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REMARKS ABOUT CONNECTED SPACES.

by

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Abstract. The main result of the paper generalize Kuratowski's Theorem "if X is a connected space, $K \subset X$ is connected and S is a component of $X \setminus K$ then $X \setminus S$ is connected" [Ku, Ch.V§46, III Theorem 5] to the case when K has finite number of components and S consists of finite number of either components or quasi components of $X \setminus K$. Some examples are also discussed.

In what follows we will use standard topological notation. For a topological space X and $A \subset X$ the symbol A^c denotes the complement of A with respect to X; i.e., $A^c = X \setminus A$. A quasi component is said to be *proper*, if it is not a component.

Theorem 1. Let X be a connected topological space, n be a natural number and let $K \subset X$ have precisely n components. If C is a union of finite number of components of K^c and Q is a union of finite number of proper quasi components of K^c, then $(C \cup Q)^c$ has at most n components.

In the proof we will need the following easy facts.

Lemma 1. Let Y be any topological space and let $k > 0$ be a natural number.

(a) If $C_1, C_2, \ldots, C_k, C_{k+1}$ are different nonempty components of Y then there exists a nonempty clopen set W in Y such that $W \cap C_i = \emptyset$ for all $i \leq k$.

(b) Let $Q_1, Q_2, \ldots, Q_k$ be quasi components of Y and let $y \in Y \setminus (Q_1 \cup Q_2 \cup \ldots \cup Q_k)$. Then there exists a clopen subset V of Y containing y and disjoint from every Q_i.

Proof. (a) By [Ku, V§46, III Theorem 6] we can find $k+1$-element open partition $\mathfrak{R}$ of Y. Then, by the Pigeon Hole Principle, there is $U \in \mathfrak{R}$ such that $C_i \cap U = \emptyset$ for every $i \leq k$, as every component can intersect at most one element of $\mathfrak{R}$.

(b) By the definition of quasi component for every $i=1,2,\ldots,k$ there exists a clopen set $V_i \subset Y$ containing y and disjoint with Q_i. Then $V = V_1 \cap V_2 \cap \ldots \cap V_k$ works.

Proof of Theorem 1. By way of contradiction let us assume that $(C \cup Q)^c$ has more than n components. Then K is contained in at most n components of $(C \cup Q)^c$ and, by Lemma 1(a), there is a nonempty clopen subset U of $(C \cup Q)^c$ such that $U \cap K = \emptyset$.

Let $x \in U$. By Lemma 1(b), there exists a clopen subset V of K^c containing x and disjoint from Q. Let us consider the space $Z = (U \cap V) \cup C \subset K^c$. Then, components forming C are also components of Z and $x \in Z \setminus C$. Hence, Z has more components that C and, by Lemma 1(a), there exists a nonempty clopen subset W of Z such that $W \cap C = \emptyset$.

We show that W is clopen in X. Since W and W^c are nonempty, this will contradict the assumption that X is connected.

First notice that

$$\mathrm{cl}(W) \subset \mathrm{cl}(V) \subset V \cup K \subset Q^c \quad \text{and} \quad \mathrm{cl}(W) \subset C^c$$

since $V \subset Q^c$ is clopen in K^c and $W \subset C^c$ is closed in $Z = (U \cap V) \cup C$. Hence,

$$\mathrm{cl}(W) \subset (V \cup K) \cap (C \cup Q)^c.$$

Moreover, $U \subset K^c$ is clopen in $(C \cup Q)^c$. So, $\mathrm{cl}(W) \subset \mathrm{cl}(U) \subset U \cup C \cup Q \subset K^c$ and

$$\mathrm{cl}(W) \subset (U \cup C \cup Q) \cap K^c.$$

But the last two displays imply that $W \subset \mathrm{cl}(W) \subset U \cap V \subset Z$ and W is closed in Z. Thus, $W = \mathrm{cl}(W)$. So, W is closed in X.

