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2^{2^ω} NONISOMORPHIC SHORT ORDERED COMMUTATIVE DOMAINS WHOSE QUOTIENT FIELDS ARE LONG

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ABSTRACT. A linearly ordered set is short if it does not contain any monotonic sequence of length ω_1, and it is long if it contains a monotonic sequence of length α for every ordinal $\alpha < (2^\omega)^+$. We prove that there exists a family $\mathbf{F}$ of power 2^{2^ω} of long ordered fields of size 2^ω that are pairwise nonisomorphic (as fields) and such that every field $F \in \mathbf{F}$ has 2^{2^ω} nonisomorphic short subdomains whose field of quotients is F. The generalization of this result for higher cardinals is also discussed. This generalizes the author's result of [Ci].

0. DEFINITIONS AND NOTATION

Our set theoretic and algebraic notation is standard and follows [Je] and [La] respectively.

Ordinals are identified with the sets of their predecessors and cardinals with initial ordinals. For an ordinal α we denote by ω_α the αth infinite cardinal, and by $\alpha + 1$ the ordinal successor of α. For a cardinal κ we denote by 2^κ the cardinality of the power set $\mathbf{P}(\kappa)$ and by κ^+ the cardinal successor of κ. Symbols $\mathbf{N}$, $\mathbf{Q}$, and $\mathbf{R}$ stand for the sets of natural, rational, and real numbers, respectively. For an ordered semigroup $G = \langle G, +, 0\rangle$ we define $G^+ = \{g \in G : g \geq 0\}$. For a commutative domain D we also use the following notation connected with the existence of radicals

$$\operatorname{rad}(D) = \{d \in D : (\forall n \in \mathbf{N})(\exists y \in D)(y^n = d)\}.$$

Let X and Y_x be linearly ordered sets such that $\mathbf{N} \subset Y_x$ for all $x \in X$. We define the support of $f \in \prod_{x\in X} Y_x$ by $\operatorname{supp}(f) = \{x \in X : f(x) \neq 0\}$ and the direct sum of the family $\{Y_x\}_{x\in X}$ by $S(X, \{Y_x\}_{x\in X}) = \{f \in \prod_{x\in X} Y_x : \operatorname{supp}(f)$ is finite$\}$. (It is usually denoted as $\sum_{x\in X} Y_x$, but in this paper such a notation would lead to very complicated formulas.) In the case when $Y_x = Y$ for all $x \in X$ we define $S(X, Y) = S(X, \{Y_x\}_{x\in X})$.

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A zero element of $S(X, \{Y_x\}_{x\in X})$ is defined as the unique element $0 \in S(X, \{Y_x\}_{x\in X})$ such that $\mathrm{supp}(0) = \varnothing$. For $x \in X$ we put $e_x \in S(X, \{Y_x\}_{x\in X})$ such that $\mathrm{supp}(e_x) = \{x\}$ and $e_x(x) = 1$. Moreover, if $0 \in X$ then every element $y \in Y_0$, $y \neq 0$, is identified with $y = f_y \in S(X, \{Y_x\}_{x\in X})$, where $\mathrm{supp}(f_y) = \{0\}$ and $f_y(0) = y$. In particular, $1 = f_1 = e_0$.

The family $S(X, \{Y_x\}_{x\in X})$ is ordered antilexicographicaly, i.e., by the formula

$$f < g \text{ if and only if } f(m) < g(m), \text{ where } m = \max\{x : f(x) \neq g(x)\}.$$

1. Main lemmas

In what follows we use the following easily verified well-known facts. (See e.g. [Ci, Lemma 1].)

Proposition 1.1. *Let $S = S(X, \{Y_x\}_{x\in X})$, where X and $Y_x \supset \mathbf{N}$, for $x \in X$, are linearly ordered sets. Then*

(1) *the linearly ordered set $\{e_x \in S : x \in X\}$ is order isomorphic with X;*

(2) *if $\langle Y_x, +\rangle$ are ordered commutative semigroups for $x \in X$ then so is S, where $(f+g)(x) = f(x) + g(x)$ for all f, $g \in S$ and $x \in X$;*

(3) *if $\langle Y_x, +\rangle$ are ordered commutative groups for $x \in X$ then so is S; moreover, if all Y_x's are divisible then so is S;*

(4) *if $\langle X, +\rangle$ is an ordered commutative semigroup, $\langle Y, +, \cdot\rangle$ is an ordered commutative domain, and $\langle Y_x, +, \cdot\rangle = \langle Y, +, \cdot\rangle$ for every $x \in X$, then S is an ordered commutative domain (sometimes denoted by $X(Y)$), where the product between p, $q \in S$ is defined by usual convolution of polynomial multiplication:*

$$(p \cdot q)(x) = \sum\{p(x_1) \cdot q(x_2) : x_1, x_2 \in X \text{ and } x_1 + x_2 = x\}.$$

We also use the following fact.

