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Density-to-deep-$\mathcal{I}$-density continuous functions

1. Preliminaries

The class of real functions continuous with respect to the deep-$\mathcal{I}$-density topology on the range and the density topology on the domain coincides with the class of all constant functions. This determines of the last class from the sixteen classes of continuous functions $C(T_1, T_2) = \{f: (\mathbb{R}, T_1) \to (\mathbb{R}, T_2)\}$, where T_i stands for ordinary, density, $\mathcal{I}$-density or deep-$\mathcal{I}$-density topology [2].

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $A, B \subset \mathbb{R}$ and $d \in \mathbb{R}$ the complement of A is denoted by A^c, the Euclidean distance between A and B by $\operatorname{dist}(A, B)$; i.e., $\operatorname{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ and we define $B - d = \{x - d \in \mathbb{R}: x \in B\}$ and $dB = \{dx \in \mathbb{R}: x \in B\}$. The families of Lebesgue measurable subsets of $\mathbb{R}$ and of subsets of $\mathbb{R}$ with Baire property are denoted by $\mathcal{L}$ and $\mathcal{B}$, while $\mathcal{N}$ and $\mathcal{I}$ stand for the ideals of Lebesgue measure zero and of first category subsets of $\mathbb{R}$. If $A \in \mathcal{L}$, we denote its Lebesgue measure by $\mathrm{m}(A)$.

To define the density topology $\mathcal{T}_{\mathcal{N}}$ and the deep-$\mathcal{I}$-density topology $\mathcal{T}_{\mathcal{D}}$ we need the following notions of density and deep-$\mathcal{I}$-density points [8, 10].

Let $A \in \mathcal{L}$. A number x, not necessarily in A, is a *density point* of A if

$$\lim_{h \to 0^+} \frac{\mathrm{m}(A \cap (x - h, x + h))}{2h} = 1. \tag{1}$$

The set of all density points of $A \in \mathcal{L}$ is denoted as $\Phi_{\mathcal{N}}(A)$. The family of sets

$$\mathcal{T}_{\mathcal{N}} = \{A \in \mathcal{L}: A \subset \Phi_{\mathcal{N}}(A)\}$$

forms a topology on $\mathbb{R}$ [8, 4] called the *density topology*.

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We say that 0 is a *deep-$\mathcal{I}$-density point* of a set $B \in \mathcal{B}$ [10] if there exists a closed set $A \subset B \cup \{0\}$ such that for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p}A\cap(-1,1)} = \chi_{(-1,1)} \ \mathcal{I}\text{-a.e.} \tag{2}$$

It is worth noticing that condition (2) is equivalent to the fact that the set

$$\liminf_{p\to\infty} n_{m_p}A = \bigcup_{q\in\mathbb{N}} \bigcap_{p\geq q} n_{m_p}A \tag{3}$$

is residual in $(-1,1)$. We say that a point b is a deep-$\mathcal{I}$-density point of $B \in \mathcal{B}$ if 0 is a deep-$\mathcal{I}$-density point of $B - b$. The set of all deep-$\mathcal{I}$-density points of $B \in \mathcal{B}$ is denoted as $\Phi_{\mathcal{D}}(B)$. The family of sets

$$\mathcal{T}_{\mathcal{D}} = \{B \in \mathcal{B} \colon B \subset \Phi_{\mathcal{D}}(B)\}$$

forms a topology on $\mathbb{R}$ called the *deep-$\mathcal{I}$-density topology* [6, 10].

