Real Analysis Exchange Vol. 17 (1991-92)

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THE PEANO CURVE AND $\mathcal{I}$-APPROMIMATE DIFFERENTIABILITY

1 Preliminaries

A function $f: \mathbb{R} \to \mathbb{R}$ is *density continuous* (*$\mathcal{I}$-density continuous*, *deep-$\mathcal{I}$-density continuous*) at the point x if it is continuous at x when the density topology ($\mathcal{I}$-density topology, deep-$\mathcal{I}$-density topology) is used on both the domain and the range [2, 3, 4, 10]. In [4] it is proved that the first coordinate of the classical Peano area-filling curve is nowhere approximately differentiable, even though it is continuous and density continuous. In this paper we generalize this result by proving in Section 3 that the same function is also $\mathcal{I}$-density and deep-$\mathcal{I}$-density continuous, even though it is nowhere $\mathcal{I}$-approximately differentiable. To prove this it is shown that a point x is a deep-$\mathcal{I}$-density point of the Baire set E if, and only if, x is an $\mathcal{I}$-density point of the unique regular open set $\tilde{E}$ such that the symmetric difference $E \triangle \tilde{E}$ is of the first category.

In Section 4 we give an example of a bounded $\mathcal{I}$-approximately continuous function that is not a derivative.

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $A, B \subset \mathbb{R}$ and $d \in \mathbb{R}$ the complement of A is denoted by A^c, the symmetric difference of A and B is denoted by $A \triangle B = (A \bigcup B) \setminus (A \cap B)$, while $B - d = \{x - d \in \mathbb{R}: x \in B\}$ and $dB = \{dx \in \mathbb{R}: x \in B\}$. The symbol $\mathcal{B}$ stands for the family of subsets of $\mathbb{R}$ which have the Baire property and $\mathcal{I}$ denotes the ideal of

[1]Received support from the University of Louisville Graduate School
AMS Subject Classification 26A21

first category subsets of $\mathbb{R}$. A statement about a subset of $\mathbb{R}$ is true $\mathcal{I}$-a.e. if the set on which it fails to be true is in $\mathcal{I}$. An open set $E \subset \mathbb{R}$ is *regular* if $E = \operatorname{int}(\operatorname{cl}(E))$. For a set $E \in \mathcal{B}$ we denote by $\tilde{E}$ the only regular open set A for which $E \triangle A \in \mathcal{I}$. The Lebesgue measure of a measurable set A is denoted by $\mathrm{m}(A)$.

If $A \in \mathcal{B}$, then 0 is an *$\mathcal{I}$-density point* of A if for every increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$\lim_{p \to \infty} \chi_{n_{m_p} A \cap (-1,1)} = \chi_{(-1,1)}, \ \mathcal{I}\text{-a.e.},$$

or, equivalently, that the set

$$\liminf_{p \to \infty} n_{m_p} A = \bigcup_{q \in \mathbb{N}} \bigcap_{p \geq q} n_{m_p} A$$

is residual in $(-1, 1)$. We say that a point a is an $\mathcal{I}$-density point of $A \in \mathcal{B}$ if 0 is an $\mathcal{I}$-density point of $A - a$. The set of all $\mathcal{I}$-density points of $A \in \mathcal{B}$ is denoted by $\Phi_{\mathcal{I}}(A)$. It is not difficult to see that $A \triangle \Phi_{\mathcal{I}}(A) \in \mathcal{I}$ for every $A \in \mathcal{B}$ [10, Theorem 3] and that

$$\Phi_{\mathcal{I}}(A) = \Phi_{\mathcal{I}}(B) \quad \text{for every } A, B \in \mathcal{B} \text{ such that } \quad A \triangle B \in \mathcal{I}. \tag{1}$$

The family of sets

$$\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} \colon A \subset \Phi_{\mathcal{I}}(A)\}$$

forms a topology on $\mathbb{R}$ called the *$\mathcal{I}$-density topology* [9, 10].

