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RESEARCH ARTICLE

Semigroups of $\mathcal{I}$-density continuous functions

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Abstract

This paper is concerned with the classes of $\mathcal{I}$-density continuous functions and deep-$\mathcal{I}$-density continuous functions as semigroups with composition as the operation. We also analyze some of their subsemigroups. It is shown that the groups of automorphisms of these semigroups and several of their subsemigroups have the inner automorphism property.

1. Preliminaries

The notation used throughout this paper is standard. In particular, R stands for the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$. The symmetric difference of sets A and B is denoted by $A \triangle B$ and the complement of a set $A \subset \mathbb{R}$ by A^c. The symbols $\mathcal{L}$ and $\mathcal{B}$ stand for the families of subsets of R which are Lebesgue measurable and have the Baire property, respectively. $\mathcal{N}$ and $\mathcal{I}$ denote the ideals of Lebesgue measure zero and of first category subsets of R. If a statement is true everywhere except for those points of a set belonging to $\mathcal{N}$, then it is true *almost everywhere (a.e.)*. If it is true everywhere except for the points of a set belonging to $\mathcal{I}$, then it is true *$\mathcal{I}$-almost everywhere ($\mathcal{I}$-a.e.)*. If $A \in \mathcal{L}$, we denote its Lebesgue measure by $\mathrm{m}(A)$.

The natural topology on R is denoted by $\mathcal{T}_{\mathcal{O}}$. The symbols $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ stand for the interior and closure of $A \subset \mathbb{R}$ with respect to $\mathcal{T}_{\mathcal{O}}$. It is also easy to see that for any set $B \in \mathcal{B}$ there is a unique open set $\tilde{B}$ such that $\tilde{B} = \mathrm{int}\left(\mathrm{cl}\left(\tilde{B}\right)\right)$ and $B = \tilde{B} \triangle I$ for some $I \in \mathcal{I}$ [18]. Any open set G for which $G = \mathrm{int}(\mathrm{cl}(G))$ is called a *regular* open set. In a sense, $\tilde{B}$ is the largest open set such that B can be written as $\tilde{B} \triangle I$ for some $I \in \mathcal{I}$.

Any of the sets $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ or $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$ is a *right interval set* at 0 if $0 < b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$. A *left interval set* at 0 is defined similarly. A set E is a right (left) interval set at $x \in \mathbb{R}$ if $E - x$ is a right (left, respectively) interval set at 0.

For a topological space $(X, \mathcal{T})$ let $\mathrm{S}(X)$ or $\mathrm{S}((X, \mathcal{T}))$ be the semigroup of all continuous selfmaps of X; i.e., continuous functions $f : X \to X$, with composition as the operation. If there is no topology defined on X, then the discrete topology is used in the definition of $\mathrm{S}(X)$ so that $\mathrm{S}(X) = X^X$.

If G is a semigroup, then $\mathrm{Aut}(G)$ denotes the group of all automorphisms of G. A subsemigroup G of $\mathrm{S}(X)$ has the *inner automorphism property* if for every automorphism $\Psi \in \mathrm{Aut}(G)$ there is an $h \in G$ such that $\Psi(g) = h \circ g \circ h^{-1}$ for every $g \in G$.

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For a topological space X the property that $S(X)$ has the inner automorphism property is strongly related to the following definitions and facts.

A collection of topological spaces is *S–admissible* if for each pair of spaces X and Y from the collection, any isomorphism $\Phi : S(X) \to S(Y)$ is induced by a homeomorphism $h : X \to Y$; i.e., $\Phi(f) = h \circ f \circ h^{-1}$ for all $f \in S(X)$. In other words, within an S–admissible class of spaces, X is homeomorphic to Y if, and only if, $S(X)$ is isomorphic to $S(Y)$, so that the topological structure of the space X is fully characterized by the algebraic structure of $S(X)$. It is easy to see that for a topological space X a necessary and sufficient condition for belonging to an S–admissible class is that the semigroup $S(X)$ has the inner automorphism property. This gives a motivation for studying the topological spaces X for which $S(X)$ has the inner automorphism property. For more information on the subject see Magill [14].

The basis for studying the inner automorphism property of subsemigroups G of $S(X)$ is given by the following theorem of Schreier [21].

Proposition 1.1. *Let X be a set and let G be a subsemigroup of $S(X)$ such that every constant mapping is in G. Then, for every $\Psi \in \mathrm{Aut}(G)$ there exists a unique bijection h of X such that $\Psi(g) = h \circ g \circ h^{-1}$ for every $g \in G$.*

Every semigroup considered in this paper contains all constant functions. In particular, if Ψ is an automorphism of a semigroup $G \subset S(X)$ containing all constant functions then the unique bijection h of X for which $\Psi(g) = h \circ g \circ h^{-1}$ for all $g \in G$ will be called the *generating bijection* of Ψ.

Following Magill [14, Definition 2.2, p. 198] we say that a topological space X is *generated* if it is T_1 and the collection of complements of level sets for its continuous selfmaps $\{(f^{-1}(\{x\}))^c : x \in X \text{ and } f \in S(X)\}$ forms a subbase for X. (Compare also [22].) It is known that the class of all generated spaces is S–admissible [14, Theorem 2.3, p. 198]. In particular, there is the following.

Proposition 1.2. *If a topological space X is generated then, $S(X)$ has the inner automorphism property.*

2. Topologies

This paper is concerned with the semigroups $\mathcal{C}_{\mathcal{I}\mathcal{I}} = S((\mathbb{R}, \mathcal{T}_{\mathcal{I}}))$ of *$\mathcal{I}$-density continuous functions*, $\mathcal{C}_{\mathcal{D}\mathcal{D}} = S((\mathbb{R}, \mathcal{T}_{\mathcal{D}}))$ of *deep-$\mathcal{I}$-density continuous functions* and $\mathcal{C}_{\mathcal{N}\mathcal{N}} = S((\mathbb{R}, \mathcal{T}_{\mathcal{N}}))$ of *density continuous functions* and some of their subsemigroups. We start with definitions of the topologies $\mathcal{T}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{D}}$. For this, we first introduce the notions of a density point, $\mathcal{I}$-density point and deep-$\mathcal{I}$-density point. (See [18, 19, 23]. For discussion, compare also [6].)