We will use also the following dual versions of the density points. We say that x is a *dispersion* (*deep-$\mathcal{I}$-dispersion*) point of A if x is a density (deep-$\mathcal{I}$-density) point of A^c. In particular, 0 is a deep-$\mathcal{I}$-dispersion point of A if there exists an open $B \supset (A \setminus \{0\})$ such that for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$(-1,1) \cap \bigcap_{q\in\mathbb{N}} \bigcup_{p\geq q} n_{m_p}B = (-1,1) \cap \limsup_{p\to\infty}(n_{m_p}B) \in \mathcal{I}. \tag{4}$$

The symbols $Const$, $\mathcal{C}$ and $C_{\mathcal{N}\mathcal{D}}$ stand for the classes of real functions that are constant, ordinary continuous and continuous with the density topology on the domain and deep-$\mathcal{I}$-density topology on the range, respectively. Baire*1 denotes the class of all functions $f \colon \mathbb{R} \to \mathbb{R}$ with the property that for every perfect set P there is its nonempty portion $Q = P \cap (a,b)$ such that f restricted to Q is continuous [7].

We say that any of the sets $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$ or $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a *right interval set of a point* $a \in \mathbb{R}$ if $a_{n+1} < b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$. In the case when $a = 0$ we simply say that it is *a right interval set.*

We need also the following two propositions. The first one can be found in [1, Lemma 2.4]. (Compare also [9, Theorem 1] and [10, Theorem 2].) The second in [2, Lemma 2.4].

Proposition 1. *If* $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ *is a right interval set such that*

(i) $\lim_{n\to\infty} (b_n - a_n)/a_n = 0$; *and*

(ii) $\lim_{n\to\infty} b_{n+1}/a_n = 0$,

then 0 *is a deep-*$\mathcal{I}$*-dispersion point of* E. *In particular,* $E^c \in \mathcal{T}_{\mathcal{D}}$.

Proposition 2. *Let* $C \subset (0,1]$ *be a closed nowhere dense set and let* $\{b_n\}_{n\in\mathbb{N}}$ *be a decreasing sequence of positive numbers such that the limit* $\lim_{n\to\infty} b_{n+1}/b_n = 0$. *Then* 0 *is a deep-*$\mathcal{I}$*-dispersion point of the set*

$$E = \bigcup_{n\in\mathbb{N}} b_n C.$$

In particular, $E^c \in \mathcal{T}_{\mathcal{D}}$.

2. Continuous functions

We will start this section with the following lemma.

Lemma 3. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a measurable function such that* $f(0) = 0$ *and let* $c \in (0,1)$. *If* $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ *is a right interval set such that, for every* $n \in \mathbb{N}$, $a_n \geq c\, b_n$ *and* $\{d_n\}_{n\in\mathbb{N}}$ *is a sequence from* $(0,1)$ *such that*

$$m(f^{-1}([a_n, b_n]) \cap (0, d_n)) \geq d_n/2, \tag{5}$$

then $f \notin C_{\mathcal{N}\mathcal{D}}$.

Proof. The sets $f^{-1}([a_n, b_n]) \cap (0,1)$ are pairwise disjoint and bounded. Therefore $\lim_{n\to\infty} \mathrm{m}(\bigcup_{k\geq n} f^{-1}([a_k, b_k]) \cap (0,1)) = 0$. From this and (5) it follows that $\lim_{n\to\infty} d_n = 0$.

Taking a subsequence, if necessary, we can assume that

$$\lim_{n\to\infty} b_{n+1}/b_n = 0.$$

We may also assume that $a_n = c\, b_n$, because decreasing a_n does not change (5).

There are two cases to consider:

(1) there is a point $x \in (c,1)$ and an $\varepsilon > 0$ such that such that for every nontrivial interval $I \subset (c,1)$ containing x and for every $k \in \mathbb{N}$ there is $n_k \geq k$ with $\mathrm{m}(f^{-1}(b_{n_k}I) \cap (0, d_{n_k})) \geq \varepsilon d_{n_k}$; and

(2) for every $x \in (c,1)$ and every $\varepsilon > 0$ there exists an interval $I \subset (c,1)$ containing x and an $k \in \mathbb{N}$ such that $\mathrm{m}(f^{-1}(b_n I) \cap (0, d_n)) < \varepsilon d_n$ for every $n \geq k$.