A point $a \in \mathbb{R}$ is a *deep-$\mathcal{I}$-density point* [10] of an $A \in \mathcal{B}$ if there exists a closed set $F \subset A \bigcup \{a\}$ such that a is an $\mathcal{I}$-density point of F. The set of all deep-$\mathcal{I}$-density points of $A \in \mathcal{B}$ is denoted by $\Phi_{\mathcal{D}}(A)$. The family of sets

$$\mathcal{T}_{\mathcal{D}} = \{A \in \mathcal{B} \colon A \subset \Phi_{\mathcal{D}}(A)\}$$

forms a topology on $\mathbb{R}$ called the *deep-$\mathcal{I}$-density topology* [5, 10].

A point x is a *dispersion* (*$\mathcal{I}$-dispersion*, *deep-$\mathcal{I}$-dispersion*) point of A if x is a density ($\mathcal{I}$-density, deep-$\mathcal{I}$-density) point of A^c.

A function $f \colon \mathbb{R} \to \mathbb{R}$ is said to be *$\mathcal{I}$-approximately differentiable at a point* x if there exists a number $\mathrm{D}^{(\mathcal{I})} f(x)$, called the *$\mathcal{I}$-approximate derivative* of f at x, such that for every $\varepsilon > 0$, x is an $\mathcal{I}$-density point of some Baire subset of

$$\left\{ t \in \mathbb{R} \colon \frac{f(t) - f(x)}{t - x} \in (\mathrm{D}^{(\mathcal{I})} f(x) - \varepsilon, \mathrm{D}^{(\mathcal{I})} f(x) + \varepsilon) \right\}.$$

(Compare also [6] and [10, Definition 8].)

We also need the following easy fact [1, Lemma 4].

Lemma 1. *If $B = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ is such that $\lim_{n\to\infty} b_n = 0$ and there exists a positive number c such that*

$$\frac{b_n - a_n}{b_n} > c,$$

for every $n \in \mathbb{N}$, then 0 is not an $\mathcal{I}$-dispersion point of B.

2 Basic Lemmas

We start this section with the following lemma.

Lemma 2. *If A is regular open, then $\Phi_{\mathcal{I}}(A) = \Phi_{\mathcal{D}}(A)$.*

Proof. The inclusion $\Phi_{\mathcal{D}}(A) \subset \Phi_{\mathcal{I}}(A)$ is obvious from the definitions.

To prove the converse inclusion let $x \in \Phi_{\mathcal{I}}(A)$. For simplicity we assume that $x = 0$. We must show that 0 is a deep-$\mathcal{I}$-density point of A.

But 0 is an $\mathcal{I}$-density point of A if, and only if, 0 is an $\mathcal{I}$-dispersion point of A^c. Without using the full strength of Theorem 1 from [7] it follows that 0 is an $\mathcal{I}$-density point of A if, and only if,

> for every nonempty interval $(a, b) \subset (-1, 1)$ there exist numbers $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there is a nonempty interval $(c, d) \subset (a, b)$ with the properties:
>
> $$|d - c| > \varepsilon \quad \text{and} \quad (c, d) \cap n\widetilde{A^c} = \emptyset.$$

Notice that in the above it is enough to consider only intervals (a, b) with rational endpoints. Moreover, by the regularity of A, $\mathrm{cl}\left(\widetilde{A^c}\right) = A^c$, and so

$$\begin{aligned}(c, d) \cap n\widetilde{A^c} = \emptyset &\iff (c, d) \cap nA^c = \emptyset \\ &\iff \frac{1}{n}(c, d) \cap A^c = \emptyset \\ &\iff \frac{1}{n}(c, d) \subset A.\end{aligned}$$

Hence, we can conclude that 0 is an $\mathcal{I}$-density point of A if, and only if,

($\star$) for every nonempty interval $I = (a, b) \subset (-1, 1)$ with rational endpoints there exists a number $\varepsilon_I > 0$ and $n_I \in \mathbb{N}$ such that for every $n \geq n_I$ there is a nonempty interval $(c, d) \subset [c, d] \subset (a, b)$ with the properties:

$$|d - c| > \varepsilon_I \quad \text{and} \quad \frac{1}{n}[c, d] \subset A.$$

Now we can construct a closed set $E \bigcup \{0\} \subset A \bigcup \{0\}$ which satisfies ($\star$). This will finish the proof.

