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Various continuities with the density, $\mathcal{I}$-density and ordinary topologies on $\mathbb{R}$

1. Preliminaries

This paper is concerned with four topologies on the real line: the ordinary topology, density topology, $\mathcal{I}$-density topology and deep-$\mathcal{I}$-density topology. Using different topologies on the domain and range, these four topologies determine sixteen different ways a real function can be continuous. These continuities are examined and their relationships are determined. In addition, they are compared with the classes of analytic, C^∞, Baire*1, and Baire 1 functions.

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$. The symmetric difference of the sets A and B is denoted by $A \triangle B$ and the complement of $A \subset \mathbb{R}$ by A^c. The symbols $\mathcal{L}$ and $\mathcal{B}$ stand for the families of subsets of $\mathbb{R}$ which are Lebesgue measurable and have the Baire property, respectively. $\mathcal{N}$ and $\mathcal{I}$ denote the ideals of Lebesgue measure zero and first category subsets of $\mathbb{R}$. If $A \in \mathcal{L}$, then its Lebesgue measure is denoted by $\mathrm{m}(A)$.

It is easy to see that for any set $B \in \mathcal{B}$ there is a unique open set $\tilde{B}$ such that $\tilde{B} = \mathrm{int}\left(\mathrm{cl}\left(\tilde{B}\right)\right)$ and $B = \tilde{B} \triangle I$ for some $I \in \mathcal{I}$. Any open set G for which $G = \mathrm{int}(\mathrm{cl}(G))$ is called a *regular* open set. In a sense, $\tilde{B}$ is the largest open set such that B can be written as $\tilde{B} \triangle I$ for some $I \in \mathcal{I}$.

The symbol $\mathrm{dist}(A, B)$ stands for the Euclidean distance between the sets A and B; i.e., $\mathrm{dist}(A, B) = \inf\{|x-y| : x \in A, y \in B\}$. For $a \in \mathbb{R}$ and $E \subset \mathbb{R}$, define $E - a = \{x - a \in \mathbb{R}: x \in E\}$ and $aE = \{ax \in \mathbb{R}: x \in E\}$.

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The symbols $\mathcal{DB}_1$, $\mathcal{C}$, $\mathcal{C}^\infty$ and $\mathcal{A}$ stand for the classes of Darboux Baire one, ordinary continuous, infinitely many times differentiable and analytic functions from $\mathbb{R}$ to $\mathbb{R}$, respectively. $\mathcal{DB}_1^*$ denotes functions in the Darboux Baire*1 class; i.e., the class of all Darboux functions $f:\mathbb{R} \to \mathbb{R}$ with the property that for every perfect set P there is its nonempty portion $Q = P \cap (a,b)$ such that f restricted to Q is continuous [12].

Any of the sets $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$ or $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a *right interval set of a point* $a \in \mathbb{R}$ if $a < b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$. In the case when $a = 0$ we simply call it a *a right interval set.* Similarly we define a *left interval set* of a point $a \in \mathbb{R}$. We say that E is an *interval set* if it is the union of a right interval set and a left interval set.

2. Topologies

The ordinary topology on $\mathbb{R}$ is denoted by $\mathcal{T}_\mathcal{O}$.

Next, the density topology, $\mathcal{I}$-density topology and deep $\mathcal{I}$-density topology are defined. For this, are needed the notions of a density point, $\mathcal{I}$-density point and deep-$\mathcal{I}$-density point. (See [13, 17]. For a discussion of these ideas read [6].)

Let $A \in \mathcal{L}$. A number x, not necessarily in A, is a *density point* of A if

$$\lim_{h\to 0^+} \frac{\mathrm{m}(A \cap (x-h, x+h))}{2h} = 1. \tag{1}$$

The set of all density points of $A \in \mathcal{L}$ is denoted by $\Phi_\mathcal{N}(A)$. It is a consequence of the Lebesgue density theorem that for a measurable set A almost every point of A is a density point of A. Therefore, whenever A is measurable, then so is $\Phi_\mathcal{N}(A)$ and $\mathrm{m}(A \triangle \Phi_\mathcal{N}(A)) = 0$ [16, p. 107]. It is also easy to see that

$$\Phi_\mathcal{N}(A) = \Phi_\mathcal{N}(B) \quad \text{for every } A, B \in \mathcal{L} \text{ such that } \quad A \triangle B \in \mathcal{N}. \tag{2}$$

The family of sets

$$\mathcal{T}_\mathcal{N} = \{A \in \mathcal{L} : A \subset \Phi_\mathcal{N}(A)\}$$

forms a topology on $\mathbb{R}$ [13, 10] called the *density topology.*

If $A \in \mathcal{B}$, then 0 is an *$\mathcal{I}$-density point* of A if for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p} A \cap (-1,1)} = \chi_{(-1,1)}, \ \mathcal{I}\text{-a.e.} \tag{3}$$

It is worth noticing that condition (3) is equivalent to the fact that the set

$$\liminf_{p\to\infty} n_{m_p} A = \bigcup_{q\in\mathbb{N}} \bigcap_{p\geq q} n_{m_p} A \tag{4}$$

is residual in $(-1,1)$. We say that a point a is an $\mathcal{I}$-density point of $A \in \mathcal{B}$ if 0 is an $\mathcal{I}$-density point of $A - a$. The set of all $\mathcal{I}$-density points of $A \in \mathcal{B}$ is denoted by $\Phi_{\mathcal{I}}(A)$. It is not difficult to see that $A \triangle \Phi_{\mathcal{I}}(A) \in \mathcal{I}$ for every $A \in \mathcal{B}$ [17, Theorem 3] and that

$$\Phi_{\mathcal{I}}(A) = \Phi_{\mathcal{I}}(B) \quad \text{for every } A, B \in \mathcal{B} \text{ such that } \quad A \triangle B \in \mathcal{I}. \tag{5}$$

The family of sets

$$\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} \colon A \subset \Phi_{\mathcal{I}}(A)\}$$

forms a topology on $\mathbb{R}$ [14, 17] called the *$\mathcal{I}$-density topology*.

Finally, a point $a \in \mathbb{R}$ is a *deep-$\mathcal{I}$-density point* [17] of an $A \in \mathcal{B}$ if there exists a closed set $F \subset A \cup \{a\}$ such that a is an $\mathcal{I}$-density point of F. The set of all deep-$\mathcal{I}$-density points of $A \in \mathcal{B}$ is denoted by $\Phi_{\mathcal{D}}(A)$. The family of sets

$$\mathcal{T}_{\mathcal{D}} = \{A \in \mathcal{B} \colon A \subset \Phi_{\mathcal{D}}(A)\}$$

forms a topology on $\mathbb{R}$ called the *deep-$\mathcal{I}$-density topology* [11, 17].

A point x is a *dispersion* (*$\mathcal{I}$-dispersion*, *deep-$\mathcal{I}$-dispersion*) point of A if x is a density ($\mathcal{I}$-density deep-$\mathcal{I}$-density) point of A^c. In particular, 0 is an $\mathcal{I}$-dispersion point of A if for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$(-1,1) \cap \bigcap_{q\in\mathbb{N}} \bigcup_{p\geq q} n_{m_p} B = (-1,1) \cap \limsup_{p\to\infty} (n_{m_p} B) \in \mathcal{I} \tag{6}$$

and x is a deep-$\mathcal{I}$-dispersion point of A if there exists an open $U \supset (A \setminus \{x\})$ such that x is an $\mathcal{I}$-dispersion point of U.

We also need the following fact that easily follows from either of [5, Lemma 2.4], [15, Theorem 1] or [17, Theorem 2].

Proposition 2.1. *If $E = \bigcup_{n\in\mathbb{N}} [a_n, b_n]$ is a right interval set such that*

(i) $\lim_{n\to\infty} (b_n - a_n)/a_n = 0$ *and*

(ii) $\lim_{n\to\infty} b_{n+1}/a_n = 0$,

then 0 *is a deep-$\mathcal{I}$-dispersion point of* E. *In particular,* $E^c \in \mathcal{T}_{\mathcal{D}}$.

The next theorem summarizes the topological properties of the topologies defined above.

Theorem 2.2. *The topologies* $\mathcal{T}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{I}}$ *and* $\mathcal{T}_{\mathcal{D}}$ *on* $\mathbb{R}$ *have the following properties:*

(i) $\mathcal{T}_{\mathcal{O}} \subset \mathcal{T}_{\mathcal{N}}$ *and* $\mathcal{T}_{\mathcal{O}} \subset \mathcal{T}_{\mathcal{D}} \subset \mathcal{T}_{\mathcal{I}}$; *in particular,* $\mathcal{T}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{I}}$ *and* $\mathcal{T}_{\mathcal{D}}$ *are Hausdorff;*

(ii) *a subset* C *of* $\mathbb{R}$ *is closed and nowhere dense with respect to* $\mathcal{T}_{\mathcal{N}}$ *if, and only if,* $C \in \mathcal{N}$; *a subset* C *of* $\mathbb{R}$ *is closed and nowhere dense with respect to* $\mathcal{T}_{\mathcal{I}}$ *if, and only if,* $C \in \mathcal{I}$;

(iii) $\mathcal{T}_{\mathcal{N}}$ *and* $\mathcal{T}_{\mathcal{D}}$ *are completely regular;* $\mathcal{T}_{\mathcal{I}}$ *is not regular;*

(iv) $\mathcal{T}_{\mathcal{N}}$ *and* $\mathcal{T}_{\mathcal{I}}$ *are not separable; on the other hand,* $\mathcal{T}_{\mathcal{D}}$ *is separable; moreover, every set* D *which is dense with respect to* $\mathcal{T}_{\mathcal{O}}$ *is also dense with respect to* $\mathcal{T}_{\mathcal{D}}$;