Case (1). Let x and ε be as in the assumption. Put $n_0 = 0$ and, by induction on $k \in \mathbb{N}$, define a closed interval $I_k \subset (x - 1/k, x + 1/k) \cap (c,1)$ and an $n_k > n_{k-1}$ such that

$$\mathrm{m}(f^{-1}(b_{n_k}I_k) \cap (0, d_{n_k})) \geq \varepsilon d_{n_k}. \tag{6}$$

Then, by Proposition 1, 0 is a deep-$\mathcal{I}$-dispersion point of the interval set $D = \bigcup_{n \in \mathbb{N}} b_{n_k} I_k$, while $d(f^{-1}(D), 0) \neq 0$, because

$$\liminf_{k \to \infty} \frac{\mathrm{m}(f^{-1}(D) \cap (0, d_{n_k}))}{d_{n_k}} \geq \varepsilon > 0.$$

Hence, $D^c \in \mathcal{T}_{\mathcal{D}}$ and $f^{-1}(D^c) \notin \mathcal{T}_{\mathcal{N}}$; i.e., $f \notin C_{\mathcal{ND}}$.

Case (2). Let $\{q_k : k \in \mathbb{N}\}$ be an enumeration of the rational numbers in $(c,1)$ and let $\delta \in (0, 1/2)$. Put $n_0 = 0$ and, by induction on $k \in \mathbb{N}$, define an open interval I_k containing q_k and a number $n_k > n_{k-1}$ such that

$$\mathrm{m}(f^{-1}(b_{n_k}I_{n_k}) \cap (0, d_{n_k})) < \frac{\delta}{2^{n_k}} d_{n_k}.$$

Let $C = [c,1] \setminus \bigcup_{k \in \mathbb{N}} I_{n_k}$. Then, C is closed and nowhere dense. By Proposition 2, 0 is a deep-$\mathcal{I}$-dispersion point of $E = \bigcup_{k \in \mathbb{N}} b_{n_k} C$. So, $E^c \in \mathcal{T}_{\mathcal{D}}$. On the other hand, 0 is not a dispersion point of $f^{-1}(E)$, as

$$\begin{aligned} \liminf_{k \to \infty} \frac{\mathrm{m}(f^{-1}(E) \cap (0, d_{n_k}))}{d_{n_k}} &\geq \liminf_{k \to \infty} \frac{\mathrm{m}(f^{-1}(b_{n_k}C) \cap (0, d_{n_k}))}{d_{n_k}} \\ &\geq \frac{1}{2} - \sum_{k=1}^{\infty} \frac{\delta}{2^{n_k}} \geq \frac{1}{2} - \delta > 0. \end{aligned}$$

Thus, $f^{-1}(E^c) \notin \mathcal{T}_{\mathcal{N}}$ and $f \notin C_{\mathcal{ND}}$.

Now we are ready for the main result of this section.

Lemma 4. $\mathcal{C} \cap \mathcal{C}_{\mathcal{ND}} = Const$.

Proof. Evidently, $Const \subset \mathcal{C} \cap \mathcal{C}_{\mathcal{ND}}$. To prove the opposite inclusion, let $f \in \mathcal{C} \setminus Const$. We will show that $f \notin \mathcal{C}_{\mathcal{ND}}$ by using Lemma 3. Let $a < b$ be such that $f(a) \neq f(b)$. We may assume that $f(a) < f(b)$ and, by the continuity of f, that $f((a,b)) = (f(a), f(b))$. We may also assume, modifying of f in a linear way, if necessary, that $f(a) = a = -1$ and $f(b) = b = 1$. Then, we obtain $f(-1) = -1$, $f(1) = 1$ and $f((-1,1)) = (-1,1)$.