To prove that W^c is closed in X, let us first notice that

$$W^c=[Z\setminus W]\cup Z^c=[Z\setminus W]\cup[U\cap V]^c=[Z\setminus W]\cup U^c\cup V^c=[Z\setminus W]\cup[(C\cup Q)^c\setminus U]\cup[K^c\setminus V]$$

But, by the fact that W, U and V are clopen in Z, $(C\cup Q)^c$ and K^c, respectively, we conclude that

$$cl[Z\setminus W]\cap W=\emptyset,$$

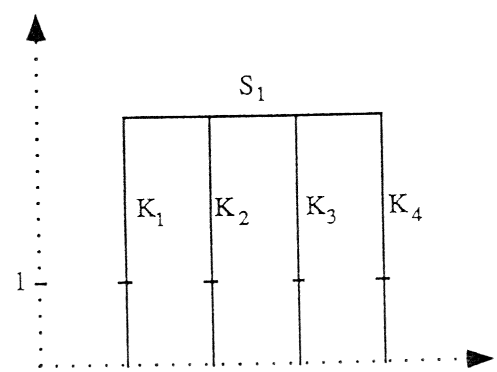
$$cl[(C\cup Q)^c\setminus U]\cap W\subset cl[(C\cup Q)^c\setminus U]\cap U=\emptyset,$$

$$cl[K^c\setminus V]\cap W\subset cl[K^c\setminus V]\cap V=\emptyset.$$

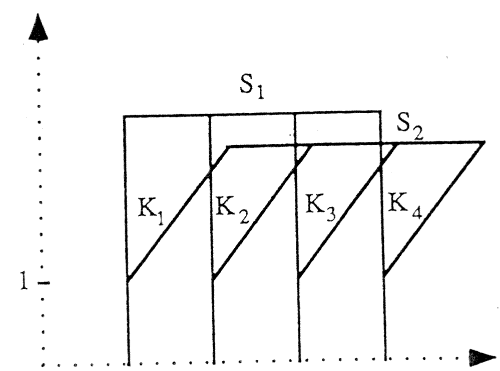
Hence $cl[W^c]\cap W=\emptyset$ i.e., W^c is closed in X.

This finishes the proof of the theorem.

It is easy to see that in Theorem 1 the space $(C\cup Q)^c$ can have precisely n components. For example, it is enough to take define $X\subset \mathbf{R}^2$ by $X = ([1,n]\times\{3\}) \cup (\{1,2,\dots,n\}\times[0,3])$, and choose $K_i=\{i\}\times(1,3)$ and $C\cup Q=\{S_1\}$, where $S_1=[1,n]\times\{3\}$. For $n=4$, see Picture 1. It is also easy to modify this example to work in the case when $C\cup Q$ has any finite number m of elements. For $m=2$ the construction is suggested in Picture 2.