Let $\{I_k\}_{k \in \mathbb{N}}$ be an enumeration of all nonempty subintervals of $(-1, 1)$ with rational endpoints and assume that $n_{I_k} \geq k$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let E_k be a union of intervals $\frac{1}{n}[c_n, d_n]$ for $n \geq n_{I_k}$, where $[c_n, d_n]$ is a subset of I_k satisfying ($\star$); i.e., such that

$$|d_n - c_n| > \varepsilon_{I_k} \quad \text{and} \quad \frac{1}{n}[c_n, d_n] \subset A.$$

Let

$$E = \bigcup_{k \in \mathbb{N}} E_k.$$

Notice that

$$E_k \subset (-\frac{1}{n_{I_k}}, \frac{1}{n_{I_k}}) \subset (-\frac{1}{k}, \frac{1}{k})$$

so that $E_k \setminus (-\frac{1}{n}, \frac{1}{n})$ intersects only finitely many closed intervals forming E_k. In particular,

$$E \setminus (-\frac{1}{k}, \frac{1}{k}) = \bigcup_{i<k} E_i \setminus (-\frac{1}{k}, \frac{1}{k}),$$

where the right-hand side intersects only finitely many closed intervals from the collection whose union forms E. This implies that $E \bigcup \{0\}$ is closed. This finishes the proof of Lemma 2.

Lemma 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{-1}(E) \in \mathcal{I}$ for every $E \in \mathcal{I}$. Then f is deep-$\mathcal{I}$-density continuous if, and only if, f is $\mathcal{I}$-density continuous.*

Proof. It is known that every $\mathcal{I}$-density continuous function is deep-$\mathcal{I}$-density continuous [3, Theorem 4.1(iv)]. To prove the converse implication choose a deep-$\mathcal{I}$-density continuous function f satisfying the assumption and let $f(x)$ be an $\mathcal{I}$-density point of $E \in \mathcal{B}$ with $f(x) \in E$. Then, by (1), $f(x)$ is an $\mathcal{I}$-density point of $\tilde{E}$ and, by the regularity of $\tilde{E}$ and Lemma 2, $f(x)$ is also a deep-$\mathcal{I}$-density point of $\tilde{E}$. Thus, x is a deep-$\mathcal{I}$-density point of $f^{-1}(\tilde{E})$. Moreover, by the assumption,

$$f^{-1}(\tilde{E}) \triangle f^{-1}(E) = f^{-1}(\tilde{E} \triangle E) \in \mathcal{I}.$$

So, by (1), x is an $\mathcal{I}$-density point of $f^{-1}(E)$.

We will need also the following characterization of a deep-$\mathcal{I}$-density point.

Lemma 4. *The following conditions are equivalent:*

(i) x is a deep-$\mathcal{I}$-density point of A;

(ii) given $s \in (0,1)$, there exists $D_s > 0$ and $R_s \in (0,1)$ such that whenever $0 < D < D_s$ and $(y-\delta, y+\delta) \subset (x-D, x+D) \setminus \{x\}$ with $2\delta/D > s$, then there is an interval $J \subset (y-\delta, y+\delta) \cap A$ with $m(J)/2\delta > R_s$.

Proof. Without any loss of generality it may be assumed that $x = 0$. In [11, Theorem (5)] Zajíček proves that 0 is an $\mathcal{I}$-density point of A if, and only if, (ii) is satisfied, where the inclusion $J \subset (y-\delta, y+\delta) \cap A$ is replaced by

$$[(y-\delta, y+\delta) \cap A] \setminus J \in \mathcal{I}. \tag{2}$$

But we can assume that A is open, since there is a closed set $F \subset A \bigcup \{0\}$ for which 0 is an $\mathcal{I}$-density point and then, by Lemma 2, 0 is also an $\mathcal{I}$-density point of $\tilde{F} = \operatorname{int}(F)$. But then (2) implies that $J \subset (y-\delta, y+\delta) \cap A$. Lemma 4 is proved.