Proposition 1.2. *Let D be an ordered commutative domain such that D contains an increasing sequence of length $2^\omega + 1$. Then, every nontrivial interval of D is long.*

Proof. It is easy to prove by transfinite induction on $\alpha < (2^\omega)^+$ that for every a, $b \in D$, $a < b$, there is an increasing sequence of type α in the interval $[a, b]$. This implies Proposition 1.2.

Let us also quote the following proposition, which is needed later. An easy proof can be found in [Ci, Lemma 5] or [Ha, p. 6].

Proposition 1.3. *If X and Y_x are short linearly ordered sets for all $x \in X$ then so is the direct sum $S(X\{Y_x\}_{x\in X})$.*

Now, let X be a linearly ordered set, let K be a field and define an ordered domain D by $D = S(S(X, \mathbf{N}), K)$. If $\mathbf{Y} = \{Y_x : x \in X\}$ is a set of different variables and

$$D' = K[\mathbf{Y}] = \bigcup\{K[Y_{x_1}, Y_{x_2}, \ldots, Y_{x_n}] : n \in \mathbf{N} \text{ and } x_1, x_2, \ldots, x_n \in X\},$$

then D and D' are isomorphic (as rings). More precisely, an isomorphism $j: D \to D'$ is defined by

$$j(f) = \sum\{f(g) \cdot (Y_{x_1})^{g(x_1)} \cdot \dots \cdot (Y_{x_n})^{g(x_n)} : g \in \operatorname{supp}(f) \text{ and } \operatorname{supp}(g) = \{x_1, x_2, \dots, x_n\}\}.$$

From the above isomorphism and the theorem that $K[X_1, \dots, X_n]$ is a unique factorization domain (see e.g. [La]) we can easily conclude

Proposition 1.4. *Let* $D = S(S(X, \mathbf{N}), K)$, *where* X *is a linearly ordered set and* K *is a field. Then*

(1) D *is a unique factorization domain*;

(2) *an element* $x \in D$ *is invertible in* D (*i.e.,* x^{-1} *exists in* D) *if and only if* $x \in K\backslash\{0\}$.

We also need the following lemma.

Lemma 1.5. *Let* X *be a linearly ordered set,* K *be a field, and let* $D = S(S(X, \mathbf{N}), K)$. *If* L *is the quotient field of* D *then* $\operatorname{rad}(L) = \operatorname{rad}(K)$.

In particular, if K *is real closed then* $\operatorname{rad}(L) = K^+$.

Proof. Inclusion "$\supset$" is obvious.

To prove the converse inclusion let $x \in \operatorname{rad}(L)$ and let us assume $x \neq 0$.

By Proposition 1.4(1), every element of $L\backslash\{0\}$ can be represented as $cp_1p_2\cdots p_k/q_1q_2\cdots q_m$, where c is invertible in D, p_1, $p_2, \dots, p_k, q_1$, $q_2, \dots, q_m$ are irreducible elements of D, and the number $k+m$ from the above representation is the smallest possible; i.e., the elements p_i and q_j are not associated for every i and j. (Recall that p, $q \in D$ are associated if there exists an invertible element d in D such that $d \cdot p = q$.)

Let $x = cp_1p_2\cdots p_k/q_1q_2\cdots q_m$ be a representation of x as above. Choose arbitrary $n > k+m$ and let $y \in L$ be such that $y^n = x$. Moreover, also let $y = da_1a_2\cdots a_s/b_1b_2\cdots b_t$ be an irreducible representation as above. Hence, $x = y^n = d^n v_1v_2\cdots v_{ns}/w_1w_2\cdots w_{nt}$, where the elements v_i and w_j are not associated for every i and j. So, $k+m = n(s+t)$, which implies that $k = m = s = t = 0$.

