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# Linear subspace of $\mathbb{R}^\lambda$ without dense totally disconnected subsets

by

**Krzysztof Ciesielski** (Morgantown, W.Va.)

**Abstract.** In [1] the author showed that if there is a cardinal $\kappa$ such that $2^\kappa = \kappa^+$ then there exists a completely regular space without dense 0-dimensional subspaces. This was a solution of a problem of Arkhangel'skiĭ. Recently Arkhangel'skiĭ asked the author whether one can generalize this result by constructing a completely regular space without dense totally disconnected subspaces, and whether such a space can have a structure of a linear space. The purpose of this paper is to show that indeed such a space can be constructed under the additional assumption that there exists a cardinal $\kappa$ such that $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$.

**1. Notation and lemmas.** The topological terminology used in this paper is standard and follows [2] with the exception that we use the term *totally disconnected* for the topological spaces which have no connected subsets with more than one point. (In [3, 2] such spaces are called hereditarily disconnected.)

The set-theoretical terminology and notation used in this paper is standard and follows [3]. In particular, ordinals are identified with their sets of predecessors and cardinals with the initial ordinals. The symbol $\omega$ denotes the first infinite ordinal as well as first infinite cardinal. $\mathcal{P}(X)$ stands for the power set of $X$ and $|X|$ is the cardinality of $X$. If $\kappa$ is a cardinal then $\kappa^+$ denotes the cardinal successor of $\kappa$ and $2^\kappa = |\mathcal{P}(\kappa)|$. $[X]^{\le\kappa}$ will denote $\{Y \subset X : |Y| \le \kappa\}$. Similarly we define $[X]^{<\kappa}$. Functions will be identified with their graphs. The class of all functions $f : X \to Y$ from a set $X$ to a set $Y$ is denoted by $Y^X$.

The space $\mathbb{R}^\lambda$ will always be considered as a linear topological space over $\mathbb{R}$ with the standard operations and the product topology. For a cardinal $\kappa$ a topological space $X$ is said to be $\kappa$-*Lindelöf* provided every open cover of $X$ has a subcover of cardinality $\le \kappa$.

We will also need the following notation. Let $\mathcal{B}_0$ denote a fixed countable base for $\mathbb{R}$, let $\mathcal{B}(A) = \{\varepsilon : D \to \mathcal{B}_0 : D \in [A]^{<\omega}\}$ and for $\varepsilon \in \mathcal{B}(A)$ let

$[\varepsilon] = \{f \in \mathbb{R}^A : (\forall a \in \mathrm{dom}(\varepsilon))(f(a) \in \varepsilon(a))\}$ be a basic open set for $\mathbb{R}^A$.

For a cardinal $\kappa$ let $H_\kappa(A)$ denote the set of all functions $g : D \to \mathbb{R}$ such that $D \in [A]^{\le\kappa}$, and let $\mathcal{F}_\kappa(A)$ be the class of all $f : H_\kappa(A) \to H_\kappa(A)$ such that $f(g) \supset g$ for all $g \in H_\kappa(A)$. For $g \in H_\kappa(A)$ let $[g] = \{f \in \mathbb{R}^A : g \subset f\}$. Moreover, for an ordinal $\xi$ with $\kappa^+ \le \xi \le \kappa^{++}$, let $\mathcal{D}_\kappa(\xi)$ be the family of all sets of the form $D_f = (\mathbb{R}^\zeta \setminus \bigcup_{g\in H_\kappa(\zeta)}[f(g)]) \times \mathbb{R}^{\kappa^{++}\setminus\zeta}$ where $\kappa^+ \le \zeta \le \xi$ and $f \in \mathcal{F}_\kappa(\zeta)$. Finally, define $\mathcal{D}_\kappa = \bigcup_{\kappa^+<\xi<\kappa^{++}} \mathcal{D}_\kappa(\xi)$.

In what follows we will use the following well known fact. For completeness sake, we sketch its proof.

Lemma 1. *For any disconnected set $S \subset \mathbb{R}^X$ there exists $\delta \in H_\omega(X)$ such that $[\delta] \cap S = \emptyset$.*

Proof. If $S$ is not dense in $\mathbb{R}^X$ then we can easily find an appropriate $\delta$.

