Analytic functions are $\mathcal{I}$-density continuous

Krzysztof Ciesielski, Lee Larson

Abstract. A real function is $\mathcal{I}$-density continuous if it is continuous with the $\mathcal{I}$-density topology on both the domain and the range. If f is analytic, then f is $\mathcal{I}$-density continuous. There exists a function which is both C^∞ and convex which is not $\mathcal{I}$-density continuous.

Keywords: analytic function, $\mathcal{I}$-density continuous, $\mathcal{I}$-density topology

Classification: 26A21

Let $\mathcal{T}_\mathcal{N}$ stand for the density topology on the real line, $\mathbb{R}$. A function $f:\mathbb{R}\to\mathbb{R}$ is *density continuous* at the point x if it is continuous at x when $\mathcal{T}_\mathcal{N}$ is used on both the domain and the range. The class of all everywhere density continuous functions is written as $\mathcal{C}_{\mathcal{N}\mathcal{N}}$. It is known that all locally convex functions are density continuous, and it follows quite easily from this that all analytic functions are in $\mathcal{C}_{\mathcal{N}\mathcal{N}}$. But, there are C^∞ functions which are not in $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ [2].

W. Wilczyński [4] introduced the $\mathcal{I}$-density topology on $\mathbb{R}$, which has many properties in common with the density topology, except that it is based upon category instead of measure. (For its definition see [4] or [3].) The $\mathcal{I}$-density topology is denoted here by $\mathcal{T}_\mathcal{I}$. The $\mathcal{I}$-density continuous functions, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, are those functions $f:\mathbb{R}\to\mathbb{R}$ which are continuous when the domain and range are both given the topology $\mathcal{T}_\mathcal{I}$.

It is natural to ask if the known properties of the density continuous functions can be proved in the case of the $\mathcal{I}$-density continuous functions. It turns out that some properties can and some cannot be proved. Theorem 7, given below, establishes that analytic functions are $\mathcal{I}$-density continuous, but the proof is necessarily different from the case of the density continuous functions because we also exhibit in Example 10, a convex and C^∞ function which is not $\mathcal{I}$-density continuous.

The notation used here is fairly standard. The set of subsets of $\mathbb{R}$ with the Baire property is written as $\mathcal{B}$. $\mathcal{I}$ stands for the ideal of first category subsets of $\mathbb{R}$. C^∞ is the set of all functions $f:\mathbb{R}\to\mathbb{R}$ which are infinitely differentiable at every point and $\mathcal{A}$ stands for the collection of all real analytic functions. A set E is a *right interval set* at a point $a\in\mathbb{R}$, if $E=\bigcup_{n\in\mathbb{N}}[a_n,b_n]$ or $E=\bigcup_{n\in\mathbb{N}}(a_n,b_n)$ where $a_n\to a$ and $a_n>b_{n+1}>a_{n+1}$ for all $n\in\mathbb{N}$. The definition of a left interval set at a is similar. The set E is an interval set at a, if it is the union of a right and left interval set at a. Any interval set at 0 is just called an interval set.

An open set S is said to be *regular*, if $S = \operatorname{int}(\operatorname{cl}(S))$. In particular, it can be shown that for any $B \in \mathcal{B}$, there is a unique regular open set, $\tilde{B}$ such that $B \triangle \tilde{B} \in \mathcal{I}$. This observation is important below because it often enables us to replace an arbitrary $B \in \mathcal{T}_{\mathcal{I}}$ by $\tilde{B}$ without losing any generality in a proof.

We begin by stating several known results which are needed below. The first is essentially the same as [5, Theorem 2].

Lemma 1. *Let $\{c_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero and, for each $n \in \mathbb{N}$, let (a_n, b_n) be an open interval centered at c_n. If*

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n - a_n}{c_n} = 0,$$

then 0 is an $\mathcal{I}$-dispersion point of

$$\bigcup_{n\in\mathbb{N}} [a_n, b_n].$$

Theorem 2. *Let B be a regular open set. The following statements are equivalent:*

(i) *0 is an $\mathcal{I}$-dispersion point of B.*

(ii) *For every increasing sequence $\{t_k\}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{k_i}\}$ such that*

$$\limsup_{i\to\infty} t_{k_i} B \cap (-1,1) \in \mathcal{I}. \tag{1}$$

(iii) *For every increasing sequence $\{t_k\}$ of positive numbers diverging to infinity and every nonempty interval $(a,b) \subset (-1,1)$ there exists a nonempty subinterval $(c,d) \subset (a,b)$ and a subsequence $\{t_{k_i}\}$ such that for every $i \in \mathbb{N}$*

$$(c,d) \cap t_{k_i} B = \emptyset.$$

Proof: The fact that (i) and (ii) are equivalent is known [3, Theorem 1].

