\chi_{\mathcal{B}}$ can be measurable and have the Baire property. (Compare this with Theorem 1.3.)

[2] Evans and Larson, 7th Real Analysis Exchange Conference, Santa Barbara, 1984. P. Kostyrko, 15th Summer Symposium on Real Analysis, Smolenice, 1991.

Before proving this theorem, we first show how it implies the existence of a uniformly antisymmetric function.

Corollary 1.2 *Let $K \subset \mathbb{N}$. If $\mathcal{P} = \{P_n : n \in K\}$ is a partition of $\mathbb{R}$ satisfying (3), then there exists a uniformly antisymmetric function $f\colon \mathbb{R} \to K$. In particular, there exists a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{N}$.*

PROOF. Let $\mathcal{P} = \{P_n : n \in K\}$ be a partition satisfying (3) and for $x \in \mathbb{R}$ define

$$f(x) = n \text{ if and only if } x \in P_n.$$

Then f is uniformly antisymmetric, since for every $x \in \mathbb{R}$ the set

$$S_x = \bigcup_{n \in \mathbb{N}} \{h > 0\colon x - h, x + h \in P_n\} = \{h > 0\colon f(x-h) = f(x+h)\}$$

is finite and $|f(x-h) - f(x+h)| \geq 1$ for every $h > 0$ such that $h \notin S_x$. Thus ($\star$) is satisfied by $d = \min S_x \cup \{1\}$. □

PROOF OF THEOREM 1.1. Let $\mathcal{B}$ be a Hamel basis for $\mathbb{R}$; i.e., a linear basis of $\mathbb{R}$ over $\mathbb{Q}$. Consider $\mathbb{Q}$ with the discrete topology and $\mathbb{Q}^{\mathcal{B}}$ with the product topology. Then, $\mathbb{Q}^{\mathcal{B}}$ is separable, being a product of continuum many separable spaces.[3] Let $D = \{d_n : n \in \mathbb{N}\}$ be a dense subset of $\mathbb{Q}^{\mathcal{B}}$.

Define a one-to-one function $E\colon \mathbb{R} \to \mathbb{Q}^{\mathcal{B}}$ by putting $E(x)(b) = q_b$, where $x = \Sigma_{b \in \mathcal{B}} q_b b$ is the unique representation of x in the Hamel basis $\mathcal{B}$. Notice that for every $x \in \mathbb{R}$ the set $\text{supp}(x) = \{b \in \mathcal{B} : E(x)(b) \neq 0\}$ is finite.

For $x \in \mathbb{R}$ let

$$[x] = \{g \in \mathbb{Q}^{\mathcal{B}}\colon g(b) = E(x)(b) \text{ for all } b \in \text{supp}(x)\}.$$

Clearly, $[x]$ is a basic open set in $\mathbb{Q}^{\mathcal{B}}$. Define

$$f(x) = \min\{n\colon d_n \in [x]\}$$

and let $P_n = f^{-1}(n)$ for every $n \in \mathbb{N}$. We will show that the partition $\mathcal{P} = \{P_n\colon n \in \mathbb{N}\}$ of $\mathbb{R}$ satisfies (3).

First notice that the following condition implies (3):

(i) for every $x \in \mathbb{R}$ there exists only a finite number of $h > 0$ such that $[x-h] \cap [x+h] \neq \emptyset$.

[3] This is the Hewitt-Marczewski-Pondiczery Theorem [3]. In fact, if $\mathcal{U}$ is a countable basis for $\mathcal{B} \subset \mathbb{R}$ then the set $D \subset \mathbb{Q}^{\mathcal{B}}$ of all functions of the form $\Sigma_{i<n} q_i \chi_{U_i}$ (where $n \in \mathbb{N}$, $q_i \in \mathbb{Q}$ and $U_i \in \mathcal{U}$) is countable and dense in $\mathbb{Q}^{\mathcal{B}}$.