Assume that $S$ is dense in $\mathbb{R}^X$ and let $U, V \subset \mathbb{R}^X$ be non-empty disjoint open sets such that $S \subset U \cup V$. Let $\{[\varepsilon_n] : n < \omega\}$ be a maximal family of non-empty disjoint basic open sets $[\varepsilon]$ such that either $[\varepsilon] \subset U$ or $[\varepsilon] \subset V$. It is countable since $\mathbb{R}^X$ has the Suslin property. Now, if $D = \bigcup_{n<\omega} \mathrm{dom}(\varepsilon_n)$, $U_0 = U \cap \bigcup_{n<\omega}[\varepsilon_n]$, $V_0 = V \cap \bigcup_{n<\omega}[\varepsilon_n]$ and $U_1$ and $V_1$ are the projections of $U_0$ and $V_0$ into $\mathbb{R}^D$, then $U_1$ and $V_1$ are non-empty, open, disjoint and, by connectedness of $\mathbb{R}^D$, there is a $\delta \in \mathrm{cl}(U_1) \cap \mathrm{cl}(V_1)$. Then $[\delta] \subset \mathrm{cl}(U_0) \cap \mathrm{cl}(V_0) \subset \mathrm{cl}(U) \cap \mathrm{cl}(V)$ and indeed $[\delta] \cap S \subset [\delta] \cap (U \cup V) = \emptyset$.

Now we are ready for our main lemma.

Lemma 2. *Assume that $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$. Then*

(1) $|\mathcal{D}_\kappa| = \kappa^{++}$;

(2) *for every totally disconnected set $S \subset \mathbb{R}^{\kappa^{++}}$ there is a $D \in \mathcal{D}_\kappa$ such that $S \subset D$*;

(3) *if $\kappa^+ < \xi < \kappa^{++}$, $\mathcal{D}_0 \subset \mathcal{D}_\kappa(\xi)$, and $|\mathcal{D}_0| \le \kappa^+$ then there is a $g \in H_{\kappa^+}(\kappa^{++})$ such that $[g] \cap \bigcup \mathcal{D}_0 = \emptyset$*;

(4) *$r(h + D) \in \mathcal{D}_\kappa$ for any $h \in \mathbb{R}^{\kappa^{++}}$, $r \in \mathbb{R} \setminus \{0\}$ and $D \in \mathcal{D}_\kappa$.*

Proof. (1) For $\kappa^+ < \xi < \kappa^{++}$ we have $|H_\kappa(\xi)| = |\xi|^\kappa|\mathbb{R}|^\kappa = \kappa^\kappa 2^\kappa = \kappa^+$, and so $|\mathcal{D}_\kappa(\xi)| = |\bigcup_{\kappa^+\le\zeta\le\xi} \mathcal{F}_\kappa(\zeta)| = \kappa^+$. Hence, $|\mathcal{D}_\kappa| \le \kappa^{++}|\mathcal{D}_\kappa(\xi)| = \kappa^{++}$.

(2) Since $S$ is totally disconnected, for every $g \in H_\kappa(\kappa^{++})$ the set $S \cap [g]$ must be disconnected or have at most one point. Since $[g]$ is connected and homeomorphic to $\mathbb{R}^{\kappa^{++}}$, by Lemma 1 we can find a countable extension $f(g) \in H_\kappa(\kappa^{++})$ of $g$ such that $S \cap [f(g)] = \emptyset$. Thus, we have defined $f \in \mathcal{F}_\kappa(\kappa^{++})$ such that $S \subset \mathbb{R}^{\kappa^{++}} \setminus \bigcup\{[f(g)] : g \in H_\kappa(\kappa^{++})\}$.