(v) *every closed subinterval of* $\mathbb{R}$ *is connected in the topologies* $\mathcal{T}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{I}}$ *and* $\mathcal{T}_{\mathcal{D}}$;

(vi) *a set* A *is compact with respect to any of the topologies* $\mathcal{T}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{I}}$ *and* $\mathcal{T}_{\mathcal{D}}$ *if, and only if, it is finite.*

Proof. (i) is obvious from the definitions. (ii) follows immediately from (2) and (5). The first part of (iii) can be found in [19] or [9] for the density topology and in [11] or [17] for the deep-$\mathcal{I}$-density topology. The second part follows from the fact that any set D which is dense with respect to $\mathcal{T}_{\mathcal{O}}$ cannot be separated in $\mathcal{T}_{\mathcal{I}}$ from any point $x \in \mathbb{R} \setminus D$ [14, Theorem 5]. The above, together with the regularity of $\mathcal{T}_{\mathcal{D}}$, also implies the second part of (iv), while its first part follows immediately from (ii). A proof of (v) can be found in [10] in the case of the density topology and in [14, Corollary 2] for the $\mathcal{I}$-density topology. The deep-$\mathcal{I}$-density topology case follows from that of $\mathcal{I}$-density topology. To argue condition (vi) we need to show that an arbitrary infinite set is not compact with respect to any of $\mathcal{T}_{\mathcal{N}}$, $\mathcal{T}_{\mathcal{I}}$ or $\mathcal{T}_{\mathcal{D}}$. So, let A be an infinite set and let us choose a set $S = \{c_n : n \in \mathbb{N}\} \subset A$ such that sequence $\{c_n\}_{n\in\mathbb{N}}$ is monotone. By (ii), S is closed with respect to $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{I}}$. It is also closed in $\mathcal{T}_{\mathcal{D}}$ if $\{c_n\}_{n\in\mathbb{N}}$ is unbounded. If $\{c_n\}_{n\in\mathbb{N}}$ is bounded, say by c, and if we additionally assume that

$$\lim_{n\to\infty} \frac{c_{n+1} - c}{c_n - c} = 0,$$

then, using Proposition 2.1, we can also show that S is closed in $\mathcal{T}_{\mathcal{D}}$. Now, the family of sets $W_m = \mathbb{R} \setminus \{c_n : n \geq m\} \in \mathcal{T}_{\mathcal{N}} \cap \mathcal{T}_{\mathcal{D}}$ forms an infinite open cover of A without a finite subcover.

In the sequel the following two lemmas are needed.

Lemma 2.3. *There exists a decreasing sequence $S = \{s_n\}_{n\in\mathbb{N}}$ converging to 0 such that 0 is not an $\mathcal{I}$-dispersion point of any set $B \in \mathcal{T}_{\mathcal{I}}$ with the property that $B \cap [s_{n+1}, s_n] \neq \emptyset$ for every $n \in \mathbb{N}$. In particular, if $B \supset S$ then $0 \notin \Phi_{\mathcal{I}}(B^c)$.*

Proof. Let

$$D_n = \{\frac{i}{2^n} : i = 1, 2, \ldots, 2^n\} \subset (0, 1]$$

and let

$$S = \bigcup_{n\in\mathbb{N}} \frac{1}{n} D_n.$$

Notice that $S \subset (0,1]$ and that $S \setminus (0, \frac{1}{n}]$ is finite for every $n \in \mathbb{N}$. Thus, S can be enumerated as a decreasing sequence converging to 0.

Now, let B satisfy our assumptions. Then, $\tilde{B}$ also satisfies them and we may assume that $B = \tilde{B}$. Moreover,

$$nB \cap \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \neq \emptyset$$

for every $i, n \in \mathbb{N}$, $i < 2^n$. In particular, for every increasing sequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ of positive integers, the set $\bigcup_{p\in\mathbb{N}} n_{m_p} B$ is dense and open. Hence, $\limsup_{p\in\mathbb{N}} n_{m_p} B$ contains a dense $\mathbf{G}_\delta$ subset of $(0, 1)$. Thus, by (6), 0 cannot be an $\mathcal{I}$-dispersion point of B.

Lemma 2.4. *Let $P \subset (0,1]$ be closed and nowhere dense and let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} b_{n+1}/b_n = 0$. Then 0 is a deep-$\mathcal{I}$-dispersion point of the set*

$$Q = \bigcup_{n\in\mathbb{N}} b_n P.$$

In particular, $Q^c \in \mathcal{T}_{\mathcal{D}}$.

Proof. According to Zajíček [20, Theorem (5)], it is enough to prove that for every $c \in (0,1)$ there exist $\varepsilon > 0$ and $\delta > 0$ such for any $x \in (0,\delta)$ there exists a closed interval $I \subset Q^c \cap (cx, x)$ such that $\mathrm{m}(I) \geq x\varepsilon$.

Let $c \in (0,1)$, $p = \min P$ and let $\delta > 0$ be such that

$$\frac{b_{n+1}}{b_n} < pc \text{ for every } n \in \mathbb{N} \text{ for which } pb_n \leq \delta. \tag{7}$$

Moreover, let $\varepsilon_0 > 0$ be a number such that for every interval $K \subset (0,1)$ of length $\geq p(1-c)/2$ there exists a closed interval $J \subset K \setminus P$ of length $\geq \varepsilon_0$. Such a number can be found, by partitioning $(0,1)$ into intervals $J_1, \ldots, J_k$, of length $< p(1-c)/4$ and defining

$$\varepsilon_0 < \min\{\sup\{b - a : [a,b] \subset J_i \setminus P\} : 1 \leq i \leq k\}.$$

Put $\varepsilon = \min\{\varepsilon_0/2, (1-c)/2\}$.

Now, let $x \in (0,\delta)$ and define

$$m = \min\{n : pb_n \leq x\}.$$

Then, by (7), $b_{m+1} < pb_m c \leq xc$. In particular,

$$Q^c \cap (cx, x) = (cx, x) \setminus b_m P.$$

Let a be the middle point of (cx, x). If $pb_m > a$ then $I = [xc, a]$ works, as $\mathrm{m}(I) = x(1-c)/2 \geq x\varepsilon$. Similarly, if $b_m < a$ then it suffices to set $I = [a, x]$.

So, let us assume that $pb_m \leq a \leq b_m$. Then $x \leq 2a \leq 2b_m$ and $pb_m < x = \frac{2}{1-c}(x-a)$, as $2(x-a) = (x - xc)$. In particular, $(x-a)/b_m > p(1-c)/2$. Thus, by the definition of ε_0, there exists a closed interval $J \subset \frac{1}{b_m}(a,x) \setminus P$ of the length $\geq \varepsilon_0$. Hence,

$$I = b_m J \subset (a,x) \setminus b_m P \subset (cx, x) \setminus b_m P = Q^c \cap (cx, x)$$

has length $\geq \varepsilon_0 b_m = 2\varepsilon_0 b_m/2 \geq x\varepsilon$. This finishes the proof.

This section is concluded with the fact that Theorem 2.2(i) lists all possible inclusions between the topologies $\mathcal{T}_\mathcal{O}$, $\mathcal{T}_\mathcal{N}$, $\mathcal{T}_\mathcal{I}$ and $\mathcal{T}_\mathcal{D}$.

Theorem 2.5. *If $\mathcal{P}(\mathbb{R})$ stands for the discrete topology on $\mathbb{R}$ then*

$$\begin{array}{ccccccc}
\mathcal{T}_\mathcal{O}\cap\mathcal{T}_\mathcal{N} & \subset & \mathcal{T}_\mathcal{D}\cap\mathcal{T}_\mathcal{N} & \subset & \mathcal{T}_\mathcal{I}\cap\mathcal{T}_\mathcal{N} & \subset & \mathcal{T}_\mathcal{N} \\
\| & & \cap & & \cap & & \cap \\
\mathcal{T}_\mathcal{O} & \subset & \mathcal{T}_\mathcal{D} & \subset & \mathcal{T}_\mathcal{I} & \subset & \mathcal{P}(\mathbb{R})
\end{array}$$

Moreover, all the inclusions are proper.

Proof. All the inclusions follow immediately from Theorem 2.2(i).

To show that the horizontal inclusions are proper, it is enough to argue for the inclusions in the first row. Thus, it is enough to prove that $\mathcal{T}_\mathcal{D}\cap\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{O}$, $\mathcal{T}_\mathcal{I}\cap\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{D}$ and $\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{I}$. To show that the vertical inclusions are proper we will show that $\mathcal{T}_\mathcal{D} \not\subset \mathcal{T}_\mathcal{N}$.

$\mathcal{T}_\mathcal{D}\cap\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{O}$. Let $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be as in Proposition 2.1. It is easy to check that 0 is also a dispersion point of E. So, $E^c \in \mathcal{T}_\mathcal{D}\cap\mathcal{T}_\mathcal{N}$ while, evidently, $E^c \notin \mathcal{T}_\mathcal{O}$.

$\mathcal{T}_\mathcal{I}\cap\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{D}$. Let $E = \mathbb{R}\setminus\mathbb{Q}$. By Theorem 2.2(ii) $\mathbb{Q}$ is closed in $\mathcal{T}_\mathcal{I}$ and $\mathcal{T}_\mathcal{N}$ and so $E \in \mathcal{T}_\mathcal{I}\cap\mathcal{T}_\mathcal{N}$. But, $\mathbb{Q}$ is also dense in $\mathcal{T}_\mathcal{D}$. Hence, $E \notin \mathcal{T}_\mathcal{D}$.

$\mathcal{T}_\mathcal{N} \not\subset \mathcal{T}_\mathcal{I}$. Let C be a nowhere dense set of positive Lebesgue measure. By the Lebesgue density theorem, $D = C \cap \Phi_\mathcal{N}(C) \in \mathcal{T}_\mathcal{N}$ and $D \neq \emptyset$. Moreover, by Theorem 2.2(ii), D is nowhere dense in $\mathcal{T}_\mathcal{I}$, so $D \notin \mathcal{T}_\mathcal{I}$.