In other words we proved that for an arbitrary $n > 0$ there exists an invertible element d in D such that $x = d^n$. But by Proposition 1.4(2), $d \in K$. Hence $x \in \operatorname{rad}(K)$.

2. Construction of the family of fields

Let $\mathbf{B}$ be a transcendence base of $\mathbf{R}$ over $\mathbf{Q}$ and let $\{B_\zeta : \zeta < 2^{2^\omega}\}$ be an enumeration without repetition of all subsets of $\mathbf{B}$ of cardinality 2^ω. Moreover, let $\mathbf{Q}(B_\zeta)$ stand for an algebraic closure of $\mathbf{Q} \cup B_\zeta$ in $\mathbf{R}$. For $\zeta < 2^{2^\omega}$ let $E_\zeta = S(S(2^\omega + 1, \mathbf{N}), Q(B_\zeta))$, an ordered commutative domain. Define K_ζ as the quotient field of E_ζ, and let F_ζ be the quotient field of $D_\zeta = S(S(\mathbf{R}, \mathbf{N}), K_\zeta)$.

We define $\mathbf{F} = \{F_\zeta : \zeta < 2^{2^\omega}\}$.

Proposition 2.1. *The field* F_ζ *is long for every* $\zeta < 2^{2^\omega}$.

Proof. Let us fix $\zeta < 2^{2^\omega}$. Applying Proposition 1.1(1) twice for pair $2^\omega + 1$ and $S(2^\omega + 1, \mathbf{N})$ and for the pair $S(2^\omega + 1, \mathbf{N})$ and E_ζ, we conclude that E_ζ has an increasing subsequence of length $2^\omega + 1$. So by Proposition 1.2, E_ζ is long. But $E_\zeta \subset K_\zeta \subset F_\zeta$. Thus, every F_ζ is long.

Proposition 2.2. *The fields* F_ζ *and* F_ξ *are nonisomorphic for every* $\zeta < \xi < 2^{2^\omega}$.

Proof. Let us assume by way of contradiction that there exists a field isomorphism $j: F_\zeta \to F_\xi$ for some $\zeta < \xi < 2^{2^\omega}$. Thus by Lemma 1.5,

$$\begin{aligned} j(\mathbf{Q}(B_\zeta)^+) &= j(\mathrm{rad}(\mathbf{Q}(B_\zeta))) = j(\mathrm{rad}(K_\zeta)) = j(\mathrm{rad}(F_\zeta)) = \mathrm{rad}(j(F_\zeta)) \\ &= \mathrm{rad}(F_\xi) = \mathrm{rad}(K_\xi) = \mathrm{rad}(\mathbf{Q}(B_\xi)) = \mathbf{Q}(B_\xi)^+. \end{aligned}$$

Hence $j(\mathbf{Q}(B_\zeta)) = \mathbf{Q}(B_\xi)$, which is a contradiction since $\mathbf{Q}(B_\zeta)$ and $\mathbf{Q}(B_\xi)$ are different real closed subfields of $\mathbf{R}$ and hence are not isomorphic. Thus F_ζ and F_ξ are nonisomorphic.

Let us also notice that in the Proposition 2.2 we can talk about fields isomorphisms rather that about ordered fields isomorphisms, since all fields $\mathbf{Q}(B_\zeta)$ are real closed and therefore, their ordered structure can be recovered from their algebraic structure.

3. Construction of the family of short commutative domains

Let us fix $\zeta < 2^{2^\omega}$. We construct 2^{2^ω} nonisomorphic short subdomains $\{P_\eta : \eta < 2^{2^\omega}\}$ of F_ζ such that F_ζ is a quotient field of each P_η.

Let $\{C_\eta : \eta < 2^{2^\omega}\}$ be an enumeration, without repetition, of all subsets of B_ζ, and let i be a bijection between $\mathbf{R}$ and E_ζ. Moreover, for $\eta < 2^{2^\omega}$ and a finite set $A \subset \mathbf{R}$ let $E_{\eta, A}$ be a subdomain of E_ζ generated by $\mathbf{Q}(C_\eta) \subset E_\zeta$ and the set $\{i(r) : r \in A\} \subset E_\zeta$. In particular, $E_{\eta, \varnothing} = \mathbf{Q}(C_\eta)$. Let $G = S(\mathbf{R}, \mathbf{N})$, and let us define

$$P_\eta = S(G, \{E_{\eta, \mathrm{supp}(g)}\}_{g \in G}) \subset D_\zeta.$$

It is not difficult to prove that P_η is closed under multiplication. In fact, this easily follows from the fact that $\mathrm{supp}(g) \cup \mathrm{supp}(h) = \mathrm{supp}(g + h)$ for all $g, h \in G$. For more details see [Ci, Lemma 3]. Thus, P_η is a commutative domain for each $\eta < 2^{2^\omega}$.