Let $A \in \mathcal{L}$. A number x, not necessarily in A, is a *density point* of A if

$$\lim_{h \to 0^+} \frac{m(A \cap (x-h, x+h))}{2h} = 1. \tag{1}$$

The set of all density points of $A \in \mathcal{L}$ we denote by $\Phi_{\mathcal{N}}(A)$. It is a straightforward consequence of the Lebesgue density theorem that for every $A \in \mathcal{L}$, $\Phi_{\mathcal{N}}(A) \triangle A \in \mathcal{N}$. The family of sets

$$\mathcal{T}_{\mathcal{N}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{N}}(A)\}$$

forms a topology on R [18, 10] called the *density topology*.

If $A \in \mathcal{B}$, we say that 0 is an *$\mathcal{I}$-density point* of A if for every increasing sequence $\{n_m\}_{m\in\mathbf{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbf{N}}$ such that

$$\lim_{p\to\infty} \chi_{\, n_{m_p} A\cap(-1,1)} = \chi_{(-1,1)} \ \mathcal{I}\text{-a.e.} \tag{2}$$

We say that a point a is an $\mathcal{I}$-density point of $A \in \mathcal{B}$ if 0 is an $\mathcal{I}$-density point of $A - a$. The set of all $\mathcal{I}$-density points of $A \in \mathcal{B}$ we denote by $\Phi_\mathcal{I}(A)$. Similar to the case with Lebesgue density, as noted above, $A\triangle\Phi_\mathcal{I}(A) \in \mathcal{I}$ for every $A \in \mathcal{B}$ [23, Theorem 3] and

$$\Phi_\mathcal{I}(A) = \Phi_\mathcal{I}(B) \quad \text{for every } A, B \in \mathcal{B} \text{ such that } \ A\triangle B \in \mathcal{I}. \tag{3}$$

The family of sets

$$\mathcal{T}_\mathcal{I} = \{A \in \mathcal{B} : A \subset \Phi_\mathcal{I}(A)\}$$

forms a topology on R [19, 23] called the *$\mathcal{I}$-density topology*.

Finally, we say that a point $a \in \mathbb{R}$ is a *deep-$\mathcal{I}$-density point* [23] of an $A \in \mathcal{B}$ if there exists a closed set $F \subset A \cup \{a\}$ such that a is an $\mathcal{I}$-density point of F. The set of all deep-$\mathcal{I}$-density points of $A \in \mathcal{B}$ is denoted $\Phi_\mathcal{D}(A)$. The family of sets

$$\mathcal{T}_\mathcal{D} = \{A \in \mathcal{B} : A \subset \Phi_\mathcal{D}(A)\}$$

forms a topology on R [11, 23] called the *deep-$\mathcal{I}$-density topology*.

The inclusion relations between the topologies $\mathcal{T}_\mathcal{O}$, $\mathcal{T}_\mathcal{N}$, $\mathcal{T}_\mathcal{I}$ and $\mathcal{T}_\mathcal{D}$ are given by the following theorem [8, Theorem 2.5].

Theorem 2.1. *If $\mathcal{P}(\mathbb{R})$ stands for the discrete topology on R then*

$$\begin{array}{ccccccc}
\mathcal{T}_\mathcal{O} \cap \mathcal{T}_\mathcal{N} & \subset & \mathcal{T}_\mathcal{D} \cap \mathcal{T}_\mathcal{N} & \subset & \mathcal{T}_\mathcal{I} \cap \mathcal{T}_\mathcal{N} & \subset & \mathcal{T}_\mathcal{N} \\
\| & & \cap & & \cap & & \cap \\
\mathcal{T}_\mathcal{O} & \subset & \mathcal{T}_\mathcal{D} & \subset & \mathcal{T}_\mathcal{I} & \subset & \mathcal{P}(\mathbb{R})
\end{array}$$

Moreover, all the inclusions are proper.

We also need the following fact. For the $\mathcal{I}$-density case it can be found in [4, Lemma 2.4], [20, Theorem 1] or [23, Theorem 2].

Proposition 2.2. *If $E = \bigcup_{n\in\mathbf{N}}[a_n, b_n]$ is a right interval set such that*

(i) $\lim_{n\to\infty} (b_n - a_n)/a_n = 0$*; and*

(ii) $\lim_{n\to\infty} b_{n+1}/a_n = 0$,

then 0 is a density and deep-$\mathcal{I}$-density point of E^c. In particular, $E^c \in \mathcal{T}_\mathcal{N} \cap \mathcal{T}_\mathcal{D}$.

3. Density Continuous Functions

Let Δ be the class of all differentiable functions from R to R. Moreover, let $\Delta^{(\mathcal{N})}$ stand for the class of all approximately differentiable functions ([2], see also [17]) and let $\Delta^{(\mathcal{N})}_\mathcal{N}$ be the class of all almost everywhere approximately differentiable functions. Often used is the group of homeomorphisms $f : (\mathbb{R}, \mathcal{T}_\mathcal{O}) \to (\mathbb{R}, \mathcal{T}_\mathcal{O})$, which is written as $\mathcal{H}$.

The classes Δ, $\Delta^{(\mathcal{N})}$, $\Delta^{(\mathcal{N})}_\mathcal{N}$ are connected to the class $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ of density continuous functions by the following theorem.