$\mathcal{T}_\mathcal{D} \not\subset \mathcal{T}_\mathcal{N}$. Let $C \subset [\frac{1}{2}, 1]$ be a closed nowhere dense set with positive Lebesgue measure and let $\{b_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} b_{n+1}/b_n = 0$. Then, by Lemma 2.4, 0 is a deep-$\mathcal{I}$-dispersion point of $E = \bigcup_{n\in\mathbb{N}} b_n C$ and $E^c \in \mathcal{T}_\mathcal{D}$. On the other hand, for every $n \in \mathbb{N}$,

$$\frac{m\left(E\cap(0,b_n)\right)}{b_n} = m\left(b_n^{-1}E\cap(0,1)\right) \geq m(C) > 0.$$

Thus, 0 is not a dispersion point of E and $E^c \notin \mathcal{T}_\mathcal{N}$.

3. Classes of functions

For $\mathcal{J},\mathcal{K} \in \{\mathcal{N},\mathcal{I},\mathcal{D},\mathcal{O}\}$ let $C_{\mathcal{J}\mathcal{K}}$ stand for the family of all continuous functions $f\colon(\mathbb{R},\mathcal{T}_{\mathcal{J}}) \to (\mathbb{R},\mathcal{T}_{\mathcal{K}})$. It is easy to see that we have sixteen different classes:

$$\begin{array}{cccc} C_{\mathcal{O}\mathcal{O}} & C_{\mathcal{D}\mathcal{O}} & C_{\mathcal{I}\mathcal{O}} & C_{\mathcal{N}\mathcal{O}} \\ C_{\mathcal{O}\mathcal{D}} & C_{\mathcal{D}\mathcal{D}} & C_{\mathcal{I}\mathcal{D}} & C_{\mathcal{N}\mathcal{D}} \\ C_{\mathcal{O}\mathcal{I}} & C_{\mathcal{D}\mathcal{I}} & C_{\mathcal{I}\mathcal{I}} & C_{\mathcal{N}\mathcal{I}} \\ C_{\mathcal{O}\mathcal{N}} & C_{\mathcal{D}\mathcal{N}} & C_{\mathcal{I}\mathcal{N}} & C_{\mathcal{N}\mathcal{N}}. \end{array}$$

Some of these classes have the following names associated with them:

$C_{\mathcal{O}\mathcal{O}} = \mathcal{C}$ – the class of (ordinary) continuous functions;

$C_{\mathcal{N}\mathcal{N}}$ – the class of density continuous functions;

$C_{\mathcal{I}\mathcal{I}}$ – the class of $\mathcal{I}$-density continuous functions;

$C_{\mathcal{D}\mathcal{D}}$ – the class of deep-$\mathcal{I}$-density continuous functions;

$C_{\mathcal{N}\mathcal{O}}$ – the class of approximately continuous functions;

$C_{\mathcal{I}\mathcal{O}}$ – the class of $\mathcal{I}$-approximately continuous functions.

We start our investigation by showing that the other classes coincide with them or with the class of all constant functions.

Theorem 3.1. *If Const stands for the class of all constant functions then*

$$C_{\mathcal{O}\mathcal{D}} = C_{\mathcal{O}\mathcal{I}} = C_{\mathcal{O}\mathcal{N}} = C_{\mathcal{D}\mathcal{I}} = C_{\mathcal{D}\mathcal{N}} = C_{\mathcal{I}\mathcal{N}} = C_{\mathcal{N}\mathcal{D}} = C_{\mathcal{N}\mathcal{I}} = Const.$$

Proof. It is obvious that the constant functions are members of all these classes, as they are universally continuous. Thus, we need only show that the functions from the classes listed above $C_{\mathcal{J}\mathcal{K}}$ are constant.

By way of contradiction, let us assume that there is a nonconstant function f in some class $C_{\mathcal{J}\mathcal{K}}$ from the theorem.

Case 1°. $C_{\mathcal{J}\mathcal{K}} \in \{C_{\mathcal{O}\mathcal{D}}, C_{\mathcal{O}\mathcal{I}}, C_{\mathcal{O}\mathcal{N}}\}$. By our assumption there are $a < b$ such that $f([a,b])$ has more than one point. But, by Theorem 2.2(vi), $f([a,b])$ is finite, because it is the continuous image of compact set. This means that $f([a,b])$ is disconnected, which is impossible, because the interval $[a,b]$ is connected in $\mathcal{T}_{\mathcal{O}}$.

Case 2°. $C_{\mathcal{JK}} \in \{C_{\mathcal{DI}}, C_{\mathcal{DN}}\}$. Let D be a countable dense subset of $(\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ such that $A = f(D)$ has more than one point. By Theorem 2.2(ii), A is closed and discrete in $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{N}}$. We will show that $B = f^{-1}(A)$ is not closed in $\mathcal{T}_{\mathcal{D}}$. First notice that A, as a discrete set having more than one element, is disconnected and thus, B is disconnected as the continuous preimage of disconnected set. So, by Theorem 2.2(v), $B \neq \mathbb{R}$. But, $B \supset D$ is dense in $\mathcal{T}_{\mathcal{D}}$, by Theorem 2.2(iv). Hence, B is not closed in $\mathcal{T}_{\mathcal{D}}$.

Case 3°. $C_{\mathcal{JK}} = C_{\mathcal{IN}}$. As in the previous case, let D be a countable dense subset of $(\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ such that $A = f(D)$ has more than one point and let $a \notin B = f^{-1}(A)$. Without any loss of generality we may assume that $a = 0 = f(a)$. Let $S = \{s_n\}_{n\in\mathbb{N}}$ be as in Lemma 2.3 and, for every $n \in \mathbb{N}$, let $d_n \in B \cap [s_{n+1}, s_n]$. Now, choose $a_n < f(d_n) < b_n$ such that 0 is a dispersion point of $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Let $U \in \mathcal{T}_{\mathcal{N}}$ be such that $0 \in U \subset E^c$. But then, by Lemma 2.3, 0 is not an $\mathcal{I}$-dispersion point of $f^{-1}(E) \subset f^{-1}(U^c)$; i.e., $f^{-1}(U) \notin \mathcal{T}_{\mathcal{I}}$.

Case 4°. $C_{\mathcal{JK}} \in \{C_{\mathcal{ND}}, C_{\mathcal{NI}}\}$. As $C_{\mathcal{NI}} \subset C_{\mathcal{ND}}$ we may assume that $C_{\mathcal{JK}} = C_{\mathcal{ND}}$. This case can be found in [3].

Theorem 3.1 shows that the classes listed above are trivial. Thus, we need no longer consider them. Similarly, we can ignore the class $C_{\mathcal{DO}}$ because of the theorem below [11, Theorem 2].

Theorem 3.2. $C_{\mathcal{IO}} = C_{\mathcal{DO}}$.

To argue the next theorem we use the following easy lemma. (See [18, Theorems 14.12 and 8.10].)

Lemma 3.3. *Let X be a completely regular topological space and Y any topological space. The following are equivalent:*

(i) *the function $f: Y \to X$ is continuous;*

(ii) *the function $g \circ f: Y \to [0,1]$ is continuous for every continuous $g: X \to [0,1]$.*

Theorem 3.4. $C_{\mathcal{DD}} = C_{\mathcal{ID}}$.

Proof. Let $f \in \mathcal{C}_{\mathcal{ID}}$. We have to prove that f is continuous as a function from $(\mathbb{R}, \mathcal{T}_{\mathcal{D}})$ to $(\mathbb{R}, \mathcal{T}_{\mathcal{D}})$. But, evidently, $g \circ f \in \mathcal{C}_{\mathcal{IO}} = \mathcal{C}_{\mathcal{DO}}$ for every $g \in \mathcal{C}_{\mathcal{DO}}$. Hence, by Lemma 3.3, $f \in \mathcal{C}_{\mathcal{DD}}$.

By the previous theorem we need no longer consider the class $\mathcal{C}_{\mathcal{ID}}$ in our discussion. The other classes will be discussed in the following sections.

4. Main Theorem

We start this section with the following list of inclusions.

Theorem 4.1. *The following relations hold.*

(i) $\mathcal{A} \subset \mathcal{C}_{\mathcal{NN}} \subset \mathcal{C}_{\mathcal{NO}} \subset \mathcal{DB}_1$;

(ii) $\mathcal{C}_{\mathcal{NN}} \subset \mathcal{DB}_1^*$;

(iii) $\mathcal{C} \subset \mathcal{C}_{\mathcal{NO}}$;

(iv) $\mathcal{A} \subset \mathcal{C}_{\mathcal{II}} \subset \mathcal{C}_{\mathcal{DD}} \subset \mathcal{C}_{\mathcal{IO}} \subset \mathcal{DB}_1$;

(v) $\mathcal{C}_{\mathcal{DD}} \subset \mathcal{DB}_1^*$;

(vi) $\mathcal{C} \subset \mathcal{C}_{\mathcal{IO}}$;

(vii) $\mathcal{A} \subset \mathcal{C}^\infty \subset \mathcal{C} \subset \mathcal{DB}_1^* \subset \mathcal{DB}_1$.

Proof. (i). The inclusions $\mathcal{A} \subset \mathcal{C}_{\mathcal{NN}}$ and $\mathcal{C}_{\mathcal{NO}} \subset \mathcal{DB}_1$ can be found in [7, Corollary 3] and [2, Theorem 5.5(b)], respectively. $\mathcal{C}_{\mathcal{NN}} \subset \mathcal{C}_{\mathcal{NO}}$ follows from Theorem 2.2(i).

(ii) can be found in [8, Theorem 3].

(iii) follows immediately from Theorem 2.2(i).