Assume that (ii) is true, but that there exists an interval $(a,b) \subset (-1,1)$ for which (iii) fails. Then every subinterval $(c,d) \subset (a,b)$ has the property that $\{k : (c,d) \cap t_k B = \emptyset\}$ is finite. From this it is apparent that $\limsup_i t_{k_i} B$ is a dense $\mathbf{G}_\delta$ subset of $(a,b) \subset (-1,1)$ for every subsequence $\{t_{k_i}\}$ of $\{t_k\}$. This contradicts (1), so (iii) must be true.

Finally, suppose that (iii) is true. Let d_n be a countable dense subset of $(-1,1)$ and suppose I_n is a sequential representation of the set $\{(d_n, d_m) : n, m \in \mathbb{N},\ d_n < d_m\}$. Applying (iii), there must exist an interval $J_1 \subset I_1$ and a subsequence $\{t_{k^1_m}\}$ of $\{t_k\}$ so that $t_{k^1_m} B \cap J_1 = \emptyset$ for all m. Proceeding inductively, for each $i \in \mathbb{N}$ there must exist an interval $J_{i+1} \subset I_{i+1}$ and a subsequence $t_{k^{i+1}_m}$ of $t_{k^i_m}$ such that $t_{k^{i+1}_m} B \cap J_{i+1} = \emptyset$ for each m. Since $\{d_n : n \in \mathbb{N}\}$ is dense in $(-1,1)$ it is clear that $\limsup_i t_{k^i_i} B \cap (-1,1) \in \mathcal{I}$, and (ii) follows. □

The following theorem is a consequence of [1, Corollary 1].

Theorem 3. *If $f: \mathbb{R} \to \mathbb{R}$ is monotone and satisfies the Lipschitz condition*

$$0 < \alpha|b-a| < |f(b) - f(a)| < \beta|b-a| < \infty$$

for all distinct a and b in some interval I, then f is $\mathcal{I}$-density continuous on I.

The first order of business is to prove that $\mathcal{A} \subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$. The following two technical lemmas are needed for the proof.

Lemma 4. *Let $f, h: [0, +\infty) \to [0, +\infty)$ be homeomorphisms such that*

$$\lim_{x \to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1.$$

Then for every $0 < c < c' < d' < d$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$,

$$f\left((\varepsilon c', \varepsilon d')\right) \subset h\left((\varepsilon c, \varepsilon d)\right).$$

Proof: Since $c/c' < 1$ and $d/d' > 1$ we can find $\delta_0 > 0$ such that for every $x \in (0, \delta_0)$

$$\frac{c}{c'} < \frac{h^{-1}(x)}{f^{-1}(x)} < \frac{d}{d'}. \tag{2}$$

Using the continuity of f^{-1} at 0 we can find $\varepsilon_0 > 0$ such that $f((0, \varepsilon_0 d)) \subset (0, \delta_0)$.

Now let $\varepsilon \in (0, \varepsilon_0)$ and $x \in f((\varepsilon c', \varepsilon d')) \subset f((0, \varepsilon_0 d)) \subset (0, \delta_0)$. So, (2) holds and $f^{-1}(x) \in (\varepsilon c', \varepsilon d')$; i.e.,

$$\varepsilon c' < f^{-1}(x) < \varepsilon d'.$$

Multiplying the above inequality by (2), we obtain

$$\varepsilon c < h^{-1}(x) < \varepsilon d,$$

which implies $x \in h\left((\varepsilon c, \varepsilon d)\right)$.

Lemma 5. *If $f, h : [0, \infty) \to [0, \infty)$ are homeomorphisms satisfying*

$$\lim_{x \to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1, \tag{3}$$

then h is right $\mathcal{I}$-density continuous at 0 iff f is right $\mathcal{I}$-density continuous at 0.