Theorem 3.1. *If $\mathcal{F}_m$ stands for the class of measurable functions, then*

$$\begin{array}{ccccccc} \Delta & \subset & \Delta^{(\mathcal{N})} & \subset & \Delta^{(\mathcal{N})}_{\mathcal{N}} & \subset & \mathcal{F}_m \\ \cup & & \cup & & \cup & & \cup \\ \Delta\cap\mathcal{C}_{\mathcal{N}\mathcal{N}} & \subset & \Delta^{(\mathcal{N})}\cap\mathcal{C}_{\mathcal{N}\mathcal{N}} & \subset & \Delta^{(\mathcal{N})}_{\mathcal{N}}\cap\mathcal{C}_{\mathcal{N}\mathcal{N}} & \subset & \mathcal{C}_{\mathcal{N}\mathcal{N}} \end{array}$$

and all the inclusions are proper.

Proof. The inclusions are obvious. The vertical inclusions are proper because there is a $\mathcal{C}^\infty$ function which is not density continuous [7, Example 1].

$\mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \Delta^{(\mathcal{N})}_{\mathcal{N}}$ follows immediately from [9, Theorem 4].

$\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \Delta^{(\mathcal{N})}$ is proved by the function $h(x) = |x|$.

$\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \Delta$ is easily justified by the function f defined by $f(x) = 0$ for $x \in E^c$ and $f(x) = (x-a_n)^2(x-b_n)^2$ for $x \in [a_n, b_n]$ and $n \in \mathbb{N}$, where $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a right interval set from Proposition 2.2. ■

The motivation to study the inner automorphism property of the classes from Theorem 3.1 arises from the following theorem of Magill [13].

Theorem 3.2. *The semigroup Δ of all differentiable functions has the inner automorphism property.*

Following this path Ostaszewski proved the following [17].

Theorem 3.3. *The semigroups $\mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ and $\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ have the inner automorphism property.*

Notice that the above theorem cannot be deduced from Proposition 1.2, because the density topology is not generated [5]. In fact, the density topology is the only known example of completely regular not generated topological space for which the semigroup of continuous selfmaps has the inner automorphism property. (Also note the remark after Theorem 5.5.)

Theorem 3.3 is proved by showing the following facts: for every generating bijection h of $\Psi \in \mathrm{Aut}(\mathcal{C}_{\mathcal{N}\mathcal{N}}) \cup \mathrm{Aut}(\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}})$ we have $h, h^{-1} \in \mathcal{H} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \Delta^{(\mathcal{N})}$ and for every generating bijection h of $\Psi \in \mathrm{Aut}(\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) \cup \mathrm{Aut}(\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}})$ we have $h, h^{-1} \in \mathcal{H} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \Delta$. In fact, if we identify the group $\mathrm{Aut}(H)$ with the set of generating bijections, we obtain

Corollary 3.4. *The following relations hold:*

$$\mathrm{Aut}(\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) = \mathrm{Aut}(\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) \subset \mathrm{Aut}(\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}) = \mathrm{Aut}(\mathcal{C}_{\mathcal{N}\mathcal{N}})$$

and the inclusion is proper.

Theorems 3.2 and 3.3 prove that the classes Δ, $\Delta \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$, $\Delta^{(\mathcal{N})}_{\mathcal{N}} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ have the inner automorphism property. For the remaining three classes the question about their inner automorphism property is moot as those classes do not form semigroups. For the class $\Delta^{(\mathcal{N})}$, the proof is given in Example 4.11. For the two other classes the result follows from the example below.

Example 3.5. There exist almost everywhere differentiable functions f and g such that $g \circ f$ is not measurable. In particular, the classes $\Delta^{(\mathcal{N})}_{\mathcal{N}}$ and $\mathcal{F}_m$ are not closed under composition.

Proof. Let $P \subset \mathbb{R}$ be any perfect nowhere dense set with positive measure, $S \subset P$ be nonmeasurable and let C be a Cantor set of measure 0. Let f be a homeomorphism such that $f^{-1}(C) = P$ and let $g = \chi_S \circ f^{-1}$. Then, f is differentiable almost everywhere, as a homeomorphism and g is differentiable almost everywhere, as $g = \chi_{f(S)}$ and $f(S) \subset C$ has measure 0. On the other hand, $g \circ f = \chi_S \circ f^{-1} \circ f = \chi_S$ is not measurable, because S is nonmeasurable. ∎

4. $\mathcal{I}$-approximate Derivative

In this section we introduce the notion of the $\mathcal{I}$-approximate derivative as category analogue of the approximate derivative. Let us recall that a function $f:\mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-approximately continuous (deep-$\mathcal{I}$-approximately continuous) if f is continuous with respect to the ordinary topology on the range and $\mathcal{I}$-density (deep-$\mathcal{I}$-density) topology on the domain. It is well known that the classes of $\mathcal{I}$-approximately continuous functions and deep-$\mathcal{I}$-approximately continuous functions coincide [11, Theorem 2].

We start with the following definitions. A function $f:\mathbb{R} \to \mathbb{R}$ is said to be *$\mathcal{I}$-approximately differentiable at a point* x if there exists a number $D^{(\mathcal{I})} f(x)$, called the *$\mathcal{I}$-approximate derivative* of f at x, such that for every $\varepsilon > 0$, x is an $\mathcal{I}$-density point of some Baire subset of

$$\left\{t \in \mathbb{R}: \frac{f(t) - f(x)}{t - x} \in (D^{(\mathcal{I})} f(x) - \varepsilon, D^{(\mathcal{I})} f(x) + \varepsilon)\right\}. \tag{4}$$

(Compare also [12] and [23, Definition 8].)

A function $f:\mathbb{R} \to \mathbb{R}$ is said to be *$\mathcal{I}$-approximately differentiable* if it is $\mathcal{I}$-approximately differentiable at every point. The class of all $\mathcal{I}$-approximately differentiable functions is denoted by $\Delta^{(\mathcal{I})}$.

A function $f:\mathbb{R} \to \mathbb{R}$ is said to be *$\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere* if there exists a set $A \in \mathcal{I}$ such that f is $\mathcal{I}$-approximately differentiable at every point of A^c. The class of all functions $\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere is denoted by $\Delta^{(\mathcal{I})}_{\mathcal{I}}$.