3 Main Theorem

Theorem 1. *There exists a continuous, density continuous and $\mathcal{I}$-density continuous function f which is nowhere approximately and $\mathcal{I}$-approximately differentiable.*

Proof. We begin by defining a version of the classical Peano area-filling curve $P : [0,1] \to [0,1] \times [0,1]$. To do this, a sequence of continuous paths P_n :

$[0,1] \to [0,1] \times [0,1]$ for $n = 0, 1, \ldots$, are defined which converge uniformly to P. This definition is facilitated by the following basic construction, which will be referred to as BCP.

Given a square $[a,b] \times [c,d]$ with one of its diagonals a parametrized constant speed path $\lambda : [\alpha, \beta] \to [a,b] \times [c,d]$, we construct a new path $\lambda_1 : [\alpha, \beta] \to [a,b] \times [c,d]$ as shown in Figure 1, where the speed of the new path is constant and three times the speed of λ.

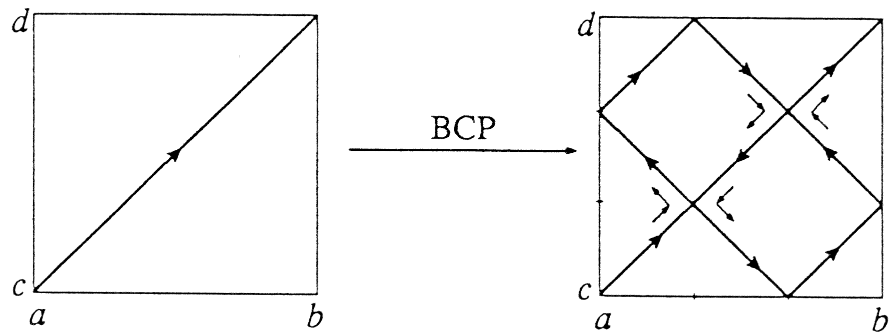

Figure 1: Basic construction BCP.

Using symmetries, BCP can be applied to either of the two diagonals of any square with either path orientation. Also, if $\|G\|_\infty = \sup_x |G(x)|$, then it is clear that

$$\|\lambda - \lambda'\|_\infty \leq \sqrt{(b-a)^2 + (d-c)^2} \tag{3}$$

for every $\lambda' : [\alpha, \beta] \to [a,b] \times [c,d]$.

To construct the Peano curve, let $P_0(t) = (t,t)$ and define P_1 by applying BCP to P_0. The image of P_1 consists of a diagonal from each of the nine squares

$$\left[\frac{i}{3}, \frac{i+1}{3}\right] \times \left[\frac{j}{3}, \frac{j+1}{3}\right], \quad i,j = 0,1,2.$$

(See Figure 1 with $a = c = 0$ and $b = d = 1$.) Construct P_2 by applying BCP to each of the diagonals of these squares as shown in Figure 2.

This process can be continued inductively in the obvious way to form the sequence P_n, $n \in \mathbb{N}$. From (3) it follows that

$$\|P_n - P_m\|_\infty \leq \sqrt{2}\, 3^{-\min(n,m)}.$$

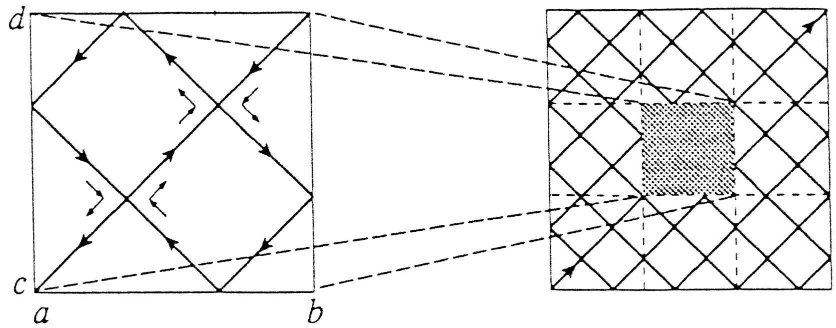

Figure 2: Construction of P_2.

This shows that P_n converges uniformly to P. It is also easy to see that the image of P is a dense, compact subset of $[0,1] \times [0,1]$, so P is an area-filling curve.