(iv). The inclusions $\mathcal{A} \subset \mathcal{C}_{\mathcal{II}}$ and $\mathcal{C}_{\mathcal{IO}} \subset \mathcal{DB}_1$ can be found in [4] and [5, Theorem 3.2], [14, Theorem 8], respectively. $\mathcal{C}_{\mathcal{DD}} \subset \mathcal{C}_{\mathcal{IO}}$ follows from Theorems 2.2(i) and 3.4. To argue the inclusion $\mathcal{C}_{\mathcal{II}} \subset \mathcal{C}_{\mathcal{DD}}$ let $f \in \mathcal{C}_{\mathcal{II}}$. Then, evidently, $g \circ f \in \mathcal{C}_{\mathcal{IO}} = \mathcal{C}_{\mathcal{DO}}$ for every $g \in \mathcal{C}_{\mathcal{DO}} = \mathcal{C}_{\mathcal{IO}}$. Hence, by Lemma 3.3, $f \in \mathcal{C}_{\mathcal{DD}}$.

(v) can be found in [5, Theorem 4.2].

(vi) follows immediately from Theorem 2.2(i).

(vii) is well known.

The inclusions between the discussed classes are summarized in the following charts. We use the convention that, for the classes of functions $\mathcal{X}$ and $\mathcal{Y}$, the symbol $\mathcal{X}\,\mathcal{Y}$ stands for the class $\mathcal{X} \cap \mathcal{Y}$.

Theorem 4.2. *The following inclusions hold*

$$
\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{IO}} & \subset & \mathcal{DB}_1 \\
 & & & & \cup & & \cup \\
\mathcal{C}_{\mathcal{II}} & \subset & \mathcal{C}_{\mathcal{DD}} & \subset & \mathcal{C}_{\mathcal{IO}}\,\mathcal{DB}_1^* & \subset & \mathcal{DB}_1^* \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C} & \subset & \mathcal{C} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}^{\infty} & \subset & \mathcal{C}^{\infty} & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}
$$

and the relativization of the above chart to (i.e., intersecting each of the element of the chart by) $\mathcal{C}_{\mathcal{NO}}$ *and* $\mathcal{C}_{\mathcal{NN}}$ *gives respectively*

$$
\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{IO}}\,\mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{C}_{\mathcal{NO}} \\
 & & & & \cup & & \cup \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{C}_{\mathcal{IO}}\,\mathcal{DB}_1^*\,\mathcal{C}_{\mathcal{NO}} & \subset & \mathcal{DB}_1^*\,\mathcal{C}_{\mathcal{NO}} \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C} & \subset & \mathcal{C} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}^{\infty} & \subset & \mathcal{C}^{\infty} & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}
$$

and

$$
\begin{array}{ccccccc}
 & & & & \mathcal{C}_{\mathcal{IO}}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{NN}} \\
 & & & & \parallel & & \parallel \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{IO}}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{NN}} \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}\,\mathcal{C}_{\mathcal{NN}} & & \\
\cup & & \cup & & \cup & & \\
\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} & \subset & \mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} & & \\
\cup & & & & & & \\
\mathcal{A} & & & & & &
\end{array}
$$

Moreover, the inclusions in the charts and the nontrivial inclusions between corresponding parts of the charts, i.e., $\mathcal{X}\,\mathcal{C}_{\mathcal{NO}} \subset \mathcal{X}$ *and* $\mathcal{Y}\,\mathcal{C}_{\mathcal{NN}} \subset \mathcal{Y}\,\mathcal{C}_{\mathcal{NO}}$ *for*

$$\mathcal{X} \in \{\mathcal{C}_{\mathcal{II}}, \mathcal{C}_{\mathcal{DD}}, \mathcal{C}_{\mathcal{IO}}\,\mathcal{DB}_1^*, \mathcal{DB}_1^*, \mathcal{C}_{\mathcal{IO}}, \mathcal{DB}_1\}$$

and

$$\mathcal{Y} \in \{\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty}, \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}^{\infty}, \mathcal{C}^{\infty}, \mathcal{C}_{\mathcal{II}}\,\mathcal{C}, \mathcal{C}_{\mathcal{DD}}\,\mathcal{C}, \mathcal{C}, \mathcal{C}_{\mathcal{II}}, \mathcal{C}_{\mathcal{DD}}, \mathcal{C}_{\mathcal{IO}}\,\mathcal{DB}_1^*, \mathcal{DB}_1^*, \},$$

are proper.

Proof. The fact that the second and third charts are appropriate relativizations follows from Theorem 4.1. Theorem 4.1 also implies all the inclusions.

To prove that the inclusions in the main charts are proper it is enough to reduce our task to the following inclusions. To argue for the inclusions between first and second rows it is enough to show that $\mathcal{C}_{\mathcal{IO}}\,\mathcal{C}_{\mathcal{NO}} \not\subset \mathcal{DB}_1^*$. For the other inclusions it is enough to consider the inclusions in the third chart. The vertical containments are proper because of $\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{A}$, $\mathcal{C}_{\mathcal{II}}\,\mathcal{C}\,\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}^{\infty}$ and $\mathcal{C}_{\mathcal{II}}\,\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}$. To prove that the horizontal inclusions are proper it is enough to show that $\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}_{\mathcal{IO}}$, $\mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}_{\mathcal{DD}}$ and $\mathcal{C}_{\mathcal{DD}}\,\mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}_{\mathcal{II}}$. To argue the additional part it is enough to show that $\mathcal{C}_{\mathcal{II}} \not\subset \mathcal{C}_{\mathcal{NO}}$ and $\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty} \not\subset \mathcal{C}_{\mathcal{NN}}$.

$\mathcal{C}_{\mathcal{IO}}\,\mathcal{C}_{\mathcal{NO}} \not\subset \mathcal{DB}_1^*$. See Example 5.11.

$\mathcal{C}_{\mathcal{II}}\,\mathcal{C}^{\infty}\,\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{A}$. See Example 5.9.

$\mathcal{C}_{\mathcal{II}}\ \mathcal{C}\ \mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}^{\infty}$. See Example 5.2.
$\mathcal{C}_{\mathcal{II}}\ \mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}$. See Example 5.3.
$\mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}_{\mathcal{IO}}$. See Example 5.4.
$\mathcal{C}^{\infty}\ \mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}_{\mathcal{DD}}$. See Example 5.7.
$\mathcal{C}_{\mathcal{DD}}\ \mathcal{C}^{\infty}\ \mathcal{C}_{\mathcal{NN}} \not\subset \mathcal{C}_{\mathcal{II}}$. See Example 6.7.
$\mathcal{C}_{\mathcal{II}} \not\subset \mathcal{C}_{\mathcal{NO}}$. See Example 5.5.
$\mathcal{C}_{\mathcal{II}}\ \mathcal{C}^{\infty} \not\subset \mathcal{C}_{\mathcal{NN}}$. See Example 6.8.
The examples needed for this proof are constructed in the next sections.

5. Easy examples

It is easy to see that linear functions are both density and $\mathcal{I}$-density continuous. (This also follows immediately from [7, Lemma 1] and Theorem 5.8.) In particular, we can notice that

Proposition 5.1. *If f is piecewise linear, then $f \in \mathcal{C}_{\mathcal{NN}}\ \mathcal{C}_{\mathcal{II}}$.*

This gives us immediately

Example 5.2. *There is an $f \in \mathcal{C}_{\mathcal{II}}\ \mathcal{C}\ \mathcal{C}_{\mathcal{NN}} \setminus \mathcal{C}^{\infty}$.*

Proof. The function $f(x) = 2x + |x|$ satisfies the conditions.

It is also easy to construct the next example.

Example 5.3. *There is an $f \in \mathcal{C}_{\mathcal{II}}\ \mathcal{C}_{\mathcal{NN}} \setminus \mathcal{C}$.*

Proof. Let $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be a right interval set as in Proposition 2.1. Then, 0 is an $\mathcal{I}$-dispersion point of E. It is also easy to see that 0 is a dispersion point of E. In particular, $E^c \in \mathcal{T}_{\mathcal{I}} \cap \mathcal{T}_{\mathcal{N}}$. Define $f\colon \mathbb{R} \to [0, 1/2]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin E \\ \frac{\operatorname{dist}(\{x\}, (a_n, b_n)^c)}{b_n - a_n} & \text{for } x \in [a_n, b_n]. \end{cases} \tag{8}$$

By Proposition 5.1, f is density and $\mathcal{I}$-density continuous at every point $x \neq 0$. Also, f is density and $\mathcal{I}$-density continuous at 0 because it is constant on $E^c \in \mathcal{T}_{\mathcal{I}} \cap \mathcal{T}_{\mathcal{N}}$, while $0 \in E^c$. Finally, f is not continuous because the value of f equals $\frac{1}{2}$ at the center of every (a_n, b_n), while $f(0) = 0$.

The next two examples are done in a manner very similar to the previous ones.

Example 5.4. *There is an $f \in \mathcal{C}_{\mathcal{N}\mathcal{N}} \setminus \mathcal{C}_{\mathcal{I}\mathcal{O}}$.*

Proof. Let $S = \{s_n\}_{n\in\mathbb{N}}$ be as in Lemma 2.3 and let $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be a right interval set such that s_n is the center of each interval $[a_n, b_n]$ and 0 is a dispersion point of E. (For example, taking $b_n - a_n = s_n/(n!)$.) Define f as in (8). Then, as in Example 5.3, f is density continuous. But f is not $\mathcal{I}$-approximately continuous at 0, as $f(0) = 0$ and 0 is not an $\mathcal{I}$-dispersion point of the open set $B = f^{-1}((1/4, \infty)) \subset f^{-1}((-\infty, 1/4))^c$ containing the set S.