We construct, by induction on $n \in \mathbb{N}$, the sequences: $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ of real numbers and sequences $\{I_n\}$ and $\{J_n\}$ of intervals. We start by putting $a_0 = c_0 = -1$, $b_0 = d_0 = 1$ and $I_0 = [-1,1]$. Then we procede inductively to obtain the following conditions:

(a) $I_n = [a_n, b_n]$;

(b) $f(c_n) = a_n$ and $f(d_n) = b_n$;

(c) $f((c_n, d_n)) = (a_n, b_n)$;

(d) $I_n \in \{[a_{n-1}, (a_{n-1}+b_{n-1})/2], [(a_{n-1}+b_{n-1})/2, b_{n-1}]\}$;

(e) $J_n = \operatorname{cl}(I_{n-1} \setminus I_n)$;

(f) $\operatorname{m}(f^{-1}(J_n) \cap [c_{n-1}, d_{n-1}]) \geq (d_{n-1} - c_{n-1})/2$.

The inductive step is self-explanatory. First, select I_n as in (d) to satisfy (f). If $I_n = [a_{n-1}, (a_{n-1}+b_{n-1})/2]$ then we put $c_n = c_{n-1}$ and $d_n = \min\{x \in [c_{n-1}, d_{n-1}] \colon f(x) = (a_{n-1}+b_{n-1})/2\}$. In the other case, proceed similarly.

Let $x \in \bigcap_{n\in\mathbb{N}}[c_n, d_n]$. Then, $f(x) \in \bigcap_{n\in\mathbb{N}} I_n$. We may assume, translating f, if necessary, that $x = 0 = f(x)$.

Evidently, $\operatorname{m}(I_n) = 2\operatorname{m}(I_{n+1})$. A simple argument shows that for every $n \in \mathbb{N}$ either $\operatorname{dist}(J_n, I_j) \geq \operatorname{m}(J_n)/4$ or $\operatorname{dist}(J_{n+1}, I_j) \geq \operatorname{m}(J_{n+1})/4$ for all $j \geq n+2$. This allows us to choose a subsequence $\{n_k\}$ such that

$$\operatorname{dist}(J_{n_k}, 0) \geq \operatorname{m}(J_{n_k})/4 \tag{7}$$

for all $k \in \mathbb{N}$. It is easy to assume that a subsequence $\{n_k\}$ is choosen in such a way that the intervals $\{J_{n_k}\}$ are monotone and on one side of 0. For simplicity we assume that $E = \bigcup_{k\in\mathbb{N}} J_{n_k}$ is a right interval set. Then, condition (7) implies that $\min J_{n_k} \geq (1/5) \max J_{n_k}$. Thus, the first part of the assumptions from Lemma 3 is satisfied for the set E with $c = 1/5$. To finish the proof we will show that the second part is satisfied as well.

First notice that for infinitely many k we have either

$$\frac{\mathrm{m}(f^{-1}(J_{n_k}) \cap [0, d_{n_k-1}])}{d_{n_k-1}} \geq \frac{\mathrm{m}(f^{-1}(J_{n_k}) \cap [c_{n_k-1}, 0])}{-c_{n_k-1}} \tag{8}$$

or the converse inequality (where, $0/0$ is considered to be 0.) Without loss of the generality we may assume that (8) holds for every k. But this, together with (f), implies that

$$\mathrm{m}(f^{-1}(J_{n_k}) \cap [0, d_{n_k-1}]) \geq d_{n_k-1}/2.$$

Thus, the assumptions of Lemma 3 are satisfied and Lemma 4 is proved.

3. General case

For the next step, the following definition and lemma are needed [5, Lemma 29.1].

A *partition* of a set E is a pairwise disjoint family $\Pi = \{E_i : i \in \Lambda\}$ such that $\bigcup_{i \in \Lambda} E_i = E$. Note that any partition Π can be associated with a function $F: E \to \Lambda$ such that $F(x) = F(y)$ if, and only if, x and y belong to the same $E_i \in \Pi$. Conversely, any function $F: E \to \Lambda$ determines a partition of E.

For a set A and $n \in \mathbb{N}$ define

$$[A]^n = \{B \subset A \colon \operatorname{card}(B) = n\}.$$

If $\Pi = \{E_i : i \in \Lambda\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogeneous* for the partition Π if, for some $i \in \Lambda$, $[H]^n \subset E_i$. That is, all n-element subsets of H are in the same piece of the partition Π.