If $P = (p_1, p_2)$, where $p_i : [0,1] \to [0,1]$, $i = 1,2$ are the coordinate functions for P, then we claim $f = p_1$ is a function satisfying the conditions of Theorem 1.

To see this, it might be helpful to see how f can be defined directly as a uniformly convergent sequence of continuous functions $f_n : [0,1] \to [0,1]$, where each f_n is the first coordinate of P_n. The first coordinate of BCP can be represented by the construction shown in Figure 3. A similar construction can be done with either diagonal via an obvious reflection. This construction is denoted BCX.

Notice that Figure 3(B) also represents $f_1 : [0,1] \to [0,1]$, if we take $a = \alpha = 0$ and $b = \beta = 1$. To form f_2 it is enough to apply BCX to each linear segment of f_1. Then, apply BCX to each linear segment of f_2 to arrive at f_3, etc.

Evidently, f is continuous, as the first coordinate of the continuous function P. Also, as proved in [4], it is density continuous and nowhere approximately differentiable.

In the rest of the proof, we will need the following easy observations.

The function P is self-similar in the sense that for every $n \in \mathbb{N}$ and every $i = 0, 1, \ldots, 9^n - 1$, there exist $l(i), r(i) \in \{0, 1, \ldots, 3^n - 1\}$ such that the

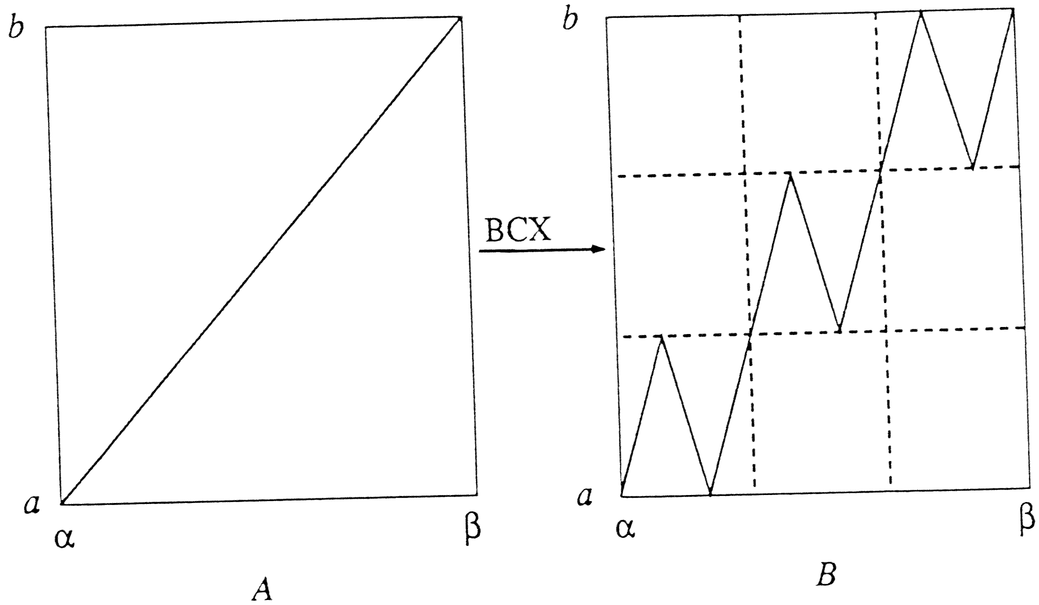

Figure 3: Basic construction BCX.

image

$$P\left(\left[\frac{i}{9^n}, \frac{i+1}{9^n}\right]\right) = \left[\frac{l(i)}{3^n}, \frac{l(i)+1}{3^n}\right] \times \left[\frac{r(i)}{3^n}, \frac{r(i)+1}{3^n}\right], \tag{4}$$

and the path followed is a scaled and reflected copy of the entire path of P in $[0,1] \times [0,1]$. Since f is the first coordinate of P, condition (4) implies also that for each integer $i \in \{0, 1, \ldots, 9^n - 1\}$, there is an integer $l(i) \in \{0, 1, \ldots, 3^n - 1\}$ such that

$$f\left(\left[\frac{i}{9^n}, \frac{i+1}{9^n}\right]\right) = \left[\frac{l(i)}{3^n}, \frac{l(i)+1}{3^n}\right]. \tag{5}$$

Also notice the following easy geometrical fact.