Example 5.5. *There is an $f \in \mathcal{C}_{\mathcal{I}\mathcal{I}} \setminus \mathcal{C}_{\mathcal{N}\mathcal{O}}$.*

Proof. Let $C \subset [\frac{1}{2}, 1]$ be a closed nowhere dense set with positive Lebesgue measure and let $\{d_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} d_{n+1}/d_n = 0$. Then, by Lemma 2.4, 0 is a deep-$\mathcal{I}$-dispersion point of $D = \bigcup_{n\in\mathbb{N}} d_n C$. Thus, there exists an open set $U \supset D$ such that 0 is an $\mathcal{I}$-dispersion point of U. Decreasing U, if necessary, we may assume that, for some sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers, $U = \bigcup_{n\in\mathbb{N}}(a_n + \varepsilon_n, b_n - \varepsilon_n)$ and 0 is a deep-$\mathcal{I}$-dispersion point of a right interval set $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$. Then, $E^c \in \mathcal{T}_\mathcal{I}$ and, as in Theorem 2.5($\mathcal{T}_\mathcal{D} \not\subset \mathcal{T}_\mathcal{N}$), 0 is not a dispersion point of U. Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin E \\ \dfrac{\operatorname{dist}(\{x\}, (a_n, b_n)^c)}{\varepsilon_n} & \text{for } x \in (a_n, b_n). \end{cases}$$

Then, as in Example 5.3, f is $\mathcal{I}$-density continuous. But f is not approximately continuous at 0, as $f(0) = 0$ and 0 is not a dispersion point of the set $f^{-1}([1, \infty)) \supset U$.

For the next example we use the following theorem [4].

Theorem 5.6. *Let $f \in C^\infty$ be such that for every $n \geq 0$*

$$f^{(n)}(0) = 0 \quad \text{and} \quad f^{(n)}((0, \varepsilon_n)) \subset (0, \infty) \quad \text{for some} \quad \varepsilon_n > 0.$$

Then f is not deep-$\mathcal{I}$-density continuous.

Example 5.7. *There is an increasing homeomorphism $f \in \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \setminus \mathcal{C}_{\mathcal{I}\mathcal{I}}$.*

Proof. Define

$$f(x) = \begin{cases} e^{-x^{-2}} & x > 0 \\ 0 & x = 0 \\ -e^{-x^{-2}} & x < 0 \end{cases} \tag{9}$$

It is known that $f \in \mathcal{C}^\infty$ and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. It is also easy to see that, for every $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that $f^{(n)}(x) > 0$ for every $x \in (0, \varepsilon_n)$. Thus, by Theorem 5.6, $f \notin \mathcal{C}_{\mathcal{I}\mathcal{I}}$. It is also known [7, Theorem 1] that functions convex on open intervals are density continuous. This easily implies that f is unilaterally density continuous at every point and so $f \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$.

Aversa and Wilczyński [1] study the homeomorphisms which preserve $\mathcal{I}$-density points. In our terminology, the homeomorphism h preserves $\mathcal{I}$-density points when $h^{-1} \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$. In particular, they proved the following theorem [1, Corollary 1], which also follows easily from Theorem 6.2.

Theorem 5.8. *If h is a homeomorphism such that h and h^{-1} fulfill a local Lipschitz condition, then h and h^{-1} are $\mathcal{I}$-density continuous.*

Now we are ready to present the next example.

Example 5.9. *There is an increasing homeomorphism $h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}} \setminus \mathcal{A}$.*

Proof. Let $h(x) = x + f(x)$, where f is from (9). The function f is $\mathcal{C}^\infty$ with a pole, so $f \in \mathcal{C}^\infty \setminus \mathcal{A}$ and also $h \in \mathcal{C}^\infty \setminus \mathcal{A}$. But, evidently, h is a homeomorphism such that h and h^{-1} fulfill a local Lipschitz condition. So, by Theorem 5.8, $h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and, by its density analog [7, Lemma 1], $h \in \mathcal{C}_{\mathcal{N}\mathcal{N}}$.

For the last example of this section we need the following obvious fact.

Proposition 5.10. *If $\{g_n\}$ is a sequence of $\mathcal{I}$-approximately (approximately) continuous functions converging uniformly to g then the function g is $\mathcal{I}$-approximately (approximately) continuous.*

Proof. This is well known for approximately continuous functions [2, Theorem 5.7]. The same easy proof also works for the $\mathcal{I}$-approximately continuous case.

Example 5.11. *There is a* $g \in \mathcal{C}_{\mathcal{IO}}\ \mathcal{C}_{\mathcal{NO}} \setminus \mathcal{DB}_1^*$.

Proof. Let $\{q_n : n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Let f be as in (8) and put

$$h_n(x) = \frac{2}{3^n} f(x - q_n).$$

Then it is easy to see that $h_n(\mathbb{R}) = [0, 1/3^n]$ and that h_n is piecewise linear at points $\neq q_n$ while $\omega(h_n, q_n) = 1/3^n$. Define $g_n = \Sigma_{i \leq n} h_i$ and let g be the limit of $\{g_n\}$. Notice that

$$|g - g_n| \leq \frac{1}{2}\frac{1}{3^n}$$

so, $\{g_n\}$ converges to g uniformly. The same argument as in Example 5.3 shows that the functions $g_n \in \mathcal{C}_{\mathcal{IO}}\ \mathcal{C}_{\mathcal{NO}}$. Thus, by Proposition 5.10, $g \in \mathcal{C}_{\mathcal{IO}}\ \mathcal{C}_{\mathcal{NO}}$. Also notice that, for every $n \in \mathbb{N}$,

$$\omega(h, q_n) = \omega(h_n + (g - g_n), q_n) \geq \omega(h_n, q_n) - \sup_{x \in \mathbb{R}}(g(x) - g_n(x)) \geq \frac{1}{3^n} - \frac{1}{3^{n+1}} > 0$$

so, g is discontinuous at every rational number. But, Baire*1 functions are continuous on a dense open set. Thus, $g \notin \mathcal{DB}_1^*$.

6. $\mathcal{C}^\infty$ deep-$\mathcal{I}$-density continuous examples

We start this section with the following technical lemma. In what follows $\mathcal{I}_0$ stands for the ideal of nowhere dense subsets of $\mathbb{R}$.

Lemma 6.1. *Let* $B \in \mathcal{B}$. *The condition that* 0 *is an* $\mathcal{I}$*-dispersion point of* B *is equivalent to any of the following:*

(i) *for every increasing sequence* $\{n_k\}_{k \in \mathbb{N}}$ *of natural numbers there exists a subsequence* $\{n_{k_p}\}_{p \in \mathbb{N}}$ *such that*

$$\limsup_{p \to \infty} \left(n_{k_p} \tilde{B}\right) \cap (-1, 1) \in \mathcal{I};$$

(ii) *for every increasing sequence* $\{n_k\}_{k \in \mathbb{N}}$ *of natural numbers there exists a subsequence* $\{n_{k_p}\}_{p \in \mathbb{N}}$ *such that*

$$\limsup_{p \to \infty} \left(n_{k_p} \tilde{B}\right) \cap (-1, 1) \in \mathcal{I}_0;$$

(iii) *for every increasing sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of natural numbers and every nonempty interval* $(a,b)\subset(-1,1)$ *there exists a nonempty subinterval* $(c,d)\subset(a,b)$ *and a subsequence* $\{n_{k_p}\}_{p\in\mathbb{N}}$ *such that for every* $p\in\mathbb{N}$

$$(c,d)\cap n_{k_p}\tilde{B}=\emptyset;$$

(iv) *for every sequence* $\{t_n\}_{n\in\mathbb{N}}$ *of positive numbers diverging to infinity and every nonempty interval* $(a,b)\subset(-1,1)$ *there exists a nonempty subinterval* $(c,d)\subset(a,b)$ *and a subsequence* $\{t_{n_k}\}_{k\in\mathbb{N}}$ *such that for every* $k\in\mathbb{N}$

$$(c,d)\cap t_{n_k}\tilde{B}=\emptyset;$$

(v) *for every sequence* $\{t_n\}_{n\in\mathbb{N}}$ *of positive numbers diverging to infinity there exists a subsequence* $\{t_{n_k}\}_{k\in\mathbb{N}}$ *such that for every nonempty interval* $(a,b)\subset(-1,1)$ *there exists a nonempty subinterval* $(c,d)\subset(a,b)$ *and* $k_0\in\mathbb{N}$ *such that for all* $k\geq k_0$

$$(c,d)\cap t_{n_k}\tilde{B}=\emptyset.$$

Proof. By (5) and (6) condition (i) is equivalent to the fact that 0 is an $\mathcal{I}$-dispersion point of B. To finish the proof, it will be shown that (i) through (iv) each imply the next item and that (v) implies (i).

(i)$\Rightarrow$(ii). Note that $\limsup_{p\to\infty}(n_{k_p}\tilde{B})\cap(-1,1)$ is a G_δ set, and a G_δ set belongs to $\mathcal{I}$ if, and only if, it belongs to $\mathcal{I}_0$.

(ii)$\Rightarrow$(iii). The sets

$$\{\bigcup_{p\geq m} n_{k_p}\tilde{B} : m\in\mathbb{N}\} \tag{10}$$

are open and their intersection is nowhere dense. If each of these open sets were dense in (a,b), then their intersection would be a dense G_δ subset of (a,b). Since this contradicts (ii), there must exist an $m_0\in\mathbb{N}$ and $(c,d)\subset(a,b)$ such that $(c,d)\cap\bigcup_{p\geq m_0} n_{k_p}\tilde{B}=\emptyset$. (iii) follows for the sequence $\{n_{k_p}\}_{p\geq m_0}$.

(iii)$\Rightarrow$(iv). There is no generality lost with the assumptions that $t_1\geq 1$, that $t_{n+1}-t_n\geq 1$ for all n and $(a,b)\subset[a,b]\subset(0,1)$. Let $p_n=\lfloor t_n\rfloor$, where $\lfloor x\rfloor$ stands for the greatest integer function. The sequence of integers $\{p_n\}$ satisfies the conditions of (iii), so there must exist a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ and a nonempty $(c,d)\subset(a,b)$ such that $p_{n_k}\tilde{B}\cap(c,d)=\emptyset$ for all $k\in\mathbb{N}$.