Proposition 3.1. P_η *is short for each* $\eta < 2^{2^\omega}$.

Proof. $\mathbf{R}$ and $\mathbf{N}$ are short so, by Proposition 1.3, $G = S(\mathbf{R}, \mathbf{N})$ is also short. Hence again by Proposition 1.3, it is enough to prove that $E_{\eta, \mathrm{supp}(g)}$ is short for every $g \in G$. But let $S = \bigcup\{\mathrm{supp}(i(r)) : r \in \mathrm{supp}(g)\} \subset S(2^\omega + 1, \mathbf{N})$. Let H be a subgroup of $S(2^\omega + 1, \mathbf{N})$ generated by S. Thus, H is at most countable, hence is short. This implies that $S(H, \mathbf{Q}(B_\zeta))$ is short. But $S(H, \mathbf{Q}(B_\zeta))$ is evidently isomorphic to the subdomain

$$D' = \{h \in S(S(2^\omega + 1, \mathbf{N}), \mathbf{Q}(B_\zeta)) : \operatorname{supp}(h) \subset H\} \subset E_\zeta$$

and $E_{\eta, \operatorname{supp}(g)} \subset D'$. Hence $E_{\eta, \operatorname{supp}(g)}$ is short for every $g \in G$ and hence so is P_η.

Proposition 3.2. *The quotient field of P_η is equal to F_ζ for every $\eta < 2^{2^\omega}$.*

Proof. Let us fix $\eta < 2^{2^\omega}$, and let F be the quotient field of P_η in F_ζ. It is enough to prove that $D_\zeta = S(G, K_\zeta) \subset F$.

First notice that for every $y \in E_\zeta$ there exists $r \in \mathbf{R}$ such that $i(r) = y$. This implies that $y \cdot e_r \in P_\eta$ and so $y = y \cdot e_r / e_r \in F$. Thus we have proved that $E_\zeta \subset F$. So K_ζ as the quotient field of E_ζ is also included in F.

This implies that for every $x \in K_\zeta$ and $g \in G$ an element $x \cdot e_g \in F$. But obviously $D_\zeta = S(G, K_\zeta)$ is generated by $\{x \cdot e_g : x \in K_\zeta \text{ and } g \in G\}$, i.e., $D_\zeta \subset F$. This finishes the proof of Proposition 3.2.

As a last step to prove our main theorem we have to prove

Proposition 3.3. *P_ξ and P_η are nonisomorphic for every $\xi < \eta < 2^{2^\omega}$.*

Proof. By Lemma 1.2 we have for every $\eta < 2^{2^\omega}$,

$$\begin{aligned} Q(C_\eta)^+ &= \operatorname{rad}(\mathbf{Q}(C_\eta)) \subset \operatorname{rad}(P_\eta) \subset \operatorname{rad}(F_\eta) \cap P_\eta \\ &= \mathbf{Q}(B_\zeta)^+ \cap P_\eta = \mathbf{Q}(B_\zeta)^+ \cap \mathbf{Q}(C_\eta) = \mathbf{Q}(C_\eta)^+. \end{aligned}$$

In other words

$$\mathbf{Q}(C_\eta)^+ = \operatorname{rad}(P_\eta) \quad \text{for every } \eta < 2^{2^\omega}.$$

By way of contradiction let us assume that there exists a field isomorphism $j \colon P_\xi \to P_\eta$ for some $\xi < \eta < 2^{2^\omega}$. Thus by the above remark,

$$j(\mathbf{Q}(C_\xi)^+) = j(\operatorname{rad}(P_\xi)) = \operatorname{rad}(P_\eta) = \mathbf{Q}(C_\eta)^+.$$

Hence $j(\mathbf{Q}(C_\xi)) = \mathbf{Q}(C_\eta)$, which is impossible since $\mathbf{Q}(C_\xi)$ and $\mathbf{Q}(C_\eta)$ are real closed subfields of $\mathbf{R}$ and are different as $C_\xi \neq C_\eta$. Thus P_ξ and P_η are nonisomorphic.