Lemma 5. (Ramsey's Theorem) *If $n, k \in \mathbb{N}$, then every finite partition $\Pi = \{E_1, E_2, \ldots, E_k\}$ of $[\mathbb{N}]^n$ has an infinite homogeneous set. In other words, for every $F: [\mathbb{N}]^n \to \{1, 2, \ldots, k\}$ there exists an infinite $H \subset \mathbb{N}$ such that F is constant on $[H]^n$.*

The next lemma combines the proofs of the theorems that density continuous functions and deep-$\mathcal{I}$-density continuous functions are in Baire*1 class [3, Theorem 3], [1, Theorem 4.2].

Lemma 6. *$C_{\mathcal{ND}} \subset$ Baire*1.*

Proof. Assume to the contrary that for some perfect set P the set

$$Z = \{x \in P: f|_P \text{ is not continuous at } x\}$$

is dense in P.

We will construct sequences: $\{x_n\}_{n\in\mathbb{N}}$ of points of P, $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of open intervals, $\{J_n\}_{n\in\mathbb{N}}$ of compact intervals, and $\{I_n\}_{n\in\mathbb{N}}$ of open intervals having the same midpoint as the corresponding J_n, and contained in that corresponding J_n. The construction is inductive, and aimed at having all the objects obtained satisfy the conditions (a) through (f) listed below.

Start by choosing $x_0 \in Z$, $(a_0, b_0) = (x_0 - 1, x_0 + 1)$ and $I_0 = J_0 = \emptyset$. Assume that for all $n \in \mathbb{N}$ and all $i \in \mathbb{N}$, $1 \le i \le n$, it holds that:

(a) $f(x_i) \in I_i \subset J_i$;

(b) $J_{i-1} \cap J_i = \emptyset$ and, for $i > 2$,

$$\mathrm{m}(J_i) \le \frac{1}{3}\min\{\mathrm{dist}(J_k, J_{k+1}): k \in \mathbb{N}, k < i-1\};$$

(c) $\mathrm{m}(J_i) < \omega(f|_P, x_i)$ and $0 < \mathrm{m}(I_i) < 2^{-i}\mathrm{m}(J_i)$;

(d) $x_i \in (a_i, b_i) \cap Z \subset [a_i, b_i] \subset (a_{i-1}, b_{i-1})$;

(e) $(b_i - a_i) < 2^{-i}$; and,

(f) $\mathrm{m}(f^{-1}(I_i) \cap (a_i, b_i)) > (1 - 2^{-i})(b_i - a_i)$.

To continue with the inductive step, note that by (c) and (d), we are able to choose

$$y \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n).$$

If $y \in Z$, then let $x_{n+1} = y$. Otherwise, $f|_P$ is continuous at y. In this case, the fact that Z is dense in P guarantees the existence of

$$x_{n+1} \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n) \cap Z.$$

Because J_n is closed and $x_{n+1} \in Z$, there is a closed interval J_{n+1} centered at $f(x_{n+1})$ such that $J_{n+1} \cap J_n = \emptyset$, $0 < \mathrm{m}(J_{n+1}) < \omega(f|_P, x_{n+1})$ and, for $i > 2$,

$$\mathrm{m}(J_i) \leq \frac{1}{3} \min\{\mathrm{dist}(J_k, J_{k+1}) \colon k \in \mathbb{N}, k < i-1\}.$$

Setting I_{n+1} to be the closed interval centered at $f(x_{n+1})$ with length equal $\mathrm{m}(J_{n+1})/2^{n+1}$, it follows that (a), (b) and (c) are true with $i = n+1$. Next, use the approximate continuity of f at x_{n+1} to find an interval $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$ containing x_{n+1} such that (d), (e) and (f) are satisfied. The induction is complete.