Since $t_{n_k}/p_{n_k} \to 1$ as $k \to \infty$, there exists a k_0 such that

$$1 \leq \frac{t_{n_k}}{p_{n_k}} < 1 + \frac{d-c}{3c}, \quad \text{for all} \quad k \geq k_0.$$

Let $J = (c + (d-c)/3, d)$ and $k \geq k_0$. Then

$$\emptyset = \frac{t_{n_k}}{p_{n_k}} \left((c,d) \cap p_{n_k} \tilde{B} \right) = \left(\frac{t_{n_k} c}{p_{n_k}}, \frac{t_{n_k} d}{p_{n_k}} \right) \cap t_{n_k} \tilde{B} \supset J \cap t_{n_k} \tilde{B}.$$

Part (iv) follows easily from this.

(iv)⇒(v). Without loss of generality we may assume that a and b are rational. Let $\{(a_s, b_s)\}_{s \in \mathbb{N}}$ be an enumeration of all such intervals, and let us choose a sequence $\{t_n\}$. The idea of the proof is an application of (iv) infinitely many times, and diagonalization.

Let $m_k^0 = k$ for $k \in \mathbb{N}$. We will, by induction on $s \in \mathbb{N}$, construct sequences $\{m_k^s\}_{k \in \mathbb{N}}$ such that, for every $s \in \mathbb{N}$, $\{m_k^s\}_{k \in \mathbb{N}}$ will be a subsequence of $\{m_k^{s-1}\}_{k \in \mathbb{N}}$, and there will be a nonempty interval (c,d) (possibly dependent on s) contained in (a_s, b_s) such that for all $k \in \mathbb{N}$

$$(c,d) \cap (t_{m_k^s} \tilde{B}) = \emptyset.$$

The construction is facilitated by (iv). Now, define $t_{n_k} = t_{m_k^k}$. Then, for every $k_0 \in \mathbb{N}$, there exists $(c,d) \subset (a_{k_0}, b_{k_0})$ such that for every $k \geq k_0$

$$(c,d) \cap (t_{n_k} \tilde{B}) = \emptyset.$$

This proves (v).

(v)⇒(i). Let $\{n_k\}_{k \in \mathbb{N}}$ be an increasing sequence of natural numbers and let $\{(a_p, b_p)\}_{p \in \mathbb{N}}$ be an enumeration of all nonempty subintervals of $(-1,1)$ with rational endpoints. Using (v), there is a subsequence $\{n_k^1\}$ of $\{n_k\}$ and a nonempty open interval $U_1 \subset (a_1, b_1)$ such that

$$U_1 \cap n_k^1 \tilde{B} = \emptyset, \quad \text{for all} \quad k \in \mathbb{N}.$$

Continuing inductively, for each $p > 1$ we can choose a subsequence $\{n_k^p\}$ of $\{n_k^{p-1}\}$ and a nonempty interval $U_p \subset (a_p, b_p)$ such that

$$U_p \cap n_k^p \tilde{B} = \emptyset, \quad \text{for all} \quad k \in \mathbb{N}. \tag{11}$$

If $n_{k_p} = n_p^p$ then (11) implies

$$\bigcup_{k\in\mathbb{N}} U_k \cap \limsup_{p\to\infty} n_{k_p}\tilde{B} = \emptyset.$$

Since $\bigcup_{k\in\mathbb{N}} U_k$ is a dense open subset of $(-1,1)$, this implies (i).

This finishes the proof of Lemma 6.1.

Notice that Lemma 6.1 can be viewed as a version of [15, Lemma 1]. (Compare also [15, Corollary 1].) In particular, an alternative prove of Lemma 6.1 is as follows. Implication "0 is an $\mathcal{I}$-dispersion point of B"⇒(v) is implicitely contained in the proof of [15, Lemma 1]. Since the implications (v)⇒(iv)⇒(iii) and (ii)⇒(i)⇒"0 is an $\mathcal{I}$-dispersion point of B" are obvious, it is enough to show (iii)⇒(ii). But it is included in this paper as a proof of (v)⇒(i).

In what follows we also need the following theorem. Notice that Theorem 5.8 that can be obtained from the following theorem by taking an arbitrary set E and $u_k = 1$.

Theorem 6.2. *Let 0 be a right $\mathcal{I}$-density point of a right interval set $E = \bigcup_{k\in\mathbb{N}}(a_k, b_k) \subset [0,1]$ and let $h\colon[-1,1]\to[-1,1]$ be such that the restricted function $h|_{(a_k,b_k)}$ is a homeomorphism for every $k\in\mathbb{N}$. Moreover, let us assume that there exist a nondecreasing sequence $\{u_k\}_{k\in\mathbb{N}}$ of positive numbers and constants $K, L > 0$ such that for every $k\in\mathbb{N}$ the functions*

$$h_k = u_k h|_{\{0\}\cup(a_k,b_k)}\colon \{0\}\cup(a_k,b_k) \to [-K,K] \quad \textit{and} \quad \left(h_k|_{(a_k,b_k)}\right)^{-1}$$

satisfy a Lipschitz condition with constant L. Then h is right $\mathcal{I}$-density continuous at 0.

Before proving the theorem let us notice that it implies the following corollary. It can be considered as an $\mathcal{I}$-density analog of the theorem that convex functions defined on open intervals are density continuous [7, Theorem 1]. Notice also that an easy modification of the Example 5.7 shows that not all convex functions defined on open intervals are $\mathcal{I}$-density continuous.

Corollary 6.3. *Let 0 be a right $\mathcal{I}$-density point of a right interval set $\bigcup_{k\in\mathbb{N}}(a_k,b_k)$ and let $h\colon[-1,1]\to[-1,1]$ be a convex function such that the restricted functions $h|_{(a_k,b_k)}$ are linear for every $k\in\mathbb{N}$. Then h is right $\mathcal{I}$-density continuous at 0.*

Proof. Let $c_k = h'((a_k + b_k)/2)$ for $k \in \mathbb{N}$. Then, by the convexity of h, the sequence $\{c_k\}$ is nonincreasing. Let, $c = \inf_{k\in\mathbb{N}} c_k$. Notice that $c \neq -\infty$, as $c = -\infty$ would contradict the convexity of h on $(-1, 1)$. If $c \neq 0$ then h and h^{-1} both satisfy a Lipschitz condition on a right neighbourhood of 0. By Theorem 5.8, this implies that h is $\mathcal{I}$-density continuous at 0. (See the remark before Theorem 6.2.) If $c = 0$, we may also assume that $h(0) = 0$. There are two cases to consider. If $P = \{k : c_k = 0\}$ is infinite, then h must be linear on a right-hand neighborhood of 0 since h is convex on $(0, 1)$. Otherwise, assume P is finite. In this case there is no generality lost in assuming that $P = \emptyset$. Evidently, the sequence $u_k = c_k^{-1}$ is nondecreasing and the derivative of $u_k h$ at points from (a_k, b_k) equals 1. Moreover, the convexity of h implies that $h_k = u_k h|_{\{0\}\cup(a_k,b_k)}: \{0\} \cup (a_k, b_k) \to [0, 1]$. An application of Theorem 6.2 finishes the proof.

We also need the following easy lemma.

Lemma 6.4. *Let $\varepsilon, M > 0$ and let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of subintervals of $[-M, M]$ such that $b_n - a_n > \varepsilon$ for every $n \in \mathbb{N}$. Then there exists a nonempty interval $(a, b) \subset [-M, M]$ and an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive numbers such that $(a, b) \subset (a_{n_k}, b_{n_k})$ for every $k \in \mathbb{N}$.*

Proof. Let c_n be the center of (a_n, b_n) for every $n \in \mathbb{N}$ and let $-M = d_0 < d_1 < \ldots < d_m = M$ be such that $d_i - d_{i-1} < \varepsilon/2$ for every $i = 1, 2, \ldots, m$. By the Pigeon Hole Principle, there exists $i \leq m$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $d_i \in (a_{n_k}, c_{n_k})$ for every $k \in \mathbb{N}$. Then the sequence $\{n_k\}_{k\in\mathbb{N}}$ and the interval $(a, b) = (d_i, d_i + \frac{\varepsilon}{2})$ suffice.

Proof of Theorem 6.2. Let h be as in the assumptions. Replacing $h(x)$ by $(h(x) - h(0))/K$, if necessary, we may assume that $h(0) = 0$ and $K = 1$.

For $k, n \in \mathbb{N}$, let

$$h_{k,n}(x) = n\, h_k\left(\frac{1}{n}x\right) = n\, u_k\, h\left(\frac{1}{n}x\right) \text{ with domain } \{0\} \cup (na_k, nb_k).$$

The functions $h_{k,n}$ and $\left(h_{k,n}|_{(na_k,nb_k)}\right)^{-1}$ also satisfy a Lipschitz condition with L as the constant.

To prove that h is right $\mathcal{I}$-density continuous at 0, let $B \in \mathcal{B}$, $0 \notin B$, be such that 0 is an $\mathcal{I}$-dispersion point of B. It is enough to prove that 0 is a right $\mathcal{I}$-dispersion point of $D = h^{-1}(B) \cap E$. Notice that $\tilde{D} = h^{-1}(\tilde{B}) \cap E$. Thus, we may assume from the beginning that both B and D are regular open sets.

It will be shown that 0 is an $\mathcal{I}$-dispersion point of $h^{-1}(B) \cap E$ by using Lemma 6.1(iii).

