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ISOMETRICALLY INVARIANT EXTENSIONS OF LEBESGUE MEASURE

KRZYSZTOF CIESIELSKI

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ABSTRACT. The purpose of this note is to give a very short prove of the theorem thta every isometrically invariant measure extending Lebesgue measure on $\mathbf{R}^n$ has a proper isometrically invariant extension, i.e., that there is no maximal isometrically invariant extension of Lebesgue measure on $\mathbf{R}^n$.

All the measures that we will consider in this note will be countable additive isometrically invariant extensions of Lebesgue measure on n-dimensional Euclidean space $\mathbf{R}^n$. By isometries we will understand bijections of $\mathbf{R}^n$ that preserve standard Euclidean distance. All the algebraic and measure theoretical terminology that will be used is standard and follows from [La, Ru] respectively.

The first construction of a proper isometrically invariant extension of Lebesgue measure goes back to Szpilrajn's paper [Sz] of 1935. In the same paper, Szpilrajn stated Sierpinski's question: "Does there exist a maximal isometrically invariant extension of Lebesgue measure on $\mathbf{R}^n$?" A negative answer to this question, i.e., the theorem "every isometrically invariant measure that extends Lebesgue measure on $\mathbf{R}^n$ has a proper isometrically invariant extension," was proved by several mathematicians under different additional assumptions and restrictions (see [Pk, Hu, Ha]). Without any assumption the theorem was proved in 1983 by Ciesielski and Pelc (see [CP]). For more historical details of this issue see also [Ci1]. The purpose of this note is to give a very short proof of the theorem that is different from that of [CP] and follows from the general technique introduced by the author in [Ci2].

Theorem. *Let $\mu\colon \mathscr{M} \to [0, \infty]$ be an isometrically invariant extension of Lebesgue measure $\mathbf{R}^n$. Then there exists a proper isometrically invariant extension of μ.*

The proof will be based on the following easy and well-known lemmas.

Lemma 1 (Szpilrajn). *Let $\mu\colon \mathscr{M} \to [0, \infty]$ be an isometrically invariant measure on $\mathbf{R}^n$. If a family $\mathscr{A}$ of subsets of $\mathbf{R}^n$ is closed under countable union,*

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closed under isometries action (i.e., $g[A] \in \mathcal{A}$ *for every* $A \in \mathcal{A}$ *and every isometry* g*) and such that every* $A \in \mathcal{A}$ *has* μ *inner measure zero, then* μ *has an isometrically invariant extension* $\nu: \mathcal{N} \to [0, \infty]$ *such that* $\mathcal{A} \subset \mathcal{N}$ *and* $\nu(A) = 0$ *for every* $A \in \mathcal{A}$ *.*

Proof. If $\mathcal{I}$ is an ideal of subsets of $\mathbf{R}^n$ generated by the family $\mathcal{A}$, and $\mathcal{N}$ stands for a σ-algebra generated by $\mathcal{M} \cup \mathcal{I}$ then all elements of $\mathcal{N}$ are of the form $(M \cup I_1) \setminus I_2$ where $M \in \mathcal{M}$ and $I_1, I_2 \in \mathcal{I}$. It is easy to see that $\nu: \mathcal{N} \to [0, \infty]$ such that $\nu((M \cup I_1) \setminus I_2) = \mu(M)$ is a well-defined isometrically invariant measure on $\mathbf{R}^n$ extending μ.

In the proof of the next lemma we use a method which goes back to Harazisvili's paper [Ha] (see also [Ci2]).