To show (i), choose $x, h \in \mathbb{R}$, $h > 0$, such that $[x-h] \cap [x+h] \neq \emptyset$. Notice that

$$E(x-h)(b) + E(x+h)(b) = 2\,E(x)(b) \quad \text{for all} \quad b \in \mathcal{B}$$

and, since $[x-h] \cap [x+h] \neq \emptyset$,

$$E(x-h)(b) = E(x+h)(b) = E(x)(b) \quad \text{for all } b \in \operatorname{supp}(x-h) \cap \operatorname{supp}(x+h).$$

Hence,

(A) $\operatorname{supp}(x-h) \cup \operatorname{supp}(x+h) = \operatorname{supp}(x)$

and for all $b \in \operatorname{supp}(x)$ exactly one of the following holds:

(B) $b \in \operatorname{supp}(x-h) \cap \operatorname{supp}(x+h)$ and $E(x-h)(b) = E(x+h)(b) = E(x)(b)$,

(C) $b \in \operatorname{supp}(x-h) \setminus \operatorname{supp}(x+h)$ and $E(x-h)(b) = 2\,E(x)(b)$, $E(x+h)(b) = 0$, or

(D) $b \in \operatorname{supp}(x+h) \setminus \operatorname{supp}(x-h)$ and $E(x+h)(b) = 2\,E(x)(b)$, $E(x-h)(b) = 0$.

It is clear that there are at most 3^n numbers $h > 0$ satisfying (A)–(D), where n is the cardinality of $\operatorname{supp}(x)$. Thus, Theorem 1.1 has been proved. □

The next theorem shows that unlike nowhere symmetrically continuous functions, the uniformly antisymmetric functions can be neither measurable nor have the Baire property. (See footnote 1.)

Theorem 1.3 *If $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly antisymmetric, then f is neither measurable nor does it have the Baire property.*

PROOF. Our proof uses the following lemma, in which a point x is said to be a *qualitative point* of a set $S \subset \mathbb{R}$, if there is a neighborhood U of x such that $U \setminus S$ is of the first category. □

Lemma 1.4 *If $d > 0$ and x is either a density point of S or a qualitative point of S, then there is $0 < h < d$ such that $-h, h \in S$.*

PROOF. If x is a density point of S, choose $0 < \varepsilon < d$ such that the Lebesgue measure of $S \cap (x-\varepsilon, x+\varepsilon)$ is greater that ε. If x is a qualitative point of S, choose $0 < \varepsilon < d$ such that $(x-\varepsilon, x+\varepsilon) \setminus S$ is of first category. In both cases it is easy to find $0 < h < \varepsilon$ such that $-h, h \in S$. □

It is well known [7] that for a Baire function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a first category set A such that every point $x \in \mathbb{R} \setminus A$ is a qualitative point of $f^{-1}((f(x) - d/2, f(x) + d/2))$ for every $d > 0$. A point h, chosen as in Lemma 1.4, contradicts $(\star)$.

The argument when f is measurable follows the same path, if we recall that a measurable function is approximately continuous almost everywhere; i.e., that almost every point x is a density point of $f^{-1}((f(x)-d/2, f(x)+d/2))$ for every $d>0$.

Corollary 1.5 *The existence of a uniformly antisymmetric function cannot be proved without using of the Axiom of Choice.*

PROOF. Shelah [8] proved that there exists a model of set theory ZF (without the Axiom of Choice) with the Dependent Choice Axiom DC (i.e., standard induction) in which every subset of $\mathbb{R}$ has the Baire property. In this model every real function is a Baire function. So, by Theorem 1.3,[4] it is not uniformly antisymmetric. □

2. Bounded antisymmetric functions

For the results in this section, we need the following generalization of ($\star$).

Let $K \subset \mathbb{R}$ be an additive subgroup of $\mathbb{R}$. A function $f\colon K \to \mathbb{R}$ is said to be *uniformly antisymmetric* provided for every $x \in K$, there exists $d>0$ such that for every $0<h<d$, $h \in K$,

$$|f(x-h)-f(x+h)| \geq d. \tag{4}$$

The next theorem has been also proved, by a different method, in [6] for the case $K=\mathbb{R}$.

Theorem 2.1 *If $K \subset \mathbb{R}$ is an uncountable additive subgroup of $\mathbb{R}$ then there is no uniformly antisymmetric function $f\colon K \to \{0,1\}$.*[5] *In particular, a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ cannot be a characteristic function.*

PROOF. By way of contradiction assume that there exists a uniformly antisymmetric function $f\colon K \to \{0,1\}$ and for every $x \in K$ let $n_x \in \mathbb{N}$ be such that for every $0<h<1/n_x$, $h \in K$,

$$|f(x-h)-f(x+h)| \geq 1/n_x.$$

Fix $n \in \mathbb{N}$ such that $L=\{x \in K : n_x = n\}$ is uncountable. Then,

$$f(x-h) \neq f(x+h) \text{ for every } x \in L \text{ and } h \in (0,1/n) \cap K. \tag{5}$$

[4]More precisely, we should notice that the Theorem 1.3 has been proved in ZF+DC.

[5]This theorem is also true in case $f\colon K \to \{0,1,2\}$. This proof will appear in future work.