So, let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of integers and let $(a, b) \subset [-1, 1]$ be a nonempty interval. We must find a nonempty interval $(c, d) \subset (a, b)$ and a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that for every $p \in \mathbb{N}$

$$(c, d) \cap n_{k_p} D = \emptyset. \tag{12}$$

First, it may be assumed that $(a, b) \subset (0, 1)$, else (12) is trivially satisfied. Since $(0, \infty) \setminus \mathrm{cl}(E)$ is a regular open set having 0 as an $\mathcal{I}$-dispersion point, Lemma 6.1(iii) implies the existence of a nonempty open interval $I \subset (a, b)$ and a subsequence $\{n'_k\}$ of $\{n_k\}$ such that $I \subset n'_k E$ for all k. There is no generality lost with the assumptions that $I = (a, b)$ and $n'_k = n_k$ for all k. Thus, with these assumptions, there exists a nondecreasing sequence of integers m_k such that $(a, b) \subset n_k(a_{m_k}, b_{m_k})$ and (a, b) is in the domain of h_{m_k,n_k} for every $k \in \mathbb{N}$.

But the functions h_{m_k,n_k} and $\left(h_{m_k,n_k}|_{(n_k a_{m_k}, n_k b_{m_k})}\right)^{-1}$ satisfy a Lipschitz condition with constant L. So, the intervals

$$h_{m_k,n_k}((a, b)) \subset h_{m_k,n_k}(\{0\} \cup (a, b)) \subset [-L, L]$$

have length at least $(b - a)/L$ and, using Lemma 6.4, we may also assume, choosing a subsequence, if necessary, that for some nonempty interval $(a', b') \subset [-L, L]$ and every $k \in \mathbb{N}$

$$(a', b') \subset h_{m_k,n_k}((a, b)).$$

Now, by Lemma 6.1(iv) used with B, the divergent sequence $\{u_{m_k} n_k\}_{k\in\mathbb{N}}$ and (a', b'), we may assume, passing to a subsequence, if necessary, that for some nonempty interval $(c', d') \subset (a', b')$ and every $k \in \mathbb{N}$

$$(c', d') \cap u_{m_k} n_k B = \emptyset,$$

which is equivalent to

$$n_k h^{-1}\left(\frac{1}{u_{m_k} n_k}(c', d')\right) \cap n_k h^{-1}(B) = \emptyset.$$

For x in the domain of $h^{-1}_{m_k,n_k}$, we have $n_k h^{-1}\left(\frac{1}{u_{m_k} n_k} x\right) = h^{-1}_{m_k,n_k}(x)$. Thus, the above is equivalent to

$$h^{-1}_{m_k,n_k}((c',d')) \cap n_k h^{-1}(B) = \emptyset.$$

But,

$$h^{-1}_{m_k,n_k}((c',d')) \subset h^{-1}_{m_k,n_k}((a',b')) \subset (a,b)$$

for every $k \in \mathbb{N}$. Using Lemma 6.4 once more and the fact that the functions h_{m_k,n_k} satisfy a Lipschitz condition with the same constant L, we may choose an increasing sequence $\{k_p\}_{p\in\mathbb{N}}$ of natural numbers and a nonempty interval (c,d) such that for every $p \in \mathbb{N}$

$$(c,d) \subset h^{-1}_{m_{k_p},n_{k_p}}((c',d')) \subset (a,b).$$

The last two conditions easily imply (12). The proof of Theorem 6.2 is finished.

The next two examples are technical and will be used to construct other examples.

Example 6.5. *Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers. For every right interval set $E = \bigcup_{n\in\mathbb{N}}[a_n,b_n] \subset [0,1]$ for which 0 is a right $\mathcal{I}$-density point there exists an $\mathcal{I}$-density and density continuous, C^∞, convex increasing homeomorphism $h\colon [0,b_1] \to [0,\infty)$ such that*

(1) *$h^{(n)}(0) = 0$ for every $n \geq 0$; and,*

(2) *for every $n \in \mathbb{N}$ there exists a positive number $d_n \leq c_n$ such that $h'(x) = d_n$ for every $x \in (a_n,b_n)$.*

Proof. Let $D = \bigcup_{n\in\mathbb{N}}(l_n,r_n) = (0,b_1] \setminus E$ and let f be a nonnegative C^∞ function such that $f(x) = 0$ for $x \in D^c$ while $f(x) > 0$ for $x \in D$. To choose such a function it is enough to define f on (l_n,r_n) by

$$f(x) = u_n e^{-(x-l_n)^{-2}-(x-r_n)^{-2}},$$

which is known to be C^∞ with poles at l_n and r_n, and in which the constants $u_n \leq c_n$ are chosen in such a way that $f^{(i)}(x) \leq \frac{1}{n}$ for every $i \leq n$ and $x \in [l_n,r_n]$.

Let $g(x) = \int_0^x f(y)\,dy$ and $h(x) = \int_0^x g(y)\,dy$.

Evidently h is a $\mathcal{C}^\infty$ increasing and convex homeomorphism satisfying (1). It is also easy to see that (2) holds, as $f(x) \le u_n \le c_n$ for all $x \in [0, b_n] \subset [0,1]$. Moreover, by Theorem 5.8, h is $\mathcal{I}$-density continuous at every point $\neq 0$ and, by Corollary 6.3, h is right $\mathcal{I}$-density continuous at 0.

The function h is density continuous, since it can be extended to a convex function on an open interval [7, Theorem 1].

Example 6.6. *Let $k < l < m < a < b < c$. There exists an $\mathcal{I}$-density continuous, $\mathcal{C}^\infty$ function $g\colon [k,c] \to \mathbb{R}$ constant on $[m,a]$ and such that $g(x) = x$ for $x \in [k,l] \cup [b,c]$.*

Proof. Without loss of generality we may assume that $m < 0 < a$. It suffices to define g on $[0,c]$, because defining g on $[k,0]$ merely involves homothetically altering the odd extension of g.

Let a_0, b_0 be such that $a < a_0 < b_0 < b$ and choose $h\colon [0, a_0 - a] \to [0,\infty)$ as in Example 6.5 in such a way that $h(a_0 - a) < b_0$. Define $g_0\colon [0,c] \to \mathbb{R}$ by putting $g_0(x) = 0$ for $x \in [0,a]$, $g_0(x) = x$ for $x \in [b_0, c]$, $g_0(x) = h(x-a)$ for $x \in [a, a_0]$ and extend g_0 on $[a_0, b_0]$ in a continuous manner as a linear function.

Evidently, g_0 is $\mathcal{I}$-density continuous. It is also $\mathcal{C}^\infty$ at all points with the exceptions of a_0 and b_0. To obtain the desired function g we will modify g_0 on $[a,b]$ by "rounding its corners" at the points a_0 and b_0. But notice that g_0 is increasing and linear on some left and right hand neighborhoods of a_0 and b_0. Thus, using an appropriate homothetic transformation of a function $L(x) + rh(x)$ where $L(x)$ is a linear function, $r \in \mathbb{R}$ and h is from Example 6.5 (or even $h(x) = \int_0^x \int_0^y \exp\left(-z^{-2} - (z-1)^{-2}\right)\,dz\,dy$) we can easily modify g_0 to be a $\mathcal{C}^\infty$ function g while keeping the property that $g'(x) > 0$ on the modification set. Thus, g is $\mathcal{I}$-density continuous.

Example 6.7. *There is an $f \in \mathcal{C}_{\mathcal{DD}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{NN}} \setminus \mathcal{C}_{\mathcal{II}}$.*

Proof. Let $g\colon [0,1] \to [0,1]$ be as in Example 6.6, constant on an interval containing 1/2 and choose constants $c_k \le \frac{1}{k}$ such that $c_k g^{(i)}(x) \le \frac{1}{k}$ for every $x \in [0,1]$ and $i \le k$. Next, choose a right interval set $E = \bigcup_{k\in\mathbb{N}}[p_k, q_k] \subset [0,1]$ for which 0 is a right $\mathcal{I}$-density point and let h and $\{d_k\}$ be choosen as in Example 6.5 for the sequence $\{c_k\}$ and the set E. Extend h onto $\mathbb{R}$ by putting

$h(x)=0$ for $x<0$ and as a linear function on $[q_1,\infty)$ by the same formula as on $[p_1,q_1]$. It is easy to see that

$$h \in \mathcal{C}_{\mathcal{DD}}\ C^{\infty}\ \mathcal{C}_{\mathcal{NN}}.$$

Notice that for every $t>0$ the function $g_{k,t}\colon [0,t]\to[0,t]$ defined by $g_{k,t}(x)=t\,d_k g(x/t)$ has the properties that

$$g_{k,t}^{(i)}(x) \le \frac{1}{k} \text{ for every } x\in[0,t] \text{ and } i\le k$$

and

$$g_{k,t}(x)=d_k x \text{ for every } x\in t\left([0,l]\cup[b,1]\right).$$

Moreover, let $S=\{s_n\}_{n\in\mathbb{N}}$ be from Lemma 2.3. Without loss of generality we may assume that $S\subset \operatorname{int}(E)$, as 0 is a right deep-$\mathcal{I}$-density point of E. Notice, that each component of E contains only finitely many points from S. Denote the component of E which contains s_n by $[p_{k(n)},q_{k(n)}]$.

We need also to choose nonempty pairwise disjoint intervals $(a_n,b_n)\subset [a_n,b_n]\subset (p_{k(n)},q_{k(n)})$, centered at s_n such that 0 is a dispersion point of $P=\bigcup_{n\in\mathbb{N}}(a_n,b_n)$ and

$$\lim_{n\to\infty}\frac{h(b_n)-h(a_n)}{h(a_n)}=0. \tag{13}$$

Define $h_n(x)=g_{k(n),b_n-a_n}(x-a_n)$ for $x\in(a_n,b_n)$, with the interval in (a_n,b_n) on which h_n is constant denoted by I_n. Notice that $s_n\in I_n$. Define f by putting $f|_{P^c}=h|_{P^c}$ and $f(x)=h_n(x)+h(a_n)$ for $x\in(a_n,b_n)$.