Now, define $\{\xi_\eta : \eta < \kappa^+\}$ by induction on $\eta < \kappa^+$ by putting $\xi_0 = \kappa^+$, $\xi_\lambda = \bigcup_{\eta<\lambda} \xi_\eta$ for limit ordinals $\lambda < \kappa^+$ and choosing $\xi_{\eta+1} < \kappa^{++}$ such that $f(H_\kappa(\xi_\eta)) \subset H_\kappa(\xi_{\eta+1})$. This can be done since $|H_\kappa(\xi_\eta)| \le \kappa^+$. Define

$\xi = \bigcup_{\eta<\kappa^+} \xi_\eta < \kappa^{++}$. Then $f|_{H_\kappa(\xi)} \in \mathcal{F}_\kappa(\xi)$ since for every $g \in H_\kappa(\xi)$ there is an $\eta < \kappa^+$ such that $g \in H_\kappa(\xi_\eta)$. Thus,

$$S \subset \mathbb{R}^{\kappa^{++}} \setminus \bigcup\{[f(g)] : g \in H_\kappa(\kappa^{++})\} \subset \mathbb{R}^{\kappa^{++}} \setminus \bigcup\{[f(g)] : g \in H_\kappa(\xi)\} \in \mathcal{D}_\kappa\,.$$

(3) Let $\{D_{f_\eta} : \eta < \kappa^+\}$ be an enumeration of $\mathcal{D}_0$ where $f_\eta \in \mathcal{F}_\kappa(\zeta_\eta)$. Put $\zeta = \sup\{\zeta_\eta : \eta < \kappa^+\}$ and construct, by induction on $\eta < \kappa^+$, an increasing (in the sense of inclusion) sequence of functions $\{g_\eta \in H_\kappa(\zeta) : \eta < \kappa^+\}$ by taking $g_\eta = f_\eta(\bigcup_{\gamma<\eta} g_\gamma)$. Thus, $[g_\eta] \cap D_{f_\eta} = \emptyset$. It is easy to see that $g = \bigcup_{\eta<\kappa^+} g_\eta \in H_{\kappa^+}(\kappa^{++})$ satisfies the requirements.

(4) It is easy to check that for $f \in \mathcal{F}_\kappa(\zeta)$ we have $r(h + D_f) = D_{f'} \in \mathcal{D}_\kappa$, where $f' \in \mathcal{F}_\kappa(\zeta)$ is defined for every $g \in H_\kappa(\zeta)$ and $\xi \in \mathrm{dom}(f(g))$ by $f'(r[g + h|_{\mathrm{dom}(g)}])(\xi) = r[f(g)(\xi) + h(\xi)]$. The function $f'$ is indeed defined on $H_\kappa(\zeta)$ since for every $g' \in H_\kappa(\zeta)$ there is a $g \in H_\kappa(\zeta)$ such that $g' = r[g + h|_{\mathrm{dom}(g)}]$.

**2. The example.** Now we are ready to prove our main theorem.

THEOREM 1. *Assume that there exists an infinite cardinal $\kappa$ such that $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$. Then there exists a linear subspace $L \subset \mathbb{R}^{\kappa^{++}}$ which does not contain any dense totally disconnected subset.*

Proof. Let $\{D_\eta : \eta < \kappa^{++}\}$ be an enumeration of $\mathcal{D}_\kappa$. We will define an increasing sequence $\{\alpha_\eta < \kappa^{++} : \eta < \kappa^{++}\}$ of ordinals and a sequence $\{g_\eta \in \mathbb{R}^{\kappa^{++}} : \eta < \kappa^{++}\}$ by induction on $\eta < \kappa^{++}$. Assume that for some $\eta < \kappa^{++}$ our construction is done for all $\zeta < \eta$. Let $L_\eta$ be a linear subspace of $\mathbb{R}^{\kappa^{++}}$ generated by $\{g_\zeta : \zeta < \eta\}$ and define

$$\mathcal{E}_\eta = \{r(h + D_\zeta) : r \in \mathbb{R} \setminus \{0\},\ h \in L_\eta,\ \zeta < \eta\}\,.$$