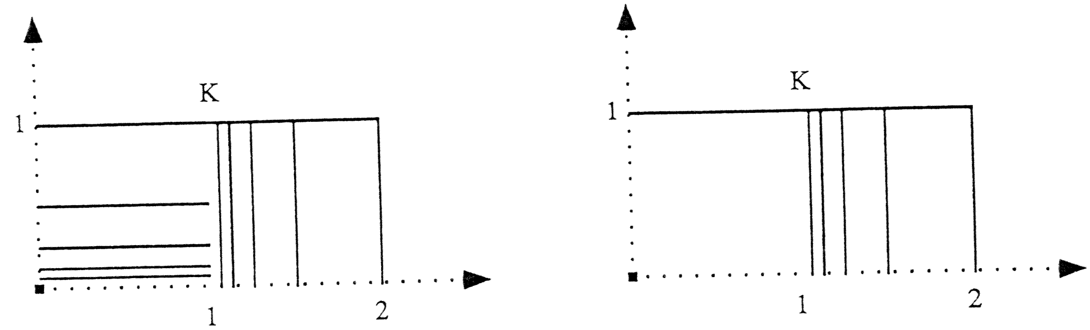


Picture 1



Picture 2

In order to obtain the conclusion of Theorem 1 we can remove from K^c an entire proper quasi component or finite number of its components. It seems to be natural to ask whether the conclusion remains valid if we remove from a proper quasi component of K^c an infinite number of components. The answer for this question is negative. To see this define $X \subset \mathbf{R}^2$ by $X = K \cup Q \cup \bigcup \{\{1+(1/2)^n\} \times [0,1] : n \in \mathbf{N}\}$, where $K = [0,2] \times \{1\}$, $Q = P \cup \{(0,0)\}$ and $P = \bigcup \{[0,1] \times \{(1/2)^n\} : n > 0, n \in \mathbf{N}\}$. It is easy to see that Q is a proper quasi component of K^c and that P^c is disconnected, while K is connected. (See Picture 3.)



Space X

Space X\P

Picture 3

Let us also notice that in the above simple examples the conclusion depends on the choice of K and $C \cup Q$. This changes, when define X as an indecomposable continuum (for definition and construction see e.g. [HY] p. 139.) In this case, we can choose an arbitrary proper subset $K \subset X$ having any finite number n of components and an arbitrary nonempty set $C \cup Q$. Then, $(C \cup Q)^c$ will have precisely n components. For $n=1$ it follows at once from Theorem 1. For $n>1$ it is the case, since K^c is connected and so, $C \cup Q = K^c$, i.e., $(C \cup Q)^c = K$. The fact that K^c is connected can be deduced from the properties of composants of indecomposable continuum that can be found in [HY] sec. 3-8. An alternative, direct proof of it goes as follows. Let $K_1, K_2, \ldots, K_n$ be nonempty different components of K. Then, $cl(K) = cl(K_1) \cup \ldots \cup cl(K_n) \neq X$, since otherwise X would be a finite union of proper subcontinua $cl(K_1), \ldots, cl(K_n)$. If $U = X \setminus cl(K)$ is disconnected, separated by nonempty open sets V_1 and V_2 then, by [Ku] V,46 Thm 7 p. 134, sets $cl(K) \cup V_i = cl(K) \cup cl(V_i)$, for $i=1,2$, would have at most n components and each of these components would be a proper subcontinuum of X. This would contradict indecomposability of X. Thus, $U=X\setminus cl(K)$ is connected. But U must be dense in X, since otherwise the sets $cl(K_1),...,cl(K_n)$ and cl(U) would form a forbidden decomposition of X. Hence, $U\subset K^c\subset X=cl(U)$ and, by connectedness of U, K^c is connected.

Theorem 1 and previous examples shows that the number of components of $(C\cup Q)^c$ is bounded by number of components of K and this upper bound is independent of the number m of components of $C\cup Q$. However, within the limit given by Theorem 1, the number of components of $(C\cup Q)^c$ can be determined by number m of components of $C\cup Q$. This is the case for a circle as can be seen from the theorem below.

Theorem 2. Let X be a continuum. The following properties are equivalent:

(1) for every $K\subset X$ having a finite number of components and for every $C\subset K^c$ containing m components of K^c the space C^c has m components.

(2) for every $K\subset X$ having a finite number of components and for every component C of K^c the space C^c is connected.

(3) for every 2-element set $K\subset X$ and for every component C of K^c the space C^c is connected.

(4) for every 2-element set $K\subset X$ the space K^c is disconnected.

(5) X is homeomorphic to a circle.

Proof. The implications (1)=>(2)=>(3) are obvious.

To prove (3)=>(4) let us assume that (4) is false. Then for some 2-element set $K\subset X$ we must have $C=K^c$, since K^c is connected. Hence $C^c=K$ is disconnected.

(4)=>(5) follows from the well known Moore's characterization of a circle (see [Ku] V,47 p. 180).

(5)=>(1) follows from the property of a circle X that for every $C\subset X$ the sets C and C^c have the same number of components.

REFERENCES

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[Ku] K. Kuratowski, *Topology vol II*, Academic Press 1968.

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