Obviously, f is C^{∞}, $\mathcal{I}$-density continuous and density continuous at any point $\neq 0$. It is also clear, by the choice of the constants c_k, that f is infinitely differentiable at 0.

To see that f is density continuous at 0 it is enough to notice that $f|_{(-\infty,0]\cup\operatorname{int}(E)}=h|_{(-\infty,0]\cup\operatorname{int}(E)}$, while $h\in\mathcal{C}_{\mathcal{NN}}$ and $(-\infty,0]\cup\operatorname{int}(E)\in\mathcal{T}_{\mathcal{N}}$.

It is also easy to see that $f\notin\mathcal{C}_{\mathcal{II}}$, as 0 is an $\mathcal{I}$-dispersion point of the countable set $f(S)$, while 0 is not an $\mathcal{I}$-dispersion point of $f^{-1}(f(S))\supset \bigcup_{n\in\mathbb{N}}I_n\supset S$.

It remains to prove that f is right deep-$\mathcal{I}$-density continuous at 0, as the left hand case is obvious. So, let 0 be an $\mathcal{I}$-dispersion point of a right interval set $B = \bigcup_{n\in\mathbb{N}}(v_n, w_n)$. It is enough to prove that 0 is an $\mathcal{I}$-dispersion point of $f^{-1}(B)$, as the sets $E \cup \{0\} \in \mathcal{T}_{\mathcal{I}}$, where E is an open interval set, form a basis of $\mathcal{T}_{\mathcal{D}}$ at 0 [11].

We claim that the lengths of some of the intervals (v_n, w_n) can be increased slightly to satisfy the condition

$$(h(a_k), h(b_k)) \cap (v_n, w_n) = \emptyset \quad \text{or} \quad (h(a_k), h(b_k)) \subset (v_n, w_n), \tag{14}$$

while still preserving the property that 0 is an $\mathcal{I}$-dispersion point of B.

To see this, define $B' = B \cup \bigcup\{(h(a_k), h(b_k)) \colon (h(a_k), h(b_k)) \cap B \neq \emptyset\}$. Evidently, B' satisfies (14). We have to prove only that 0 is an $\mathcal{I}$-dispersion point of B'.

So, let $(a, b) \subset (0, 1)$ and $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. Using Lemma 6.1(iii), we can find an increasing subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ and a nonempty interval $(c, d) \subset (a, b)$ such that

$$(c, d) \cap n_{k_p} B = \emptyset \quad \text{for all} \quad p \in \mathbb{N}.$$

According to (13), there is an $p_0 \in \mathbb{N}$ such that

$$\frac{h(b_{n_{k_p}}) - h(a_{n_{k_p}})}{h(a_{n_{k_p}})} < \frac{d - c}{3} \quad \text{for all} \quad p \geq p_0.$$

But, then it is clear that

$$(c + \frac{d-c}{3}, d - \frac{d-c}{3}) \cap n_{k_p} B' = \emptyset \quad \text{for all} \quad p \geq p_0,$$

so, by Lemma 6.1(iii), 0 is an $\mathcal{I}$-dispersion point of B'.

But (14) implies that $h^{-1}(B) = f^{-1}(B)$, so 0 is a deep-$\mathcal{I}$-dispersion point of $f^{-1}(B)$, as h is deep-$\mathcal{I}$-density continuous.

Example 6.8. *There is an $f \in \mathcal{C}_{\mathcal{I}\mathcal{I}}\ C^\infty \setminus \mathcal{C}_{\mathcal{N}\mathcal{N}}$.*

Proof. We will proceed in a method similar to Example 6.7.

Let $g \colon [0, 1] \to [0, 1]$ be as in Example 6.6 such that g is constant on $[1/3, 2/3]$. Choose constants $c_k \leq \frac{1}{k}$ such that $c_k g^{(i)}(x) \leq \frac{1}{k}$ for every $x \in [0, 1]$ and $i \leq k$. Next, choose a right interval set $E = \bigcup_{k\in\mathbb{N}}[p_k, q_k] \subset [0, 1]$ for which 0 is a right density and deep-$\mathcal{I}$-density point. Let h and $\{d_k\}$ be choosen as in Example 6.5 for the sequence $\{c_k\}$ and set E. Extend h onto $\mathbb{R}$ by putting $h(x) = 0$ for $x < 0$ and as a linear function on $[q_1, \infty)$ by the same formula as on $[p_1, q_1]$. It is easy to see that

$$h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}\ \mathcal{C}^\infty\ \mathcal{C}_{\mathcal{N}\mathcal{N}}.$$

Notice that for every $t > 0$ the function $g_{k,t}: [0, t] \to [0, t]$ defined by $g_{k,t}(x) = t\, d_k g(x/t)$ has the properties that

$$g_{k,t}^{(i)}(x) \leq \frac{1}{k} \text{ for every } x \in [0, t] \text{ and } i \leq k \tag{15}$$

and

$$g_{k,t}(x) = d_k x \text{ for every } x \in t\left([0, l] \cup [b, 1]\right). \tag{16}$$

Moreover, let $D = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be a right interval set such that $D^c \in \mathcal{T}_{\mathcal{D}} \setminus \mathcal{T}_{\mathcal{N}}$. (See Theorem 2.5.) Without loss of generality we may assume that $D \subset \operatorname{int}(E)$, as 0 is a right density and deep-$\mathcal{I}$-density point of E. Notice, that each component of E contains only finitely many intervals $[a_n, b_n]$. Denote the component of E which contains $[a_n, b_n]$ by $[p_{k(n)}, q_{k(n)}]$.

Let $h_n = g_{k(n), b_n - a_n}$, with the interval in $(0, b_n - a_n)$ on which h_n is constant denoted by I_n. Define f by putting $f|_{(\bigcup_{n\in\mathbb{N}}(a_n,b_n))^c} = h|_{(\bigcup_{n\in\mathbb{N}}(a_n,b_n))^c}$ and $f(x) = h_n(x - a_n) + h(a_n)$ for $x \in [a_n, b_n]$.

Obviously f is $\mathcal{C}^\infty$, $\mathcal{I}$-density continuous and density continuous at any point $\neq 0$. It is also clear, by the choice of the constants c_k, that f is infinitely differentiable at 0. Also, f is $\mathcal{I}$-density continuous at 0 as $f|_{D^c} = h|_{D^c}$, while $h \in \mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $D^c \in \mathcal{T}_{\mathcal{I}}$.

Finally, f is not density continuous at 0, as 0 is a dispersion point of the countable set $f(\bigcup_{n\in\mathbb{N}} I_n)$, while 0 is not a dispersion point of a set $f^{-1}(f(\bigcup_{n\in\mathbb{N}} I_n)) = \bigcup_{n\in\mathbb{N}} I_n$. This last statement is correct, as $\bigcup_{n\in\mathbb{N}} I_n$ is obtained by taking the middle third from every component of $D = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$.

References

[1] V. Aversa and W. Wilczyński. Homeomorphisms preserving $\mathcal{I}$-density points. *Boll. Un. Mat. Ital.*, B(7)1:275–285, 1987.

[2] A. M. Bruckner. *Differentiation of Real Functions.* Lecture Notes in Mathematics 659. Springer-Verlag, 1978.

[3] Krzysztof Ciesielski. Density-to-deep-$\mathcal{I}$-density continuous functions. *Real Anal. Exch.* this volume.

[4] Krzysztof Ciesielski and Lee Larson. Analytic functions are $\mathcal{I}$-density continuous. (submitted).

[5] Krzysztof Ciesielski and Lee Larson. Baire classification of $\mathcal{I}$-approximately and $\mathcal{I}$-density continuous functions. (submitted).

[6] Krzysztof Ciesielski and Lee Larson. Refinements of the density and $\mathcal{I}$-density topologies. (submitted).

[7] Krzysztof Ciesielski and Lee Larson. The space of density continuous functions. *Acta Math. Hung.*, to appear.

[8] Krzysztof Ciesielski, Lee Larson, and Krzysztof Ostaszewski. Density continuity versus continuity. *Forum Mathematicum*, 2:265–275, 1990.

[9] C. Goffman, C. J. Neugebauer, and T. Nishiura. The density topology and approximate continuity. *Duke Math. J.*, 28:497–506, 1961.

[10] Casper Goffman and Daniel Waterman. Approximately continuous transformations. *Proc. Amer. Math. Soc.*, 12:116–121, 1961.

[11] E. Łazarow. The coarsest topology for $\mathcal{I}$-approximately continuous functions. *Comment. Math. Univ. Caroli.*, 27(4):695–704, 1986.

[12] R. J. O'Malley. Baire*1 Darboux functions. *Proc. Amer. Math. Soc.*, 60:187–192, 1976.

[13] J. C. Oxtoby. *Measure and Category.* Springer-Verlag, 1971.

[14] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński. A category analogue of the density topology. *Fund. Math.*, 75:167–173, 1985.

[15] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński. Remarks on $\mathcal{I}$-density and $\mathcal{I}$-approximately continuous functions. *Comm. Math. Univ. Carolinae*, 26(3):553–563, 1985.

[16] Richard L. Wheeden and Antoni Zygmund. *Measure and Integral.* Marcel Dekker, Inc., 1977.

[17] W. Wilczyński. A category analogue of the density topology, approximate continuity, and the approximate derivative. *Real Anal. Exch.*, 10:241–265, 1984-85.

[18] S. Willard. *General Topology.* Springer, 1971.

[19] Z. Zachorski. Sur la première dérivée. *Trans. Amer. Math. Soc.*, 69:1–54, 1950.

[20] L. Zajíček. Alternative definitions of the J-density topology. *Acta Univ. Carolinae-Mat. et Phys.*, 28(1):57–61, 1987.

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