Similarly, the classes of functions that are deep-$\mathcal{I}$-approximately or $\mathcal{I}$-approximately continuous $\mathcal{I}$-almost everywhere may be defined.

It is easy to introduce the idea of a function $f:\mathbb{R} \to \mathbb{R}$ being deep-$\mathcal{I}$-approximately differentiable at point x by requiring that x is a deep-$\mathcal{I}$-density point of a set from (4). However it can be proved that for the $\mathcal{I}$-approximately continuous functions the notions of $\mathcal{I}$-approximate differentiability and deep-$\mathcal{I}$-approximate differentiability at a point coincide. Thus, there is no good reason to pursue this notion.

We start our investigation with the following propositions.

Proposition 4.1. *If a function $f:\mathbb{R} \to \mathbb{R}$ is $\mathcal{I}$-approximately differentiable at x then f is $\mathcal{I}$-approximately continuous at x.*

Proof. Choose $\varepsilon > 0$. We must prove that x is an $\mathcal{I}$-density point of the set

$$\{t \in \mathbb{R}: |f(t) - f(x)| \in (-\varepsilon, \varepsilon)\}.$$

But, if we choose $\delta > 0$ such that $\delta\,(D^{(\mathcal{I})} f(x) - \varepsilon, D^{(\mathcal{I})} f(x) + \varepsilon) \subset (-\varepsilon, \varepsilon)$, then

$$\{t \in \mathbb{R}: |f(t) - f(x)| \in (-\varepsilon, \varepsilon)\} \supset$$

$$\{t \in \mathbb{R}: |t-x| < \delta \;\&\; |f(t)-f(x)| \in \delta\,(D^{(\mathcal{I})}f(x)-\varepsilon,\, D^{(\mathcal{I})}f(x)+\varepsilon)\} \supset$$

$$\{t \in \mathbb{R}: |t-x| < \delta \;\&\; |f(t)-f(x)| \in |t-x|\,(D^{(\mathcal{I})}f(x)-\varepsilon,\, D^{(\mathcal{I})}f(x)+\varepsilon)\} =$$

$$\left\{t \in \mathbb{R}: \frac{|f(t)-f(x)|}{|t-x|} - D^{(\mathcal{I})}f(x) \in (-\varepsilon,\varepsilon)\right\} \cap (x-\delta, x+\delta)$$

and, by assumption, x is an $\mathcal{I}$-density point of some Baire subset of the last set. This finishes the proof. ∎

Proposition 4.2. *If $f:\mathbb{R}\to\mathbb{R}$ is $\mathcal{I}$-approximately differentiable $\mathcal{I}$-almost everywhere, then f is a Baire function.*

Proof. Poreda, Wagner-Bojakowska and Wilczyński proved that a function $f:\mathbb{R}\to\mathbb{R}$ is $\mathcal{I}$-approximately continuous $\mathcal{I}$-almost everywhere if, and only if, f has the Baire property [19, Theorem 7]. Use of Proposition 4.1 completes the proof. ∎

In the sequel we also need the following two examples.

Example 4.3. For every $a < b \le c < d$ there exists a differentiable $\mathcal{I}$-density continuous and density continuous function $f:\mathbb{R}\to[0,1]$ such that $f(x)=0$ for $x \in (a,d)^c$ and $f(x)=1$ for $x\in[b,c]$.

Proof. We have to define f only on (a,b) and (c,d). So, define

$$f(x) = \frac{(x-a)^2(x-2b+a)^2}{(b-a)^4}$$

for $x \in (a,b)$ and

$$f(x) = \frac{(x-d)^2(x-2c+d)^2}{(c-d)^4}$$

for $x\in(c,d)$. It is easy to see that f is differentiable and, because it is everywhere unilaterally a polynomial, it is also density and $\mathcal{I}$-density continuous [7, Corollary 3], [3]. ∎

The previous example easily implies the existence of the following example.

Example 4.4. There exists an approximately and $\mathcal{I}$-approximately differentiable, density continuous and $\mathcal{I}$-density continuous function which is not continuous. In particular,

$$\Delta^{(\mathcal{N})} \cap \Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{N}\mathcal{N}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}.$$

Proof. Let $E = \bigcup_{n\in\mathbb{N}}[a_n,d_n]$ be an interval set as in Proposition 2.2. For each n, let f_n be as in Example 4.3 with $[a,d]=[a_n,d_n]$. Define $f(x)=0$ on E^c and $f=f_n$ on $[a_n,d_n]$. The rest is obvious. ∎

Now we are ready to prove the next theorem.

Theorem 4.5. *If $\mathcal{F}_c$ stands for the Baire functions, then*

$$\begin{array}{ccccccc}
\Delta & \subset & \Delta^{(\mathcal{I})} & \subset & \Delta_{\mathcal{I}}^{(\mathcal{I})} & \subset & \mathcal{F}_c \\
\cup & & \cup & & \cup & & \cup \\
\Delta \cap \mathcal{C}_{\mathcal{DD}} & \subset & \Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{DD}} & \subset & \Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{DD}} & \subseteq & \mathcal{C}_{\mathcal{DD}} \\
\cup & & \cup & & \cup & & \cup \\
\Delta \cap \mathcal{C}_{\mathcal{II}} & \subset & \Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}} & \subset & \Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}} & \subseteq & \mathcal{C}_{\mathcal{II}}
\end{array}$$

and all the inclusions with the symbol $\subset$ are proper.

Proof. The inclusion $\mathcal{C}_{\mathcal{II}} \subset \mathcal{C}_{\mathcal{DD}}$ can be found in [8, Theorem 4.1(iv)]. The other inclusions are obvious.

The vertical inclusions are proper because $\Delta \cap \mathcal{C}_{\mathcal{DD}} \not\subset \mathcal{C}_{\mathcal{II}}$ [8, Example 6.7] and $\Delta \not\subset \mathcal{C}_{\mathcal{DD}}$ [8, Example 5.7].