For every $t \in \mathbb{N}$, $t > 1$, and nonempty interval $(a, b) \subset [0, 1]$ there are $i, n \in \mathbb{N}$ such that

$$K = \left[\frac{i}{t^n}, \frac{i+1}{t^n}\right] \subset (a, b) \quad \text{and} \quad \frac{\mathrm{m}(K)}{b-a} \geq \frac{1}{2t} \tag{6}$$

To see this, let n be the smallest natural number such that

$$1/t^n < (b-a)/2.$$

Thus, $2/t^{n-1} \geq (b-a)$ and there exists i such that $i/t^n \in (a,(b+a)/2)$. Hence, $K = [i/t^n, (i+1)/t^n] \subset (a,b)$ and $\mathrm{m}(K)/(b-a) \geq (1/t^n)/(2/t^{n-1}) = 1/2t$. This finishes the proof of (6).

Notice that (5) implies $f^{-1}(E)$ is nowhere dense for every nowhere dense set E. So,

$$f^{-1}(E) \in \mathcal{I} \text{ for every } E \in \mathcal{I}.$$

Thus, by Lemma 3, to show that f is $\mathcal{I}$-density continuous it is enough to prove that f is deep-$\mathcal{I}$-density continuous.

Let $x \in [0,1]$ and let $A \subset \mathbb{R} \setminus \{f(x)\}$ be a set such that $f(x)$ is a deep-$\mathcal{I}$-density point of A. It must be shown that x is a deep-$\mathcal{I}$-density point of $f^{-1}(A)$. This will be done with the aid of Lemma 4.

Let $s = 1/9^k \in (0,1)$. We must find $D_s > 0$ and $R_s \in (0,1)$ such that whenever $0 < D < D_s$ and an interval $I \subset (x-D, x+D) \setminus \{x\}$ with $\mathrm{m}(I)/D > s$, then there is an interval $J \subset I \cap f^{-1}(A)$ with

$$\frac{\mathrm{m}(J)}{\mathrm{m}(I)} > R_s. \tag{7}$$

Let $s' = s/9^3$. Using Lemma 4 with A and $f(x)$, there exists $D_{s'} > 0$ and $R_{s'} = 1/3^l \in (0,1)$ such that

- whenever $0 < D' \leq D_{s'}$ and an interval

  $$I' \subset (f(x) - D', f(x) + D') \setminus \{f(x)\}$$

  with $\mathrm{m}(I')/D' \geq s'$, then there is an interval $J' \subset I' \cap A$ with

  $$\mathrm{m}(J')/\mathrm{m}(I') > R_{s'} \tag{8}$$

Let $D_s > 0$ be such that

$$|f(x) - f(y)| < D_{s'} \text{ for } |x-y| < D_s \tag{9}$$

and let $R_s = 1/9^{l+5}$. Let $0 < D < D_s$ and choose an interval $I \subset (x-D, x+D) \setminus \{x\}$ with $\mathrm{m}(I)/D > s$. We will find an interval $J \subset I \cap f^{-1}(A)$ with $\mathrm{m}(J)/\mathrm{m}(I) > R_s$.

Assume that $I \subset (x, x+D)$. The other case is similar.

Using (6), we can find $I_0 = [j/9^{n-1}, (j+1)/9^{n-1}] \subset I$ such that

$$\frac{\mathrm{m}(I_0)}{\mathrm{m}(I)} \geq \frac{1}{18}. \tag{10}$$

Moreover, using (4), it is easy to find $I_1 = [i/9^n, (i+1)/9^n] \subset I_0$ such that $f(x) \notin f(I_1)$. Thus,

$$\frac{\mathrm{m}(I_1)}{D} = \frac{1}{9}\frac{\mathrm{m}(I_0)}{D} \geq \frac{1}{9}\frac{1}{18}\frac{\mathrm{m}(I)}{D} > \frac{s}{9^3} = s'.$$

In particular, there exist $p = (s')^{-1}$ contiguous intervals $I^1, I^2, \ldots, I^p$ of length $1/9^n$, one of which is I_1 and such that $x \in I^1 \cup I^2 \cup \ldots \cup I^p$.