Proof: Without loss of generality we may assume that both functions are increasing, as the decreasing case is essentially the same.

So assume that h is right $\mathcal{I}$-density continuous at 0. It will be shown that f is right $\mathcal{I}$-density continuous at 0. This will finish the proof, as the converse implication follows by exchanging f with h.

Let us choose $B \in \mathcal{B}$, $0 \notin B$, which has 0 as an $\mathcal{I}$-dispersion point. We will use Theorem 2 to prove that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$.

First, notice that since f and h are both homeomorphisms, we may assume that B is a regular open set. Choose a divergent increasing sequence of positive real numbers $\{t_k\}_{k\in\mathbb{N}}$ and a nonempty interval $(a,b) \subset (0,1)$. Since 0 is a right $\mathcal{I}$-dispersion point of $h^{-1}(B)$, there exists a nonempty interval $(c,d) \subset (a,b)$ and a subsequence $\{t_{k_p}\}_{p\in\mathbb{N}}$ of $\{t_k\}_{k\in\mathbb{N}}$ such that for every $p \in \mathbb{N}$

$$(c,d) \cap t_{k_p} h^{-1}(B) = \emptyset.$$

But this last condition is equivalent to

$$h\left(\left(\frac{1}{t_{k_p}}c, \frac{1}{t_{k_p}}d\right)\right) \cap B = \emptyset.$$

Now let $0 < c < c' < d' < d$. Then, by Lemma 4,

$$f\left(\frac{1}{t_{k_p}}c', \frac{1}{t_{k_p}}d'\right) \subset h\left(\frac{1}{t_{k_p}}c, \frac{1}{t_{k_p}}d\right)$$

for almost all $p \in \mathbb{N}$. This implies that for almost all $p \in \mathbb{N}$

$$f\left(\left(\frac{1}{t_{k_p}}c', \frac{1}{t_{k_p}}d'\right)\right) \cap B = \emptyset,$$

or

$$(c', d') \cap t_{k_p} f^{-1}(B) = \emptyset.$$

This finishes the proof of Lemma 5. □

The following theorem, which is interesting in its own right, is also needed in what follows. Its analogue for ordinary density continuity is also known to be true [2].

Theorem 6. *For any $\alpha \in \mathbb{R}$, the function $f(x) = x^\alpha$ is $\mathcal{I}$-density continuous on its domain.*

PROOF: If $x \neq 0$ and $f(x)$ exists, then it is clear that on a neighborhood of x, f satisfies the conditions of Theorem 3, so f is $\mathcal{I}$-density continuous at x.

Suppose $x = 0$ and $\alpha > 0$. It suffices to show f is right $\mathcal{I}$-density continuous at 0. Let $B \in \mathcal{B}$ such that 0 is an $\mathcal{I}$-dispersion point of B. It must be shown that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$.

To do this, first note that f is a homeomorphism on $(0,\infty)$, so $f^{-1}(S) \in \mathcal{I}$ whenever $S \in \mathcal{I}$ and there is no generality lost with the assumption that B is a regular open set. Choose any nonempty interval $(a,b) \subset (0,1)$ and an increasing sequence $\{s_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity. Let $(a',b') = f((a,b))$ and define the increasing sequence

$$t_k = \frac{1}{f(1/s_k)} \to \infty.$$

Using Theorem 2, there exists an interval $(c',d') \subset (a',b')$ and a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that

$$(c',d') \cap t_{k_i} B = \emptyset \quad \text{for all} \quad i \in \mathbb{N}.$$

Suppose that $(c,d) = f^{-1}((c',d'))$. Then a straightforward calculation shows

$$\begin{aligned}
\emptyset &= f^{-1}\left((c',d') \cap t_{k_i} B\right) \\
&= (c,d) \cap f^{-1}\left(\frac{1}{f(1/s_{k_i})} B\right) \\
&= (c,d) \cap \left(s_{k_i}^{-\alpha} B\right)^{-1/\alpha} \\
&= (c,d) \cap s_{k_i}(B)^{-1/\alpha} \\
&= (c,d) \cap s_{k_i} f^{-1}(B).
\end{aligned}$$

From Theorem 2, we see that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$, and the theorem follows. □

Theorem 7. $\mathcal{A} \subset \mathcal{C}_{\mathcal{II}}$.