The fact that the horizontal inclusions of the form $\subset$ are proper follows from $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}} \not\subset \Delta$ (Example 4.4) and $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}} \not\subset \Delta^{(\mathcal{I})}$ which is proved by $f(x) = |x|$.

This finishes the proof. ∎

It remains open whether the inclusions of the form $\subseteq$ in the previous theorem are proper. In particular, a positive answer to this question would follow from a positive answer for the following problem.

Problem 4.6. Does there exist an $\mathcal{I}$-density continuous function which is nowhere $\mathcal{I}$-approximately differentiable?

To prove the next theorem, the following two lemmas are needed. The first one is a version of the chain rule. The analogous theorem for approximate derivatives can be found in [16].

Lemma 4.7. *If $f, g\colon \mathbb{R} \to \mathbb{R}$ are such that $\Delta^{(\mathcal{I})} f(x_0)$ and $\Delta^{(\mathcal{I})} g(f(x_0))$ exist then $\Delta^{(\mathcal{I})}(g \circ f)(x_0)$ also exists and*

$$\Delta^{(\mathcal{I})}(g \circ f)(x_0) = \Delta^{(\mathcal{I})} g(f(x_0))\, \Delta^{(\mathcal{I})} f(x_0). \tag{5}$$

Proof. First assume that $\Delta^{(\mathcal{I})} g(f(x_0)) > 0$ and $\Delta^{(\mathcal{I})} f(x_0) > 0$. Put $z_0 = f(x_0)$ and let u_1, u_2, v_1, v_2 be arbitrary positive numbers such that $0 < u_1 < \Delta^{(\mathcal{I})} f(x_0) < u_2$ and $0 < v_1 < \Delta^{(\mathcal{I})} g(z_0) < v_2$. Then, there exist sets $U, V \in \mathcal{T}_{\mathcal{I}}$, $x_0 \in U$, $z_0 \in V$, such that

$$u_1 \le \frac{f(x) - f(x_0)}{x - x_0} \le u_2 \tag{6}$$

for all $x \in U$ and

$$v_1 \le \frac{g(z) - g(z_0)}{z - z_0} \le v_2 \tag{7}$$

for all $z \in V$. Since, by Proposition 4.1, f is $\mathcal{I}$-density continuous at x_0, there exists $W \in \mathcal{T}_{\mathcal{I}}$ such that $x_0 \in W \subset f^{-1}(V)$. In particular, by (7), for $x \in W$ we have $z = f(x) \in V$ and

$$v_1 \le \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \le v_2. \tag{8}$$

Multiplying (6) by (8) we obtain

$$u_1v_1 < \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} < u_2v_2;$$

i.e.,

$$u_1v_1 < \frac{g(f(x)) - g(f(x_0))}{x - x_0} < u_2v_2,$$

for every $x \in U \cap W \in \mathcal{T}_{\mathcal{I}}$. But,

$$u_1v_1 < \Delta^{(\mathcal{I})}g(f(x_0))\,\Delta^{(\mathcal{I})}f(x_0) < u_2v_2$$

where the numbers u_1v_1 and u_2v_2 can be choosen as close to the number $\Delta^{(\mathcal{I})}g(f(x_0))$ $\Delta^{(\mathcal{I})}f(x_0)$ as we wish. This implies (5).

The remaining cases, when $\Delta^{(\mathcal{I})}g(f(x_0)) \leq 0$ or $\Delta^{(\mathcal{I})}f(x_0) \leq 0$, are very similar, modulo some little technical problems with the signs of the inequalities. This finishes the proof. ∎

Lemma 4.8. *If $f \in \mathcal{C}_{\mathcal{I}\mathcal{I}} \cap \Delta_{\mathcal{I}}^{(\mathcal{I})}$ and $g \in \Delta_{\mathcal{I}}^{(\mathcal{I})}$, then $g \circ f \in \Delta_{\mathcal{I}}^{(\mathcal{I})}$.*

Proof. Let $A, B \in \mathcal{I}$ be such that f is $\mathcal{I}$-approximately differentiable on A^c and g is $\mathcal{I}$-approximately differentiable on B^c.

By Lemma 4.7 the function $g \circ f$ is $\mathcal{I}$-approximately differentiable on the set $D = A^c \cap f^{-1}(B^c)$. Moreover, for every x, if $E_x = f^{-1}(x)$ then $g \circ f$ is $\mathcal{I}$-approximately differentiable on the set $E_x \cap \tilde{E}_x \in \mathcal{T}_{\mathcal{I}}$, as $g \circ f$ is constant on this set. Thus, if $E = \bigcup_{x \in \mathbb{R}}(E_x \cap \tilde{E}_x)$, then $g \circ f$ is $\mathcal{I}$-approximately differentiable on $D \cup E$.

To finish the proof it is enough to prove that $(D \cup E)^c \in \mathcal{I}$; in other words that every x is an $\mathcal{I}$-density point of $D \cup E$.

So, let $x \in \mathbb{R}$. Then, $f(x)$ is an $\mathcal{I}$-density point of $B^c \cup \{f(x)\}$, because B is discrete in $\mathcal{T}_{\mathcal{I}}$. So, since f is $\mathcal{I}$-density continuous, x is an $\mathcal{I}$-density point of

$$\begin{aligned} f^{-1}(B^c \cup \{f(x)\}) &= f^{-1}(B^c) \cup f^{-1}(f(x)) \\ &= f^{-1}(B^c) \cup E_{f(x)} \\ &= f^{-1}(B^c) \cup (E_{f(x)} \cap \tilde{E}_{f(x)}) \cup (E_{f(x)} \setminus \tilde{E}_{f(x)}). \end{aligned}$$

But $E_{f(x)} \setminus \tilde{E}_{f(x)} \in \mathcal{I}$ and $A \in \mathcal{I}$. Hence, x is an $\mathcal{I}$-density point of $A^c \cap (f^{-1}(B^c) \cup (E_{f(x)} \cap \tilde{E}_{f(x)})) \subset D \cup E$. This finishes the proof. ∎

As a corollary we obtain

Theorem 4.9. *The classes Δ, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ and $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ are closed under composition. In particular, they form semigroups.*

Proof. The classes Δ, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ are obviously closed under composition. Therefore the same is true of the classes $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$.