This finishes the proof of our main theorem.

Theorem 1. *There exists a family $\mathbf{F}$ of power 2^{2^ω} of long ordered fields of size 2^ω that are pairwise nonisomorphic and such that every field F from $\mathbf{F}$ has 2^{2^ω} nonisomorphic short subdomains whose field of quotients is F.*

4. Generalizations for higher cardinals

Let κ be an infinite cardinal. We say that a linearly ordered set is κ-short if it does not contain any monotonic sequence of length κ^+, and it is κ-long if it contains a monotone sequence of length α for every ordinal $\alpha < (2^\kappa)^+$.

The following theorem is a generalization of Theorem 1 for arbitrary κ replacing ω.

Theorem 2. *Let κ be an infinite cardinal. Then there exists a family $\mathbf{F}$ of power 2^{2^κ} of κ-long ordered fields of size 2^κ that are pairwise nonisomorphic and such that every field $F \in \mathbf{F}$ has 2^{2^κ} nonisomorphic κ-short subdomains whose field of quotients is F.*

The proof of the Theorem 2 is based on the same concept as that of Theorem 1. However, it also needs some new ideas and thus we sketch its proof.

First we need the following propositions:

Propositions 4.1. *There exists a family $\mathbf{X}$ of 2^{2^κ} nonisomorphic linear orderings of cardinality 2^κ each containing an increasing sequence of length $2^\kappa + 1$.*

Proof. Let $P = (2^\kappa + 1) \times [0, 1]$ be ordered lexicographically. For every $A \subset 2^\kappa$ let $X_A = \{(\xi, r) \in P : (\xi \in A \cup \{2^\kappa\}) \Rightarrow (r = 1)\} \subset P$. It is not difficult to see that $\mathbf{X} = \{X_A : A \subset 2^\kappa\}$ satisfies our requirements.

For linearly ordered sets X and Y let $X \oplus Y$ be the linearly ordered set $X \cup Y$ such that the orderings on X and Y are preserved and $x < y$ for every $x \in X$ and $y \in Y$. Let us also notice that the family $\mathbf{X}$ from Proposition 4.1 has the following stronger property:

Proposition 4.2. *Let Y_1 and Y_2 be linearly ordered sets, and let $X_1, X_2 \in \mathbf{X}$ where $\mathbf{X}$ satisfies the conclusion of Proposition 4.1. If $j: X_1 \oplus Y_1 \to X_2 \oplus Y_2$ is an order isomorphism then $X_1 = X_2$ and $j(Y_1) = Y_2$.*

We also need the following theorem of Todorcevic [To], which is an answer to a problem asked in an earlier version of this paper.

Theorem 3. *For every infinite cardinal κ there exists a κ-short linearly ordered set Y of size 2^κ with 2^{2^κ} nonisomorphic rigid linear suborderings of power 2^κ.*

For the sake of completeness and the fact that [To] may not be widely available, we reproduce the proof of Theorem 3 in §5 of this paper.

Proof of Theorem 2. Let us fix an infinite cardinal, let Y be a κ-short linearly ordered set of cardinality 2^κ satisfying Theorem 3, and let $\mathbf{X}$ be from Proposition 4.1.

For $X \in \mathbf{X}$ let us define $G(X) = S(X \oplus Y, \mathbf{Q})$ and let

$$K_X = \{f \in \mathbf{R}^{G(X)} : \operatorname{supp}(f) \text{ is decreasingly well ordered in type } < \omega_1\}.$$

Lemma 4.3. *K_X is a real closed κ-long ordered field of cardinality 2^κ for every $X \in \mathbf{X}$.*

Proof. Let us fix $X \in \mathbf{X}$. By Proposition 1.1(3) $G(X)$ is a divisible group. Thus K_X is a real closed field (see [A1, Corollary 3.2]). By the choice of X it also contains increasing sequence of length $2^\kappa + 1$, and the same argument as for Proposition 1.2 shows that K_X is κ-long. The cardinality of K_X is obvious.

Lemma 4.4. *The fields* K_{X_1} *and* K_{X_2} *are nonisomorphic for different* X_1, $X_2 \in \mathbf{X}$.