Let

$$\{x\} = \bigcap_{n \in \mathbb{N}} [a_n, b_n].$$

We show that there is an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers such that

(1) $f(x)$ is a deep-$\mathcal{I}$-dispersion point of $\bigcup_{i \in \mathbb{N}} I_{n_i}$, and

(2) x is not a dispersion point of $f^{-1}\left(\bigcup_{i \in \mathbb{N}} I_{n_i}\right)$.

This implies that $f \notin C_{\mathcal{ND}}$.

First notice that x is not a dispersion point of $f^{-1}\left(\bigcup_{i \in \mathbb{N}} I_{n_i}\right)$ for every sequence $\{n_i\}_{i \in \mathbb{N}}$ as, by condition (f),

$$\lim_{i \to \infty} \frac{\mathrm{m}(f^{-1}(I_{n_i}) \cap (a_{n_i}, b_{n_i}))}{\mathrm{m}((a_{n_i}, b_{n_i}))} = 1.$$

To find an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers such that condition (2) is satisfied we will consider two cases.

Case 1°. There exists an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers such that the J_{n_i} are pairwise disjoint.

By taking a subsequence of $\{n_i\}_{i \in \mathbb{N}}$, if necessary, it may be assumed that

$$\bigcup_{i \in \mathbb{N}} J_{n_i}$$

is either a right or left interval set. For simplicity, assume it is a right interval set.

Let $J_{n_i} = [c_i, d_i]$ and $I_{n_i} = (\alpha_i, \beta_i)$. Then

$$f(x) = 0 < d_{i+1} < c_i < \alpha_i < \beta_i < d_i$$

for all i. Condition (c) states that

$$\frac{\beta_i - \alpha_i}{d_i - c_i} = \frac{\mathrm{m}(I_{n_i})}{\mathrm{m}(J_{n_i})} < \frac{1}{2^{n_i}}.$$

Let z_n be the common center of I_n and J_n, for $n \geq 0$. Then

$$\lim_{i\to\infty} \frac{\beta_i - \alpha_i}{\beta_i} \leq \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{z_{n_i}} \leq \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{z_{n_i} - c_i} = 2 \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{d_i - c_i} = 0.$$

The above allows us to choose a subsequence of $\{n_i\}_{i\in\mathbb{N}}$ satisfying the assumptions of Proposition 1.

Case 2^o. There is no pairwise disjoint subsequence $\{J_{n_i}\}_{i\in\mathbb{N}}$ of the sequence $\{J_n\}_{n\in\mathbb{N}}$.

Let us first consider the subsequence $\{J_{2n+1}\}_{n\in\mathbb{N}}$, indexed by the odd numbers, of the sequence $\{J_n\}_{n\in\mathbb{N}}$. Define a partition function $F\colon [\mathbb{N}]^2 \to \{0,1\}$ by

$$F(\{n,m\}) = 1 \quad \text{if, and only if,} \quad J_{2n+1} \cap J_{2m+1} \neq \emptyset.$$

By Lemma 5 (Ramsey's Theorem) there exists an infinite homogeneous subset $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$; i.e., a sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that for some $k \in \{0,1\}$, $F(\{n_i, n_j\}) = k$ for all positive integers $i \neq j$. But $k = 0$ would contradict the definition of the case 2^o, which is currently considered. Thus $k = 1$; i.e.,

$$J_{2n_i+1} \cap J_{2n_j+1} \neq \emptyset \tag{9}$$

for all nonnegative integers $i \neq j$.