For the classes $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ the conclusion follows immediately from Lemma 4.7, while for the class $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ it follows from Lemma 4.8. ∎

Problem 4.10. Are the classes $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{DD}}$ and $\Delta_{\mathcal{I}}^{(\mathcal{I})}$ closed under composition? If so, do they have the inner automorphism property?

We finish this section by showing that the remaining classes $\Delta^{(\mathcal{I})}$ and $\mathcal{F}_c$ of Theorem 4.5 are not closed under composition.

Example 4.11. There exists a differentiable function f and an approximately and $\mathcal{I}$-approximately differentiable function g such that $g \circ f$ is neither approximately nor $\mathcal{I}$-approximately continuous. In particular, the classes $\Delta^{(\mathcal{N})}$ and $\Delta^{(\mathcal{I})}$ are not closed under composition.

Proof. Let $D = \bigcup_{n \in \mathbb{N}} (p_n, q_n) = (0, \infty) \setminus E$ where E is from Proposition 2.2. Let us define $h: \mathbb{R} \to \mathbb{R}$ by putting $h(x) = 0$ for $x \in D^c$ and

$$h(x) = u_n (x - p_n)(x - q_n)$$

for $x \in (p_n, q_n)$, where $u_n > 0$. Define f by $f(x) = \int_0^x h(y)\,dy$. It is easy to see that f is differentiable and constant on each interval contained in D^c. Decreasing the constants u_n, if necessary, we may assume that $\lim_{n\to\infty} f(p_{n+1})/f(p_n) = 0$.

Let us choose intervals (a_n, b_n) centered at $f(p_n)$ such that the assumptions from Proposition 2.2 are satisfied and let $E = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$. Then, $E^c \in \mathcal{T}_{\mathcal{N}} \cap \mathcal{T}_{\mathcal{D}}$. Let us define $g(x) = 0$ for $x \in E^c$ and

$$g(x) = v_n (x - a_n)^2 (x - b_n)^2$$

for $x \in (a_n, b_n)$, where the v_n are choosen in such a way that $g(f(p_n)) = 1$. Then g is differentiable at every point $\neq 0$ and g is constantly equal 0 on $E^c \in \mathcal{T}_{\mathcal{I}} \cap \mathcal{T}_{\mathcal{N}}$. Hence, $h \in \Delta^{(\mathcal{N})} \cap \Delta^{(\mathcal{I})}$. On the other hand, $g \circ f = 1$ on the set D^c while, $g \circ f(0) = 0$. Thus, $g \circ f \notin \Delta^{(\mathcal{N})} \cup \Delta^{(\mathcal{I})}$. ∎

Example 4.12. The class $\mathcal{F}_c$ is not closed under composition.

Proof. Let h be an embedding of the irrationals $\mathbb{R} \setminus \mathbb{Q}$ into the Cantor set $C \subset [0,1]$. Put $f(q) = 2$ for $q \in \mathbb{Q}$ and $f(x) = h(x)$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, $f \in \mathcal{F}_c$. Moreover, let $S \subset \mathbb{R} \setminus \mathbb{Q}$ be a set without the Baire property and let g be the characteristic function of $f(S)$; i.e., $g = \chi_{f(S)}$. Then $g \in \mathcal{F}_c$, since $f(S) \subset C$ and $g \circ f = \chi_S \notin \mathcal{F}_c$. ∎

5. Semigroups of $\mathcal{I}$-density Continuous Functions

For the reminder of this paper let $\mathcal{E}$ stand for any of the semigroups of Theorem 4.9; e.g., one of the classes $\mathcal{C}_{\mathcal{II}}$, $\mathcal{C}_{\mathcal{DD}}$, $\Delta \cap \mathcal{C}_{\mathcal{II}}$, $\Delta \cap \mathcal{C}_{\mathcal{DD}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{DD}}$ or $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}}$. In particular, $\Delta \cap \mathcal{C}_{\mathcal{II}} \subset \mathcal{E} \subset \mathcal{C}_{\mathcal{DD}}$. It will be shown that these semigroups, with the possible exception of $\Delta_{\mathcal{I}}^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}}$, have the inner automorphism property.

We start with the following lemma.

Lemma 5.1. *If h is a generating bijection of an automorphism Ψ of $\mathcal{E}$, then h is $\mathcal{I}$-approximately continuous.*

Proof. Let $r \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. We will show that h is $\mathcal{I}$-approximately continuous at r.

Let $f: \mathbb{R} \to \mathbb{R}$ be as in Example 4.3 for $a = h(r) - \varepsilon$, $b = c = h(r)$ and $d = h(r) + \varepsilon$. Let $g: \mathbb{R} \to \mathbb{R}$ with $g(x) = f(x) + h(r)$. It is easy to see that $g \in \Delta \cap \mathcal{C}_{\mathcal{II}} \subset \mathcal{E}$.

Ψ is an automorphism of $\mathcal{E}$, thus there is an $\alpha \in \mathcal{E} \subset \mathcal{C}_{\mathcal{DD}}$ such that $\Psi(\alpha) = g$. We have $h \circ \alpha = g \circ h$. Note that $h(\alpha(r)) = g(h(r)) = f(h(r)) + h(r) = 1 + h(r)$. Therefore, $h(\alpha(r)) \neq h(r)$. But, h is a bijection, so $\alpha(r) \neq r$.