Proof: Let $h \in \mathcal{A}$. It is enough to prove that h is $\mathcal{I}$-density continuous at 0. We prove that h is right $\mathcal{I}$-density continuous at 0. The left-hand argument is similar.

Let $h(x) = \sum_{n=0}^{\infty} a_n x^n$. We can assume that $a_0 = 0$. Since the $\mathcal{I}$-density topology is closed under homothetic transformations of its open sets, we can also assume that for $i = \min\{n : a_n \neq 0\}$ we have $a_i = 1$. Now let $f(x) = x^i$. Because h is analytic, h^{-1} exists on some right neighborhood of 0. Let us assume that h^{-1} is positive on this neighborhood, the other case being similar. Then

$$\begin{aligned}
1 = \lim_{x\to 0^+} \frac{h(x)}{x^i} &= \lim_{x\to 0^+} \frac{h(h^{-1}(x))}{(h^{-1}(x))^i} \\
&= \lim_{x\to 0^+} \left(\frac{x^{\frac{1}{i}}}{h^{-1}(x)}\right)^i \\
&= \left(\lim_{x\to 0^+} \frac{f^{-1}(x)}{h^{-1}(x)}\right)^i.
\end{aligned}$$

Hence,

$$\lim_{x\to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1$$

and, by Lemma 5 and Theorem 6, h is $\mathcal{I}$-density continuous at 0. □

After seeing that $\mathcal{A} \subset \mathcal{C}_{\mathcal{I}\mathcal{I}}$, it is natural to ask whether the same can be claimed for C^∞. This turns out not to be true. The lemma and theorem given below are used to establish this fact.

Lemma 8. *Let $f \in \mathcal{C}^\infty$ be such that for every $n \geq 0$*

$$f^{(n)}(0) = 0 \quad \textit{and} \quad f^{(n)}((0,\varepsilon_n)) \subset (0,\infty), \quad \textit{for some} \;\; \varepsilon_n > 0.$$

Then

$$\lim_{x\to 0^+} \frac{f(ax)}{f(x)} = 0,$$

for every $a \in (0,1)$.

PROOF: Let $a \in (0,1)$ and $n \in \mathbb{N}$. Moreover, let us choose $\varepsilon > 0$ such that $0 < \varepsilon < \varepsilon_k$ for every $k \leq n+1$. In particular, $f^{(n)}$ is increasing on $(0,\varepsilon)$, and so

$$\left| \frac{f^{(n)}(a\xi)}{f^{(n)}(\xi)} \right| < 1 \quad \text{for every} \;\; \xi \in (0,\varepsilon).$$

Now let $x \in (0,\varepsilon)$ and let $g(x) = f(ax)$. Using Cauchy's Theorem n-times we can find $\xi \in (0,x)$ such that

$$\left| \frac{f(ax)}{f(x)} \right| = \left| \frac{g(x)}{f(x)} \right| = \left| \frac{g^{(n)}(\xi)}{f^{(n)}(\xi)} \right| = |a^n| \left| \frac{f^{(n)}(a\xi)}{f^{(n)}(\xi)} \right| < a^n.$$

Thus,

$$\lim_{x\to 0^+} \frac{f(ax)}{f(x)} = 0.$$

Theorem 9. *Let $f \in \mathcal{C}^\infty$ be such that for every $n \geq 0$*

$$f^{(n)}(0) = 0 \quad \textit{and} \quad f^{(n)}((0,\varepsilon_n)) \subset (0,\infty) \quad \textit{for some} \;\; \varepsilon_n > 0.$$

Then f is not $\mathcal{I}$-density continuous.