Now let us repeat the Ramsey-type argument, which was used above, for the even-numbered counterparts of $\{J_{2n_i+1}\}_{i\in\mathbb{N}}$. Define $G\colon [\mathbb{N}]^2 \to \{0,1\}$ by

$$G(\{i,j\}) = 1 \quad \text{if, and only if,} \quad J_{2n_i} \cap J_{2n_j} \neq \emptyset.$$

By Lemma 5 (Ramsey's Theorem) there exists a subsequence $\{n_{i_s}\}_{s\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that

$$J_{2n_{i_s}} \cap J_{2n_{i_t}} \neq \emptyset \tag{10}$$

for all nonnegative integers $s \neq t$, while condition (9) is still preserved, or more precisely

$$J_{2n_{i_s}+1} \cap J_{2n_{i_t}+1} \neq \emptyset \tag{11}$$

for $s \neq t$. Define $\varepsilon = \operatorname{dist}\left(J_{2n_{i_0}}, J_{2n_{i_0}+1}\right)$. By (b), $\varepsilon > 0$. Moreover, by (b), (10) and (11)

$$B_0 = \bigcup_{s \in \mathbb{N}} J_{2n_{i_s}} \subset \left\{x \colon \operatorname{dist}\left(x, J_{2n_{i_0}}\right) < \frac{\varepsilon}{3}\right\}$$

and

$$B_1 = \bigcup_{s \in \mathbb{N}} J_{2n_{i_s}+1} \subset \left\{x \colon \operatorname{dist}\left(x, J_{2n_{i_0}+1}\right) < \frac{\varepsilon}{3}\right\}.$$

Hence

$$\operatorname{dist}(B_0, B_1) \geq \frac{\varepsilon}{3} > 0.$$

Note that

$$S_0 = \bigcup_{s \geq 0} I_{2n_{i_s}} \subset B_0$$

and

$$S_1 = \bigcup_{s \geq 0} I_{2n_{i_s}+1} \subset B_1.$$

Thus $\operatorname{dist}(S_0, S_1) > 0$, which implies that either

$$\operatorname{dist}(f(x), S_0) > 0$$

or

$$\operatorname{dist}(f(x), S_1) > 0.$$

This clearly means that $f(x)$ is an $\mathcal{I}$-dispersion point of either S_0 or S_1.

This finishes the proof of Lemma 6.

Now, we are ready to prove our main theorem.

Theorem 7. $C_{\mathcal{ND}} = Const.$

Proof. Evidently, $Const \subset C_{\mathcal{ND}}$. To prove the other inclusion, let $f \in C_{\mathcal{ND}}$. By Lemma 6 the set

$$U = \operatorname{int}(\{x \in \mathbb{R} \colon f \text{ is continuous at } x\})$$

is dense. Notice that U^c does not have any isolated points, because the approximately continuous function f has the Darboux property. Thus, the set $P = U^c$ is perfect. We prove that $P = \emptyset$. By way of contradictions let us assume that $P \neq \emptyset$ and let $\{(a_n, b_n): n \in \mathbb{N}\}$ be an enumeration of all components of U. Notice that, by Lemma 4, f is constant on any interval (a_n, b_n) and, by the Darboux property, also on $[a_n, b_n]$.

Now, let us use Lemma 6 for f and P. Then, there is a nonempty portion $Q = P \cap (c, d)$ on which f is continuous. The set P is nowhere dense, so there exists an n such that $(c, d) \cap (a_n, b_n) \neq \emptyset$. Then, $(c, d) \cup (a_n, b_n)$ is an interval properly containing (a_n, b_n). We will obtain a contradiction with the assumption that (a_n, b_n) is a component of U by showing that f is continuous on $J = (c, d) \cup (a_n, b_n)$. So, let $x \in J$. If $x \in U$, then evidently f is continuous at x. If $x \in P$, then choose a sequence $\{x_i\}_{i \in \mathbb{N}}$ converging to x and define

$$y_i = \begin{cases} x_i & \text{if } x_i \in P \\ a_n & \text{for } x_i \in (a_n, b_n). \end{cases}$$

Then, $y_i \in P$, $f(x_i) = f(y_i)$ for $i \in \mathbb{N}$ and $\lim_{i \to \infty} y_i = x$. Moreover,

$$\lim_{i \to \infty} f(x_i) = \lim_{i \to \infty} f(y_i) = f(x)$$

as $f|_P$ is continuous at x. Hence, f is continuous at x.

This finishes the proof of Theorem 7.

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