Choose $x, y, z \in L$ such that $x < y < z < x + 1/n$ and let

$$a = x - y + z, \quad b = x + y - z, \quad c = -x + y + z.$$

Then, for $h = z - y \in (0, 1/n) \cap K$ we have $a = x + h$ and $b = x - h$. Hence, by (5), $f(a) \neq f(b)$. Similarly, $f(b) \neq f(c)$ and $f(c) \neq f(a)$. But this is impossible, since the points a, b and c are distinct and f attains only two values. This contradiction finishes the proof of Theorem 2.1. □

The next theorem is the main step in the proof that the assumption that K is uncountable in Theorem 2.1 is essential.

Theorem 2.2 *There is a partition $\mathcal{P} = \{A_0, A_1\}$ of the set D_+ of positive dyadic numbers, $k/2^n$, $k, n \in \mathbb{N}$, such that for every $x \in D_+$ the set*

$$D_x = \bigcup_{i \in \{0,1\}} \{h \in D_+ \cap (0, x) \colon x - h, x + h \in A_i\} \tag{6}$$

is finite.

PROOF. Put

$$A_0 = \left\{ \frac{4k+1}{2^n} \colon k, n \in \mathbb{N} \right\}, \quad A_1 = \left\{ \frac{4k+3}{2^n} \colon k, n \in \mathbb{N} \right\}$$

and define

$$D_n = \left\{ \frac{2k+1}{2^n} \colon k \in \mathbb{N} \right\}$$

for every $n \in \mathbb{N}$. It is easy to see that the sets D_n form a partition of D_+. Notice also that for $m < n$

$$\frac{4k+1}{2^n} + \frac{4l+1}{2^n} = \frac{2(k+l)+1}{2^{n-1}} \in D_{n-1} \text{ and } \frac{4k+1}{2^m} + \frac{4l+1}{2^n} \in D_n.$$

Hence, for $x \in D_n$ we have $2x \in D_{n-1}$ and if $a, b \in A_0$ are such that $a + b = 2x$, then $a, b \in \bigcup_{m \leq n} D_m$. Similarly we can show that for $x \in D_n$, if $a, b \in A_1$ are such that $a + b = 2x$, then $a, b \in \bigcup_{m \leq n} D_m$. In particular, if $x \in D_n$, $a = x - h$, $b = x + h$ and $a, b \in A_i$ for $i \in \{0, 1\}$, then $a, b \in \bigcup_{m \leq n} D_m$. But, since $a, b > 0$ it is easy to see that there are only finitely many pairs $\langle a, b \rangle$ such that $a + b = 2x$ and $a, b \in \bigcup_{m \leq n} D_m$. So, the set D_x is finite. □

Corollary 2.3 *If D is the set of all dyadic numbers $k/2^n$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, then there exists a uniformly antisymmetric function $f \colon D \to \{0, 1\}$. In particular, the uncountability of K in Theorem 2.1 is essential.*

PROOF. As in Corollary 1.2, we can use the partition of Theorem 2.2 to show the existence of a uniformly antisymmetric function $f\colon D_+ \to \{0,1\}$ by putting $f(x) = i$ for $x \in A_i$. Extend this function to D by defining $f(0)$ arbitrarily and $f(-d) = 1 - f(d)$ for $d \in D_+$. It is easy to see that such a function is uniformly antisymmetric. □

The following problems remain open.[6]

Problem 1 *Is there a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ such that its range $f[\mathbb{R}]$ is (a) finite? (b) bounded?*

Problem 2 *Is there a finite partition $\mathcal{P}$ of $\mathbb{R}$ such that for every $x \in \mathbb{R}$ the set*

$$\bigcup_{P\in\mathcal{P}} \{h > 0\colon x - h, x + h \in P\}$$

is finite?[7]

Notice that Corollary 1.2 shows that a positive answer for Problem 2 implies a positive answer for Problem 1 and that, by Theorem 2.1 and Corollary 1.2, a partition $\mathcal{P}$ from Problem 2 must have at least 3 elements.

Problem 3 *Is there a countable subfield K of $\mathbb{R}$ for which we can find a uniformly antisymmetric function $f\colon K \to \{0,1\}$? What about $K = \mathbb{Q}$?*[8]

3. Generalizations

The following theorem generalizes some of the results given above. Part (ii) was inspired by discussions with Paul Erdös. The implication (i)⇒(ii) is due to Sierpiński [9]. The equivalence of (i) and (ii) is also implicitly contained in Freiling [4].