By Lemma 2(4), $\mathcal{E}_\eta \subset \mathcal{D}_\kappa$ and it is easy to see that $|\mathcal{E}_\eta| \le \kappa^+$. Hence, by Lemma 2(3), there exists $g \in H_{\kappa^+}(\kappa^{++})$ such that $[g] \cap \bigcup \mathcal{E}_\eta = \emptyset$. We can also find $\alpha_\eta < \kappa^{++}$ such that $g \in H_{\kappa^+}(\alpha_\eta)$ and $\alpha_\zeta < \alpha_\eta$ for all $\zeta < \eta$. Define $g_\eta \in \mathbb{R}^{\kappa^{++}}$ by taking $g_\eta \supset g$, $g_\eta(\alpha_\eta) = 1$ and $g_\eta(\xi) = 0$ for $\xi > \alpha_\eta$, and notice that $g_\eta \notin \bigcup \mathcal{E}_\eta$ since $g_\eta \in [g]$. Define $L$ to be the linear subspace of $\mathbb{R}^{\kappa^{++}}$ generated by $\{g_\eta : \eta < \kappa^{++}\}$.

To see that $L$ satisfies the assertion of the theorem first notice that $L = \bigcup_{\eta<\kappa^{++}} L_\eta$. If $g \in L$ then there are $\eta_1 < \ldots < \eta_n < \kappa^{++}$ and non-zero real numbers $r_1, \ldots, r_n$ such that $g = r_1 g_{\eta_1} + \ldots + r_n g_{\eta_n}$. Then $g(\alpha_{\eta_n}) = r_n \neq 0$ while $g(\xi) = 0$ for $\xi > \alpha_{\eta_n}$. Hence, for the function $z_\eta$ defined by $z_\eta(\xi) = 0$ for all $\alpha_\eta \le \xi < \kappa^{++}$ we have $L \cap [z_\eta] = L_\eta \neq L$. But for every $D \subset [L]^{\le\kappa^+}$, there is an $\eta < \kappa^{++}$ such that $D \subset L_\eta$. Since every set $[z_\eta]$ is closed in $\mathbb{R}^{\kappa^{++}}$, we conclude that $L$ does not have a dense subset of cardinality $\kappa^+$.

On the other hand, we will show that $L \cap D_\xi \subset L_\xi$ for every $\xi < \kappa^{++}$. This will finish the proof since $|L_\xi| \le \kappa^+$ and, by Lemma 2(2), every totally disconnected set in $\mathbb{R}^{\kappa^{++}}$ is a subset of some $D_\xi$.

So let $g = h + rg_\eta \in L \setminus L_\xi$, where $h \in L_\eta$, $\eta \ge \xi$ and $r \in \mathbb{R} \setminus \{0\}$. Then $r^{-1}(-h + D_\xi) \in \mathcal{E}_\eta$, and so $g_\eta \notin r^{-1}(-h + D_\xi)$. Hence, indeed, $g = h + rg_\eta \notin D_\xi$.

This finishes the proof of Theorem 1.

**3. Remarks.** The example from [1] mentioned in the abstract is hereditarily $\kappa$-Lindelöf if the assumption $2^\kappa = \kappa^+$ is used in the construction. In particular, under the Continuum Hypothesis the space is hereditarily Lindelöf, and hence also normal. By the similar method we can generalize the example from Theorem 1 to be hereditarily $\kappa^+$-Lindelöf. However, the following problem remains open.

PROBLEM 1. *Does there exist* (*at least consistently with* ZFC) *a linear topological space without dense totally disconnected subspaces which is normal*? *Lindelöf*? *hereditarily Lindelöf*?

Let us also mention that the set-theoretical assumption in Theorem 1 can be weakened to the following: there exists an infinite cardinal $\lambda$ such that $2^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. The proof remains essentially the same.

We finish the paper by quoting yet another problem of Arkhangel'skiĭ (private communication) concerning the same subject.

PROBLEM 2. *Does there exist a completely regular topological space* $X$ *such that* $C_{\rm p}(X)$ *has no dense* 0-*dimensional* (*or totally disconnected*) *subspace*, *where* $C_{\rm p}(X)$ *stands for the space of all continuous functions* $f : X \to \mathbb{R}$ *with the topology of pointwise convergence*?

## References

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DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA 26506-6310
U.S.A.

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