PROOF: We start with a proof that f is not right $\mathcal{I}$-density continuous at 0. Let $D_n = \{\frac{i}{2^n} : i = 1, 2, \dots, 2^n\}$ for $n \in \mathbb{N}$. First notice that if a sequence $\{n_k\}_{k\in\mathbb{N}}$ is such that

$$n_{k+1} > 2^k n_k \quad \text{for every} \;\; k \in \mathbb{N}, \tag{4}$$

then

$$\min \frac{1}{n_k} D_k = \frac{1}{n_k}\frac{1}{2^k} > \frac{1}{n_{k+1}} = \max \frac{1}{n_{k+1}} D_{k+1}.$$

This means that if $\{s_i\}_{i>1}$ is a decreasing ordering of $D = \bigcup_{k\in\mathbb{N}} \frac{1}{n_k} D_k$, then

$$\frac{1}{n_k} D_k = \{s_i \colon 2^k \le i < 2^{k+1}\}.$$

We also define a sequence $\{n_k\}_{k\in\mathbb{N}}$ by induction on k such that it will satisfy condition (4) and for every $k > 0$

$$\frac{f(s_i)}{f(s_{i-1})} \le \frac{1}{k} \quad \text{for} \quad 2^k \le i < 2^{k+1}. \tag{5}$$

Put $n_1 = 1$ and assume that n_{k-1} has already been chosen for some $k > 1$. Choose $n_k > 2^{k-1} n_{k-1}$ such that

$$\frac{f(\frac{2^k-1}{2^k}x)}{f(x)} < \frac{1}{k}, \quad \text{for all} \quad x \in (0, \frac{1}{n_k}).$$

Such a choice is possible by Lemma 8. Then, the above condition obviously implies condition (5) for $2^k < i < 2^{k+1}$. Increasing n_k, if necessary, we can also obtain condition (5) for $i = 2^k$. This finishes the construction of D.

Now let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint intervals such that every interval (a_n, b_n) is centered at $c_n = f(s_n)$ and that

$$\lim_{n\to\infty} \frac{b_n - a_n}{c_n} = 0.$$

By (5),

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$$

so, by Lemma 1, 0 is an $\mathcal{I}$-dispersion point of the interval set

$$E = \bigcup_{n\in\mathbb{N}} (a_n, b_n).$$

On the other hand, we notice that for every subsequence $\{n_{k_i}\}_{i\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$, the set

$$\bigcup_{i\in\mathbb{N}} n_{k_i} f^{-1}(E) \supset \bigcup_{i\in\mathbb{N}} D_{k_i}$$

is dense and open in $[0,1]$. So, 0 is not a right $\mathcal{I}$-dispersion point of $f^{-1}(E)$ and f is not $\mathcal{I}$-density continuous at 0. $\square$

Example 10. *There exists a convex C^∞ function that is not $\mathcal{I}$-density continuous.*

PROOF: Define $g\colon (-\infty, 0.5) \to \mathbb{R}$ by

$$g(x) = \begin{cases} e^{-x^{-2}} & x \in (0, 1/2) \\ 0 & x \in (-\infty, 0] \end{cases}$$

Examining the second derivative of g it is easy to see that g is convex on $(-\infty, 1/2)$. It is well-known that $f \in C^\infty$ and that $f^{(n)}(0) = 0$ for all n. Repeated differentiation of f makes it apparent that for each n there is an $\varepsilon_n > 0$ such that $f^{(n)}(x) > 0$ whenever $0 < x < \varepsilon_n$. Now an application of Theorem 9 finishes the argument. □

It is also not difficult to see that the function described in Theorem 9 does not preserve $\mathcal{I}$-density points. In particular, the function g from Example 10 does not preserve $\mathcal{I}$-density points.

References

[1] Aversa V., Wilczyński W., *Homeomorphisms preserving $\mathcal{I}$-density points*, Boll. Un. Mat. Ital. **B(7)1** (1987), 275–285.

[2] Ciesielski K., Larson L., *The space of density continuous functions*, Acta Math. Hung. **58** (1991), 289–296.

[3] Poreda W., Wagner-Bojakowska E., Wilczyński W., *A category analogue of the density topology*, Fund. Math. **75** (1985), 167–173.

[4] Wilczyński W., *A generalization of the density topology*, Real Anal. Exchange **8(1)** (1982–83), 16–20.

[5] ______, *A category analogue of the density topology, approximate continuity, and the approximate derivative*, Real Anal. Exchange **10** (1984–85), 241–265.

Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA

Department of Mathematics, University of Louisville, Louisville, KY 40292, USA

(*Received* July 8, 1991, *revised* April 15, 1993)