Theorem 3.1 *The following conditions are equivalent.*

(i) *The Continuum Hypothesis.*

[6] The preliminary versions of this paper have been circulated since the Fall of 1992. Since then, several problems have been solved, as mentioned in the footnotes.

[7] A negative answer has been given by P. Komjáth and S. Shelah [5]. J. Baumgartner [1] also showed independently the somewhat stronger result that for every finite partition $\mathcal{P}$ of $\mathbb{R}$ and a cardinal number $\kappa < \mathfrak{c}$ there exists $x \in \mathbb{R}$ such that the set $\bigcup_{P\in\mathcal{P}}\{h > 0\colon x - h, x + h \in P\}$ has cardinality $\geq \kappa$.

[8] P. Komjáth and S. Shelah [5] showed that a uniformly antisymmetric function $f\colon K \to \{0,1\}$ exists for every countable subfield K of $\mathbb{R}$. They also proved that there exists a function $f\colon \mathbb{Q} \to \{0,1,2,3\}$ such that the set S_x is finite for every $x \in \mathbb{Q}$. Such a function is uniformly antisymmetric.

(ii) *There exists a partition $\mathcal{P} = \{A_0, A_1\}$ of $\mathbb{R}$ such that for every $x \in \mathbb{R}$ the set*

$$T_x = \bigcup_{i \in \{0,1\}} \{h > 0 \colon x - h, x + h \in A_i\} \tag{7}$$

is at most countable.

(iii) *There exists a function $f \colon \mathbb{R} \to \{0,1\}$ such that for every $x \in \mathbb{R}$ there is a $d > 0$ with the property that*

$$|f(x-h) - f(x+h)| \geq d \text{ for all but countable many } h \in (0,d). \tag{8}$$

PROOF. (i)$\Rightarrow$(ii). Let $\mathcal{B} = \{b_\zeta \colon \zeta < \omega_1\}$ be a Hamel basis. For $x \in \mathbb{R} \setminus \{0\}$ let $q(x) = q_n$ where $x = q_1 b_{\zeta_1} + \ldots + q_n b_{\zeta_n}$ is the unique representation of x in the basis $\mathcal{B}$, with $\zeta_1 < \ldots < \zeta_n$ and $q_i \neq 0$. Put

$$x \in A_0 \text{ if and only if } q(x) > 0$$

and $A_1 = \mathbb{R} \setminus A_0$. We will show that the partition $\mathcal{P} = \{A_0, A_1\}$ satisfies (7).

For $\xi < \omega_1$ let K_ξ be the linear subspace of $\mathbb{R}$ generated by $\{b_\zeta \colon \zeta \leq \xi\}$. Notice that $\mathbb{R} = \bigcup_{\xi < \omega_1} K_\xi$ and that, by the Continuum Hypothesis, every K_ξ is countable. We will show that if $x \in K_\xi$ then $T_x \subset K_\xi$. This will finish the proof.

But it is easy to see that for $h \in \mathbb{R} \setminus K_\xi$ and $x \in K_\xi$,

$$x \in A_0 \text{ if and only if } x + h \in A_0.$$

Since h and $-h$ cannot belong to the same A_i for $h \neq 0$, we have $T_x \subset K_\xi$.

(ii)$\Rightarrow$(iii). The proof of this implication is the same as the one for Corollary 1.2, if we put $f(x) = i$ for $x \in A_i$.

(iii)$\Rightarrow$(i). Let f be as in (iii) and for every $x \in \mathbb{R}$ let $n_x \in \mathbb{N}$ be such that for every $0 < h < 1/n_x$,

$$|f(x-h) - f(x+h)| \geq 1/n_x \text{ for all but countable many } 0 < h < d \tag{9}$$

and let C_x be a countable set of exceptional points for (9); i.e.,

$$C_x = \{h < 1/n_x \colon f(x-h) \neq f(x+h)\}.$$

By way of contradiction, assume that the Continuum Hypothesis fails and let B be a linearly independent subset of $\mathbb{R}$ over $\mathbb{Q}$ of cardinality ω_2. Choose $K \subset B$ of cardinality ω_2 such that for some $n \in \mathbb{N}$ we have $n_x = n$ for all

$x \in K$. Let U be an open interval of length less than $1/n$ such that the set $L = K \cap U$ has cardinality ω_2. Then, in particular,

$$(10) \qquad f(x-h) \neq f(x+h) \quad \text{for every} \quad x \in L \quad \text{and} \quad h \in (0, 1/n) \setminus C_x,$$

and

$$(11) \qquad |x-y| < 1/n \quad \text{for every} \quad x, y \in L.$$

Define, by transfinite induction, a sequence $\{t_\xi \in L : \xi < \omega_2\}$ such that

$$(12) \qquad t_\xi \notin T_\xi \quad \text{for every} \quad \xi < \omega_2,$$

where T_ξ is the smallest linear subspace of $\mathbb{R}$ containing $\{t_\zeta : \zeta < \xi\}$ and such that

$$(13) \qquad C_x \subset T_\xi \quad \text{for every} \quad x \in T_\xi.$$

Evidently every set T_ξ has cardinality $\leq \omega_1$, so the induction can be done easily.