The function α is deep-$\mathcal{I}$-density continuous. Thus, there exists a set $U \in \mathcal{T}_{\mathcal{D}}$ such that $\alpha(x) \neq r$ for all $x \in U$. Then, for all $x \in U$, $f(h(x)) + h(r) = g(h(x)) = h(\alpha(x)) \neq h(r)$, which implies $f(h(x)) \neq 0$; i.e., $h(x) \in (a, d) = (h(r) - \varepsilon, h(r) + \varepsilon)$. So $|h(x) - h(r)| < \varepsilon$ for $x \in U$. This means precisely that h is $\mathcal{I}$-approximately continuous. ∎

Corollary 5.2. *If h is a generating bijection of an automorphism Ψ of $\mathcal{E}$, then h is a homeomorphism.*

Proof. h is an $\mathcal{I}$-approximately continuous bijection of R. Since h is also a Darboux Baire one function, it must be a homeomorphism. ∎

As a next step we need the following

Lemma 5.3. *Let $h: \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism which is not $\mathcal{I}$-density continuous at 0. Then there exists a function $f \in \Delta \cap \mathcal{C}_{\mathcal{II}} \subset \mathcal{E}$ such that $h \circ f \circ h^{-1}$ is not deep-$\mathcal{I}$-density continuous.*

Proof. Without any loss of generality we may assume that $h(0) = 0$.

It is very easy to verify that any deep-$\mathcal{I}$-density continuous homeomorphism is also $\mathcal{I}$-density continuous. (Compare e.g. [23, Theorem 23].) Thus, h is also not deep-$\mathcal{I}$-density continuous at 0. Again, without any loss of generality we may further assume that h is not right deep-$\mathcal{I}$-density continuous at 0. The left-hand side argument is essentially the same.

The sets $E \cup (-E) \cup \{0\}$, where E is an open right interval set, form basis for $\mathcal{T}_{\mathcal{D}}$ at 0 [11]. Thus, there exists a right interval set $E = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ such that 0 is an $\mathcal{I}$-density point of E^c, while 0 is not a right $\mathcal{I}$-density point of the complement of

$$D = \bigcup_{n \in \mathbb{N}} [h^{-1}(a_n), h^{-1}(b_n)].$$

Using the definition of deep-$\mathcal{I}$-density point we can easily choose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $h^{-1}(a_n) < \alpha_n < \beta_n < h^{-1}(b_n)$ for $n \in \mathbb{N}$ and 0 is still not a right $\mathcal{I}$-density point of the complement of

$$\bigcup_{n \in \mathbb{N}} [\alpha_n, \beta_n]$$

while, evidently, 0 is an $\mathcal{I}$-density point of the complement of

$$\bigcup_{n \in \mathbb{N}} [h(\alpha_n), h(\beta_n)] \subset \bigcup_{n \in \mathbb{N}} [a_n, b_n].$$

For each $n \in \mathbb{N}$, let $f_n: \mathbb{R} \to [0, 1]$ be a function from Example 4.3 for $h^{-1}(a_n) < \alpha_n < \beta_n < h^{-1}(b_n)$. Define $f(x) = 0$ for $x \in E^c$ and $f(x) = c_n f_n(x)$ for $x \in [a_n, b_n]$ and $n \in \mathbb{N}$, where $c_n > 0$ are choosen in such a way that

$$\lim_{n \to \infty} \frac{h(c_{n+1})}{h(c_n)} = 0. \tag{9}$$

Then it is easy to see that f is differentiable and $\mathcal{I}$-density continuous.

On the other hand

$$T = (h \circ f \circ h^{-1})\left(\bigcup_{n\in\mathbf{N}}(\alpha_n,\beta_n)\right) = \bigcup_{n\in\mathbf{N}} h(c_n).$$

Now, by (9) and Proposition 2.2, it is easy to see that there is a right interval set $S \supset T$ such that 0 is a deep-$\mathcal{I}$-density point of S^c, while 0 is not an $\mathcal{I}$-density point of $(\bigcup_{n\in\mathbf{N}}[\alpha_n,\beta_n])^c$. Hence, $h \circ f \circ h^{-1}$ is not deep-$\mathcal{I}$-density continuous. ■

Corollary 5.4. *If h is a generating bijection of an automorphism of $\mathcal{E}$, then h and h^{-1} are $\mathcal{I}$-density continuous homeomorphisms of R.*

Proof. By Corollary 5.2, h is a homeomorphism of R. So, by Lemma 5.3 it must be $\mathcal{I}$-density continuous. The statement for h^{-1} follows from the fact that h^{-1} is a generating bijection for Ψ^{-1}. ■

As an immediate corollary we obtain the following.

Theorem 5.5. *The semigroups $\mathcal{C}_{\mathcal{II}}$ and $\mathcal{C}_{\mathcal{DD}}$ have the inner automorphism property. Moreover,*

$$\mathrm{Aut}(\mathcal{C}_{\mathcal{II}}) = \mathrm{Aut}(\mathcal{C}_{\mathcal{DD}}).$$

Let us notice that we can also deduce that $\mathcal{C}_{\mathcal{DD}}$ has the inner automorphism property by using Proposition 1.2, as the deep-$\mathcal{I}$-density topology $\mathcal{T}_{\mathcal{D}}$ is generated. This follows immediatelly from the fact that the function which demonstrates the complete regularity of the $\mathcal{T}_{\mathcal{D}}$ [11, Theorem 5] is also deep-$\mathcal{I}$-density continuous[3]. However, Proposition 1.2 cannot be used in the case of the $\mathcal{I}$-density topology, because $\mathcal{T}_{\mathcal{I}}$ is not generated. This follows from the fact that for $\mathcal{T}_{\mathcal{I}}$ to be generated, the family $\mathcal{F} = \{(f^{-1}(\{x\}))^c : x \in \mathbb{R} \text{ and } f\colon(\mathbb{R}, \mathcal{T}_{\mathcal{I}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{I}})\}$ would have to form a subbase of $\mathcal{T}_{\mathcal{I}}$. This is not the case since even the family $\{(f^{-1}(\{1\}))^c : f\colon(\mathbb{R}, \mathcal{T}_{\mathcal{I}}) \to [0,1]\} \subset \mathcal{F}$ does not form a subbase for $\mathcal{T}_{\mathcal{I}}$, because $\mathcal{T}_{\mathcal{I}}$ is not regular [19, Theorem 5].