Define

$$D' = \max\{|f(x) - f(i/9^n)|, |f(x) - f((i+1)/9^n)|\} > 0$$

and $I' = f(I_1)$. By (9) we see that $D' < D_{s'}$ and, by (4), $f(i/9^n)$ and $f((i+1)/9^n)$ are the end points of I' so that $I' \subset [f(x)-D', f(x)+D'] \setminus \{f(x)\}$. Moreover, since $x, i/9^n, (i+1)/9^n \in I^1 \cup I^2 \cup \ldots \cup I^p$ then, by (5), we have

$$D' \leq \mathrm{m}\left(f\left(\bigcup_{j=1}^{p} I^j\right)\right) \leq \sum_{j=1}^{p} \mathrm{m}(f(I^j)) = p\mathrm{m}(I').$$

Hence,

$$\frac{\mathrm{m}(I')}{D'} \geq \frac{\mathrm{m}(I')}{p\mathrm{m}(I')} = p^{-1} = s'.$$

Thus, by (8), there is an interval $J' \subset I' \cap A$ such that $\mathrm{m}(J')/\mathrm{m}(I') > R_{s'}$.

Using (6), we can find an interval

$$J_1' = [j_0/3^m, (j_0+1)/3^m] \subset J'$$

such that $\mathrm{m}(J_1')/\mathrm{m}(J') \geq 1/6 > 1/9$. Hence,

$$\frac{\mathrm{m}(J_1')}{\mathrm{m}(f(I_1))} = \frac{\mathrm{m}(J_1')}{\mathrm{m}(I')} = \frac{\mathrm{m}(J_1')}{\mathrm{m}(J')}\frac{\mathrm{m}(J')}{\mathrm{m}(I')} > \frac{1}{9}R_{s'} = \frac{1}{3^{l+2}}$$

and $J_1' = [j_0/3^m, (j_0+1)/3^m] \subset f(I_1) = f([i/9^n, (i+1)/9^n])$. But now condition (4) implies easily that there exists an interval

$$J = [j/9^m, (j+1)/9^m] \subset I_1 = [i/9^n, (i+1)/9^n]$$

such that $f(J) = J_1'$ and

$$\frac{\mathrm{m}(J)}{\mathrm{m}(I_1)} > \left(\frac{1}{3^{l+2}}\right)^2 = \frac{1}{9^{l+2}}.$$

Hence, by (10),

$$\frac{\mathrm{m}(J)}{\mathrm{m}(I)} \geq \frac{\mathrm{m}(J)}{18\mathrm{m}(I_0)} = \frac{1}{9}\frac{\mathrm{m}(J)}{18\mathrm{m}(I_1)} > \frac{1}{9^3}\frac{1}{9^{l+2}} = R_s.$$

Condition (7) is proved. This finishes the proof that f is $\mathcal{I}$-density continuous.

To see that f is not $\mathcal{I}$-approximately differentiable at a point $x \in [0,1]$ let us do the following construction for each $n \in \mathbb{N}$. Choose $i \in \mathbb{N}$ such that $x \in [i/9^n, (i+1)/9^n]$. Then, by (5), $f([i/9^n, (i+1)/9^n]) = [j/3^n, (j+1)/3^n]$ for some $j \in \mathbb{N}$. It is also not difficult to see that condition (4) implies that

$$\left\{f\left(\left[\frac{9i}{9^{n+1}}, \frac{9i+1}{9^{n+1}}\right]\right), f\left(\left[\frac{9i+8}{9^{n+1}}, \frac{9i+9}{9^{n+1}}\right]\right)\right\}$$
$$= \left\{\left[\frac{3j}{3^{n+1}}, \frac{3j+1}{3^{n+1}}\right], \left[\frac{3j+2}{3^{n+1}}, \frac{3j+3}{3^{n+1}}\right]\right\}.$$