Now, put $x = t_0$, $z = t_{\omega_1}$ and, for $0 < \xi < \omega_1$ consider the numbers $|-x+t_\xi|$. All these numbers are different, so there is $0 < \eta < \omega_1$ such that

$$(14) \qquad |-x+t_\eta| \notin C_z.$$

Put $y = t_\eta$ and proceed as in Theorem 2.1. Define

$$(15) \qquad a = x-y+z, \quad b = x+y-z, \quad c = -x+y+z.$$

Then, $(a+b)/2 = x$, $(b+c)/2 = y$ and $(c+a)/2 = z$. We will show that

$$(16) \qquad f(a) \neq f(b), \quad f(b) \neq f(c), \quad f(c) \neq f(a)$$

which will give us the desired contradiction, since the points a, b and c are distinct and f admits only two values.

To prove (16), notice first that it follows from (10) as long as

$$(17) \qquad |a-x| < 1/n, \quad |b-y| < 1/n, \quad |c-z| < 1/n$$

and

$$(18) \qquad |a-x| \notin C_x, \quad |b-y| \notin C_y, \quad |c-z| \notin C_z.$$

But (17) follows easily from (15) and (11). Finally, (18) can be proved as follows.

$|a-x| \notin C_x$, since otherwise we would have $z-y=a-x \in T_{\omega_1}$ and $z \in y+T_{\omega_1} \subset T_{\omega_1}$, contradicting $z=t_{\omega_1} \notin T_{\omega_1}$.

$|b-y| \notin C_y$, since otherwise we would have $z-x=y-b \in T_{\omega_1}$ and $z \in x+T_{\omega_1} \subset T_{\omega_1}$, contradicting $z=t_{\omega_1} \notin T_{\omega_1}$.

$|c-z| \notin C_z$, since otherwise we would have $|-x+t_\eta|=|-x+y|=|c-z| \in C_z$, contradicting (14).

This finishes the proof of Theorem 3.1. □

We finish this paper with the following open problem.

Problem 4 *Does there exist a partition $\mathcal{P}=\{P_n : n \in \mathbb{N}\}$ of $\mathbb{R}$ such that the set*

$$S_x = \bigcup_{n \in \mathbb{N}} \{h>0 : x-h, x+h \in P_n\}$$

has at most one element.[9]

References

[1] James E. Baumgartner, A letter to K. Ciesielski, May 1993.

[2] E. F. Collingwood, A. J. Lohwater, The Theory of Cluster Sets, Cambridge University Press, Cambridge, 1966.

[3] R. Engelking, General Topology, Polish Scientific Publishers, Warsaw, 1977.

[4] Chris Freiling, *A converse to a theorem of Sierpinski on almost symmetric sets*, Real Analysis Exchange **15** (1989-90), 760–767.

[5] P. Komjáth and S. Shelah, *On uniformly antisymmetric functions*, Real Analysis Exchange **19(1)** (1993-94), 218–225.

[6] P. Kostyrko, *There is no strongly locally antisymmetric set*, Real Analysis Exchange **17** (1991-92), 423–425.

[7] K. Kuratowski, Topology, Vol. I, Academic Press, New York, NY, 1968.

[8] S. Shelah, *Can you take Solovay's innaccessible away?*, Israel J. Math. **48** (1984), 1-47.

[9] W. Sierpiński, *Sur une fonction non mesurable partout presque symetrique*, Acta Litt. Scient (Szeged) **8** (1936), 1-6. *Oeuvres Choisies* t. **III**, 277-281.

[9] P. Komjáth and S. Shelah [5] proved that a positive answer for this question is equivalent to the Continuum Hypothesis. They also proved that if $\mathfrak{c} \geq \aleph_n$ then for every partition $\mathcal{P}=\{P_n : n \in \mathbb{N}\}$ of $\mathbb{R}$ there exists $x \in \mathbb{R}$ such that the set S_x has $\geq 2^n-1$ elements.