To discuss the inner automorphism properties of the other semigroups we need the following two lemmas.

Lemma 5.6. *The notions of $\mathcal{I}$-approximate derivative, approximate derivative and ordinary derivative coincide on the class of all homeomorphisms. In other words, if $f\colon\mathbb{R}\to\mathbb{R}$ is a homeomorphism and $x \in \mathbb{R}$ then $f'(x) = \Delta^{(\mathcal{I})}f(x) = \Delta^{(\mathcal{N})}f(x)$ whenever any piece of this equation exists.*

Proof. See Łazarow and Wilczyński [12], also Wilczyński [23, Theorem 38]. ■

Lemma 5.7. *Let Ψ be an automorphism of $\Delta \cap \mathcal{C}_{\mathcal{II}}$, $\Delta \cap \mathcal{C}_{\mathcal{DD}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{II}}$ or $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{DD}}$ and let h be its generating bijection. Then h is differentiable; i.e., $h \in \Delta$.*

[3] If $0 \in A \in \mathcal{T}_{\mathcal{D}}$, then there exist right interval sets $E \subset V \subset A$ composed of closed and open intervals, respectively, such that 0 is a deep-$\mathcal{I}$-density point of a closed set $P = \mathrm{cl}((-E) \cup E)$ which is a subset of $U = (-V) \cup V \cup \{0\}$. Lazarow's function is defined by $f(0) = 1$ and $f(x) = (\mathrm{dist}(x,U^c))/(\mathrm{dist}(x,U^c) + \mathrm{dist}(x,P))$ for $x \neq 0$. Because f is piecewise linear on a neighborhood of every point $x \neq 0$, it must be density continuous at x. It is deep-$\mathcal{I}$-density continuous at 0 because f is constant on $U \in \mathcal{T}_{\mathcal{D}}$.

Proof. It is already known that h is a homeomorphism. Thus h is differentiable almost everywhere [2]. Let x_0 be a point of differentiability of h and let $x \in \mathbb{R}$ be any other point. Define $f(t) = t + (x - x_0)$. Then $f \in \Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, so it belongs to all of the semigroups under consideration. For any positive δ we have

$$\begin{aligned}\frac{h(x+\delta)-h(x)}{\delta} &= \frac{h\circ f(x_0+\delta)-h\circ f(x_0)}{\delta} \\ &= \frac{\Psi(f)\circ h(x_0+\delta)-\Psi(f)\circ h(x_0)}{\delta}.\end{aligned}$$

If Ψ is an automorphism of $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ then $\Psi(f)$ is differentiable, and since h is differentiable at x_0, the quotient in (10) converges to $\Psi(f)'(h(x_0))\,h'(x_0)$.

If Ψ is an automorphism of $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ or $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ then $\Psi(f)$ is $\mathcal{I}$-approximately differentiable. So, by Lemma 4.7, $\Psi(f)\circ h$ is $\mathcal{I}$-approximately differentiable at x_0. Hence, condition (10) guarantees that $D^{(\mathcal{I})}h(x)$ exists and is equal $D^{(\mathcal{I})}(\Psi(f)\circ h)(x_0)$. But, h is a homeomorphism. Thus, by Lemma 5.6, h must be differentiable at x as well. This ends the proof. ■

As an immediate corollary we obtain

Theorem 5.8. *The semigroups* $\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, $\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$, $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ *and* $\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}$ *have the inner automorphism property. Moreover,*

$$\begin{aligned}\mathrm{Aut}(\Delta \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}) = \mathrm{Aut}(\Delta \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}) = \mathrm{Aut}(\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}) = \mathrm{Aut}(\Delta^{(\mathcal{I})} \cap \mathcal{C}_{\mathcal{D}\mathcal{D}}) \\ \subset \mathrm{Aut}(\mathcal{C}_{\mathcal{I}\mathcal{I}}) = \mathrm{Aut}(\mathcal{C}_{\mathcal{D}\mathcal{D}})\end{aligned}$$

and the inclusion is proper.

We are not able to prove or disprove that the semigroup $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ has the inner automorphism property. We proved that each automorphism of $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ is generated by a homeomorphism h of the real line which must be differentiable almost everywhere. However, to be an element of $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$, h would need to be differentiable outside of a set of first category. In general, a homeomorphism of the real line need not be $\mathcal{I}$-a.e. differentiable. In fact, Belna, Cargo, Evans and Humke [1] show that there exists a strictly increasing homeomorphism $h : [0,1] \to [0,1]$ and $\tau \in (0,1)$ such that

$$\underline{D}^{-}h(x) = \underline{D}^{+}h(x) = \tau \text{ and } \overline{D}^{-}h(x) = \overline{D}^{+}h(x) = \infty$$

for a residual set of points $x \in [0,1]$. On the other hand, there are certain restrictions on the "severity" of the nondifferentiability of h on sets of the second category. Neugebauer [15] shows that a continuous function f has $\underline{D}^{-}f(x) = \overline{D}^{+}f(x)$ and $\overline{D}^{-}f(x) = \overline{D}^{+}f(x)$ on a residual set.

We are not able to find a satisfactory answer to our question. Therefore, the following remains.

Problem 5.9. Let h be a homeomorphism of R such that for every $\mathcal{I}$-density continuous, $\mathcal{I}$-a.e. $\mathcal{I}$-approximately differentiable f, $h \circ f \circ h^{-1}$ is also $\mathcal{I}$-density continuous, and $\mathcal{I}$-a.e. $\mathcal{I}$-approximately differentiable. Is h differentiable on a residual set?

Clearly, an affirmative answer to this problem is equivalent to the fact that $\Delta^{(\mathcal{I})}_{\mathcal{I}} \cap \mathcal{C}_{\mathcal{I}\mathcal{I}}$ has the inner automorphism property.

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