Lemma 2. *Let* $\mathbf{R}^n = \bigcup\{N_k: k = 0, 1, 2, \dots\}$*. If each* N_k *satisfies the condition*

(*) *for every countable set* $\{g_r: r = 0, 1, 2, \dots\}$ *of isometries there is an uncountable set* H *of isometries such that for every distinct* $h_1, h_2 \in H$

$$h_1\left(\bigcup\{g_r[N_k]: r = 0, 1, 2, \dots\}\right) \cap h_2\left(\bigcup\{g_r[N_k]: r = 0, 1, 2, \dots\}\right) = \varnothing$$

then every isometrically invariant extension $\mu: \mathcal{M} \to [0, \infty]$ *of Lebesgue measure on* $\mathbf{R}^n$ *has a proper isometrically invariant extension.*

Proof. Let $\mu: \mathcal{M} \to [0, \infty]$ be an isometrically invariant extension of Lebesgue measure on $\mathbf{R}^n$. Define

$$\mathcal{A}_k = \left\{\bigcup\{g_r[N_k]: r = 0, 1, 2, \dots\}: \text{ where all } g_r\text{'s areisometries of } \mathbf{R}^n\right\}.$$

If $M \in \mathcal{M}$ is a subset of $A \in \mathcal{A}_k$ then $h_1[M] \cap h_2[M] = \varnothing$ for every distinct h_1, h_2 from H. But $\mu(h[M]) = \mu(M)$ for every h from H. Moreover, measure μ is σ-finite as an extension of Lebesgue measure. This implies that $\mu(M) = 0$ and so A has μ inner measure zero.

Thus we proved that every $\mathcal{A}_k$ satisfies the assumptions of Lemma 1. Hence for each $k = 0, 1, 2, \dots$ there is an isometrically invariant extension ν_k of μ such that $\nu_k(N_k) = 0$. But all N_k's cannot have μ measure zero. So some ν_k must be a proper extension of μ.

The following lemma is an elementary geometrical fact and will be left without the proof.

Lemma 3. *Every isometry of* $\mathbf{R}^n$ *can be represented as a superposition* $t \circ L$ *where* t *is a translation by a vector* $(t_1, t_2, \dots, t_n)$ *and* L *is a linear transformation of* $\mathbf{R}^n$ *represented by some* $n \times n$ *matrix* (a_{ij}).

Proof of the theorem. By Lemma 2 it is enough to construct N_k's such that $\mathbf{R}^n = \bigcup\{N_k: k = 0, 1, 2, \dots\}$ and each N_k satisfies condition $(*)$.

Let $\mathscr{B}$ be a transcendence base of $\mathbf{R}$ over $\mathbf{Q}$ and let us represent $\mathscr{B}$ as $\mathscr{B} = \bigcup\{\mathscr{B}_k : k = 0, 1, 2, \dots\}$ where $\mathscr{B}_0 \subset \mathscr{B}_1 \subset \mathscr{B}_2 \subset \cdots$ and $\mathscr{B}_{k+1} \setminus \mathscr{B}_k$ is uncountable. Define

$$N_k = [\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}_k))]^n ,$$

where $\mathbf{Q}(\mathscr{B}_k)$ is a field generated by $\mathbf{Q}$ and $\mathscr{B}_k$ and $\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}_k))$ is an algebraic closure of $\mathbf{Q}(\mathscr{B}_k)$ in $\mathbf{R}$. We have to prove that N_k's satisfy $(*)$.

So let us choose k and a countable set $\{g_r : r = 0, 1, 2, \dots\}$ of isometries. There exists a countable set $\mathscr{A} \subset \mathscr{B}$ such that all g_r's are defined over $\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{A}))$, i.e., that for each g_r the coefficients t_i's and a_{ij}'s from Lemma 3 are in $\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{A}))$. Let $L = \mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{A} \cup \mathscr{B}_k))$. Then

$$\bigcup\{g_r[N_k] : r = 0, 1, 2, \dots\} \subset L^n .$$

Define

$$H = \{t_\alpha : \alpha \in \mathscr{B}_{k+1} \setminus (\mathscr{A} \cup \mathscr{B}_k)\} ,$$

where t_α is a translation by a vector $(\alpha, 0, 0, \dots, 0)$. Then H is uncountable and for distinct α, $\beta \in H$,

$$t_\alpha\left(\bigcup\{g_r[N_k] : r = 0, 1, 2 \dots\}\right) \cap t_\beta\left(\bigcup\{g_r[N_k] : r = 0, 1, 2, \dots\}\right) \subset t_\alpha(L^n) \cap t_\beta(L^n) = \varnothing$$

as $\alpha - \beta \notin L$. This finishes the proof of the theorem.

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Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506