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The Density Topology is not Generated

The density continuous functions, $\mathcal{C}_D$, are the real functions $f\colon(\mathbf{R},\tau_d)\to(\mathbf{R},\tau_d)$ which are continuous when the density topology

$$\tau_d = \{A \subset \mathbf{R}\colon \text{ every } a \in A \text{ is a density point of } A\}$$

is used on both the domain and the range. (For more details on the density topology see [6] or [7].) It has recently been shown that the density continuous functions do not form a vector space and there are monotone, and even C^∞ functions which are not density continuous [2]. On the other hand, all locally convex functions are density continuous [2] and density continuous functions are in the class Baire*1 [3].

The purpose of this note is to answer a question posed by Krzysztof Ostaszewski [5] related to the properties of the set of density continuous functions, $\mathcal{C}_D$, when viewed as a semigroup. This question is:

Query Is the density topology generated?

It turns out that this question can be answered negatively using a characterization of the level sets of a density continuous function.

A topological space (X,τ) is *generated* if, whenever τ' is another topology on X, with the property that the set of continuous selfmaps

$$f : (X,\tau')\to(X,\tau')$$

contains the set of continuous selfmaps $f : (X,\tau)\to(X,\tau)$, then it is also true that $\tau' \supset \tau$. The generated spaces are characterized by the following theorem of Warndof [8]. (Compare also [4, Definition 2.2, p. 198].)

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Theorem 1 *A Hausdorff topological space (X, τ) is generated if, and only if, the class of complements of level sets of its continuous selfmaps is a subbase for τ.*

Therefore, to show that a topology is not generated, it suffices to show that the level sets of the continuous selfmaps under that topology do not form a subbase for the closed sets of that topology. Our argument is based upon the following facts [1].

Theorem 2 *C_D is a lattice.*

Theorem 3 *The associated sets of density continuous functions, i.e., the sets in the form $\mathbf{R} \setminus f^{-1}(a)$ for $f \in C_D$ and $a \in \mathbf{R}$, are precisely the density open sets which are in $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$.*

In what follows $\text{int}(A)$ and $\overline{A}$ stand for the interior and closure of $A \subset \mathbf{R}$ with respect to the ordinary topology on $\mathbf{R}$.

Lemma 1 *If $f \in C_D$ and $a \in [-\infty, \infty)$, then $\text{int}(\{f > a\})$ is dense in $\{f > a\}$.*

Proof. Let $G = \{f > a\}$. Assume that $\text{int}(G)$ is not dense in G. Then there is an open interval I such that $I \cap G \neq \emptyset$, but $I \cap \text{int}(G) = \emptyset$. Since both G and G^c are $\mathbf{G}_\delta$ sets according to Theorem 3, the Baire category theorem implies that G must be nowhere dense in I. We see that $\overline{I} \cap \overline{G}$ is a nowhere dense perfect subset of $\overline{I}$. It is clear that G is dense in $\overline{I} \cap \overline{G}$.

Let J be a component of $\overline{I} \cap (\overline{G})^c$. Since G is density open, we see that $\overline{J} \subset G^c$. Using the fact that $\overline{G}$ is nowhere dense in I, this implies that G^c is dense in $\overline{G} \cap I$.

But, we have established that both $G \cap \overline{I}$ and $G^c \cap \overline{I}$ are dense, disjoint, $\mathbf{G}_\delta$ subsets of $\overline{G} \cap I$, which violates the Baire category theorem. This contradiction proves Lemma 1.

Theorem 4 *The density topology on $\mathbf{R}$ is not generated.*

Proof. Let $\tau = \{\mathbf{R} \setminus f^{-1}(0) : f \in C_D\}$. It suffices to show that τ is not a subbase for the density topology. To do this, we note that since Theorem

2 shows C_D is a lattice, τ is closed under finite intersections. Therefore, it suffices to show that τ is not a basis for the density topology.

Let $E = \mathbf{R} \setminus \mathbf{Q}$. It follows at once from Lemma 1 that E cannot be written as a union of elements from τ because $\text{int}(E) = \emptyset$. But, E has full measure in $\mathbf{R}$, so it is open in the density topology. This contradiction proves the theorem.

By the above theorem the reals equipped with the density topology is an example of a completely regular not generated topological space whose semigroup of continuous selfmaps has the inner authomorphism property [5]. It is the only such example known to the authors. In particular, the implication in the following theorem of Magill [4]

> If a completely regular space X is generated, then X has the inner automorphism property.

cannot be reversed.

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