Proof. Let us assume that there exists a field isomorphism $j: K_{X_1} \to K_{X_2}$ for some X_1, $X_2 \in \mathbf{X}$. We prove that this implies $X_1 = X_2$.

We start with the following:

Claim. There exists an ordered group isomorphism $i: S(X_1 \oplus Y, \mathbf{Q}) \to S(X_2 \oplus Y, \mathbf{Q})$.

Proof of Claim. By definition $G(X_k) = S(X_k \oplus Y, \mathbf{Q})$. Thus we have to find an ordered group isomorphism $i: G(X_1) \to G(X_2)$.

Let $\sim$ be the equivalence relation defined by $a \sim b$ if and only if there exist natural numbers n and m such that $a < n \cdot b$ and $b < m \cdot a$, with the equivalence class of a denoted by $[a]$. Then, $K_{X_k}/_\sim = \{[a] : a \in (K_{X_k})^+, a > 0\}$ is an ordered group.

Next, notice that j preserves the orderings of K_{X_1} and K_{X_2} as these fields are real closed. Moreover, $G(X_k)$ is isomorphic, as an ordered group, with the group $K_{X_k}/_\sim$. But an isomorphism between K_{X_1} and K_{X_2} induces an isomorphism between $K_{X_1}/_\sim$ and $K_{X_2}/_\sim$. This finishes the proof of the claim.

Since $i: S(X_1 \oplus Y, \mathbf{Q}) \to S(X_2 \oplus Y, \mathbf{Q})$ is an ordered group isomorphism, we can repeat the proof the claim to show the existence of an order isomorphism between $S(X_k \oplus Y, \mathbf{Q})/_\sim$ and $X_k \oplus Y$ for $k = 1, 2$ and between $S(X_1 \oplus Y, \mathbf{Q})/_\sim$ and $S(X_2 \oplus Y, \mathbf{Q})/_\sim$ where $\sim$ is defined similarly as in the claim. This gives us an order isomorphism between $X_1 \oplus Y$ and $X_2 \oplus Y$. But by Proposition 4.2, this implies that $X_1 = X_2$. This finishes the proof of Lemma 4.4.

Now for $X \in \mathbf{X}$ let F_X be a quotient field of $S(S(Y, \mathbf{N}), K_X)$ and let us define

$$\mathbf{F} = \{F_X : X \in \mathbf{X}\}.$$

Evidently $F_X \supset K_X$ is κ-long. Also, if F_{X_1} and F_{X_2} are isomorphic for some X_1, $X_2 \in \mathbf{X}$ then, by Lemmas 1.5 and 4.3, K_{X_1} and K_{X_2} are isomorphic as well. But by Lemma 4.4, this implies that $X_1 = X_2$. In other words, different fields from $\mathbf{F}$ are nonisomorphic.

Now let us fix $X \in \mathbf{X}$ and put $F = F_X$. To finish the proof of Theorem 2 we have to find 2^{2^κ} nonisomorphic κ-short subdomains of F for which a field of quotient is F.

Let $\mathbf{Z}$ be a family of 2^{2^κ} nonisomorphic subsets of Y of cardinality 2^κ. For $Z \in \mathbf{Z}$ let

$$L_Z = \{f \in \mathbf{R}^{S(Z, \mathbf{Q})} : \text{supp}(f) \text{ is decreasingly well ordered in type } < \omega_1\}.$$

As in Lemma 4.3 we conclude that L_Z is a real closed field of cardinality 2^κ. Notice also that if $T = \{f \in L_Z : \text{supp}(f)$ has at most one element$\}$ then L_Z

can be identified with decreasing sequences from $T^{<\omega_1}$ where $T^{<\omega_1}$ is ordered lexicographically. Thus, similarly as in Lemma 4.5, we can show that L_Z is κ-short. Moreover, up to isomorphism, $L_Z \subset K_X$ as $S(Z, \mathbf{Q})$ is isomorphic to a subgroup of $S(X \oplus Y, \mathbf{Q})$.