This implies, in particular, that for every $y \in [9i/9^{n+1}, (9i+1)/9^{n+1}]$ and $y' \in [(9i+8)/9^{n+1}, (9i+9)/9^{n+1}]$ we have

$$\frac{|f(y)-f(y')|}{|y-y'|} \geq \frac{1/3^{n+1}}{1/9^n} = 3^{n-1}.$$

Hence, an easy geometrical argument implies that for one of the intervals $[9i/9^{n+1}, (9i+1)/9^{n+1}]$ or $[(9i+8)/9^{n+1}, (9i+9)/9^{n+1}]$, which we denote by $[a_n, b_n]$, we have $x \notin [a_n, b_n]$ and

$$\frac{|f(y)-f(x)|}{|y-x|} \geq 3^{n-1} \text{ for every } y \in [a_n, b_n].$$

But, by Lemma 1, x is not an $\mathcal{I}$-dispersion point of $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Thus, for every $\mathcal{I}$-density open set U containing x, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there is an $y \in (x-\varepsilon, x+\varepsilon) \cap U \cap \bigcup_{m>n}[a_m, b_m]$ for which

$$\frac{|f(y)-f(x)|}{|y-x|} \geq 3^n.$$

This implies that f is not $\mathcal{I}$-approximately differentiable. Also notice that the construction of the intervals $[a_n, b_n]$ given above also implies that f is not approximately differentiable. This finishes the proof of Theorem 1.

4 Derivatives and $\mathcal{I}$-approximate continuity

In this section we show that the well-known fact that every bounded approximately countinuous function is a derivative is not true for the bounded $\mathcal{I}$-approximately continuous functions.

Example 1. *There exists a bounded $\mathcal{I}$-density continuous function which is not a derivative.*

Proof. Let $P \subset (0,1]$ be a nowhere dense closed set with positive measure. Choose a sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers satisfying $\lim_{k\to\infty} n_k/n_{k+1} = 0$ and define

$$A = \bigcup_{k\in\mathbb{N}} \frac{1}{n_k} P.$$

Then, by [3, Lemma 2.4], 0 is a deep-$\mathcal{I}$-dispersion point of A. Hence, there exists a closed set $B \bigcup \{0\} \subset A^c$ such that 0 is an $\mathcal{I}$-density point of B. Moreover, it can be assumed that

$$B = \bigcup_{k\in\mathbb{N}} [a_k, b_k] \bigcup [c_k, d_k],$$

where $a_k < b_k < a_{k+1} < 0 < d_{k+1} < c_k < d_k$ [5, 8].

On the other hand, for all $k \in \mathbb{N}$,

$$\frac{\mathrm{m}(B^c \cap (0, 1/n_k))}{1/n_k} \geq \frac{\mathrm{m}(A \cap (0, 1/n_k))}{1/n_k} > \mathrm{m}(P) > 0, \qquad (11)$$

so 0 is not a dispersion point of B^c.

Define the function f on $A \bigcup B$ by

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in B \end{cases}$$

and extend f on elsewhere in such a way that it is piecewise linear on $(0, \infty)$ and bounded by 1. Since 0 is an $\mathcal{I}$-dispersion point of B^c, it is apparent that f is $\mathcal{I}$-density continuous. On the other hand, f cannot be a derivative. To see this, suppose F is any primitive function for f and define

$$G(x) = \int_0^x f.$$

Since f is continuous on $\mathbb{R} \setminus \{0\}$, we see that $F - G$ must be constant on both $(-\infty, 0)$ and $(0, \infty)$. Since both F and G are continuous, this implies that $F - G$ is constant on $\mathbb{R}$ and therefore G is differentiable on $\mathbb{R}$. But, this is impossible since, by (11),

$$\begin{aligned} D^{-}G(0) &= 0 < \mathrm{m}(P) \\ &\leq \liminf_{k\to\infty} \frac{\mathrm{m}(E \cap (0, 1/n_k))}{1/n_k} \\ &\leq \limsup_{k\to\infty} \frac{G(1/n_k)}{1/n_k} \leq \overline{D}^{+}G(0). \end{aligned}$$

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Received July 15, 1991