Let i be a bijection between Y and F, and for a finite set $A \subset Y$ let E_A be the subdomain of F generated by $L_Z \subset F$ and the set $\{i(r) : r \in A\} \subset F$. In particular $E_\varnothing = L_Z$. Let us define

$$S_Z = S(S(Y, \mathbf{N}), \{E_{\mathrm{supp}(g)}\}_{g \in S(Y, \mathbf{N})}) \subset F.$$

As Proposition 1.3 is true also for κ-short linear orderings we easily conclude that $S(Y, \mathbf{N})$ is κ-short. Also, as $E_{\mathrm{supp}(g)}$ is generated by κ-short D_Z and finite numbers of elements, it is not difficult to prove that $E_{\mathrm{supp}(g)}$ is κ-short for every $g \in S(Y, N)$. Hence S_Z is κ-short.

The proofs that S_Z is a subdomain of F and that the quotient field of S_Z is F are similar to that presented in §3 for $\kappa = \omega$ and are left to the reader.

Finally, let S_{Z_1} and S_{Z_2} be isomorphic for some $Z_1, Z_2 \in \mathbf{Z}$. Then, as in Proposition 3.3, we can prove that L_{Z_1} and L_{Z_2} are isomorphic. But now, repeating the argument from Lemma 4.4, we conclude that Z_1 and Z_2 are isomorphic, so $Z_1 = Z_2$.

This finishes the proof of the Theorem 2.

5. Proof of the Theorem 3

The purpose of this section is to reproduce the general construction of [To] from which Theorem 3 follows. Familarity with [Je, §1.7] or [Ku, §2.6] is assumed.

Fix a regular uncountable cardinal θ that is not inaccessible, and let $\lambda < \theta$ be the minimal cardinal with the property $2^\lambda \geq \theta$. Let $\lim(\omega)$ be the class of all ordinals of cofinality ω. For each δ in $\lim(\omega) \cap \theta$ fix a strictly increasing function $x_\delta \colon \omega \to \delta$ cofinal in δ. Let $\ll$ be a linear ordering of θ such that $(\theta, \ll)$ is isomorphic to a subset of $(\{0, 1\}^\lambda, \leq_{\mathrm{lex}})$ where $\leq_{\mathrm{lex}}$ stands for the lexicographical order. As in Proposition 1.3 we argue that $(\theta, \ll)$ is λ-short.

Now for each $S \subset \lim(\omega) \cap \theta$ we define

$$L(S) = \{x_\delta : \delta \in S\}$$

ordered lexicographically with respect to $\ll$, i.e.,

$$x_\delta < x_\gamma \text{ if and only if } x_\delta(n) \ll x_\gamma(n), \text{ where } n = \min\{m : x_\delta(m) \neq x_\gamma(m)\}.$$

It is easy to see that

Proposition 5.1. *$L(S)$ is λ-short for every $S \subset \lim(\omega) \cap \theta$.*

We also need the following propositions:

Proposition 5.2. *If $S \subset \lim(\omega) \cap \theta$ is stationary then there is $r(S) \subset S$ with $S \backslash r(S)$ being nonstationary such that every nontrivial interval of $L(r(S))$ contains x_δ for a stationary set of δ's in $r(S)$.*

Proof. Let S_0 be the set of those $\delta \in S$ such that for some natural number n the set $\{\gamma \in S : x_{\delta|n} \subset x_\gamma\}$ is nonstationary. First let us show that S_0 is nonstationary.

So by way of contradiction, let us assume that S_0 is stationary. Thus, there exists a stationary set $T \subset S$ such that the number n from the above definition is the same for every $\delta \in T$. Now using the Pressing-Down Lemma n-times (see [Ku, Lemma 6.15, p. 80]) for the regressive functions $f_i(x_\delta) = x_\delta(i)$ we can find a stationary set $T' \subset T$ such that $x_{\delta|n} = x_{\gamma|n}$ for every δ, $\gamma \in T'$. But this implies that for $\delta \in T' \subset S_0$ the set $\{\gamma \in S : x_{\delta|n} \subset x_\gamma\}$ contains T', i.e., is stationary. This gives the desired contradiction.

Now for δ, $\gamma \in S \backslash S_0$ let $I(\delta, \gamma)$ denote the set of those $\alpha \in S \backslash S_0$ such that x_α belongs to the open interval in $L(S \backslash S_0)$ with endpoints x_δ and x_γ. Notice that if $I(\delta, \gamma)$ is nonstationary for some δ, $\gamma \in S \backslash S_0$ then $I(\delta, \gamma)$ is empty. Otherwise, if $x_\alpha \in I(\delta, \gamma)$ then for some natural number n the set $\{\gamma \in S \backslash S_0 : x_{\alpha|n} \subset x_\gamma\}$ is a subset of the interval with endpoints x_δ and x_γ, i.e., $x_\alpha \in S_0$.

Let S_1 be the set of those $\gamma \in S \backslash S_0$ such that for some $\delta < \gamma$, $\delta \in S \backslash S_0$, the set $\{\alpha : x_\alpha \in I(\delta, \gamma)\}$ is nonstationary. If S_1 is nonstationary then $r(S) = S \backslash (S_0 \cup S_1)$ satisfies the required properties. So by way of contradiction, let us assume that S_1 is stationary, and let f be the regressive function f on S_1 defined by $f(\gamma) = \delta$ where $\delta < \gamma$ is such that $I(\delta, \gamma)$ is empty. By the Pressing-Down Lemma there exists a stationary set $T \subset S_1$ such that f is constant on T. But this is impossible because at most two of the intervals $I(f(\gamma), \gamma)$ for $\gamma \in T$ would be empty. This finishes the proof of Proposition 5.2.

Proposition 5.3. *If $S \subset \lim(\omega) \cap \theta$ is stationary then $L(S)$ is not isomorphic to a subset of $(\{0, 1\}^\lambda, \leq_{\text{lex}})$.*

Proof. By the minimality of $\lambda < \theta$ and the fact that θ is regular, $2^{<\lambda} < \lambda$. Thus, $(\{0, 1\}^\lambda, \leq_{\text{lex}})$ does not contain λ disjoint intervals. On the other hand the set $\{x_{\delta|n} : \delta \in S \text{ and } n \in \mathbf{N}\}$ has cardinality $\theta \geq \lambda$ and, by Proposition 5.2, for a stationary set of δ's the set $\{x_\gamma : x_{\delta|n} \subset x_\gamma\}$ contains a nonempty interval. Thus, $L(S)$ contains λ disjoint nonempty intervals.

Proposition 5.4. *If there is an order embedding of $L(S_0)$ into $L(S_1)$ then $S_0 \backslash S_1$ is nonstationary in θ.*

Proof. Assume that $S = S_0 \backslash S_1$ is stationary and fix strictly increasing $f : L(S_0) \to L(S_1)$. Since f is one-to-one, the Pressing-Down Lemma implies that the set of those δ in S for which $f(x_\delta) = x_\gamma$ and $\delta \geq \gamma$ (so $\delta > \gamma$, since $\delta \neq \gamma$) is nonstationary.

So we may find a stationary set $S' \subset S$ such that for every δ in S' if $f(x_\delta) = x_\gamma$ then $\delta < \gamma$. Moreover, we can also assume, choosing a subset of S', if necessary, that for some natural number n if $f(x_\delta) = x_\gamma$ then $x_\gamma(n-1) < \delta \leq x_\gamma(n)$. Furthermore, applying the Pressing-Down Lemma n-times, we can find a stationary set $S'' \subset S'$ and a function $t\colon n \to \theta$ such that $t \subset x_\gamma$ for every γ such that $\delta \in S''$ and $f(x_\delta) = x_\gamma$. Moreover, we may assume that if $\delta < \delta'$ are in S'' and if $f(x_\delta) = x_\gamma$ then $\delta \leq x_\gamma(n) < \gamma < \delta'$. But this means that the lexicographical ordering of $f(L(S''))$ is determined by the n th coordinate and therefore, is isomorphic with the subset of $(\{0, 1\}^\lambda, \leq_{\text{lex}})$, which contradicts Proposition 5.3.

As an immediate corollary from Propositions 5.3 and 5.4 we obtain the following conclusion.

Proposition 5.5. *Let $S \subset \lim(\omega) \cap \theta$ be stationary in θ. If $r(S) \subset S$ is as in Proposition 5.2 then $L(r(S))$ is rigid.*

Now notice that there exist a family $\mathbf{S}_0$ of size 2^θ of stationary subsets of $\lim(\omega) \cap \theta$ such that symmetric difference $S_0 \Delta S_1$ is stationary for all distinct S_0, $S_1 \in \mathbf{S}_0$. This follows from the fact that $\lim(\omega) \cap \theta$ is the union of θ disjoint stationary sets (see [Je, Lemma 7.6, p. 59]). Put $Y_\theta = L(\lim(\omega) \cap \theta)$ and let $\mathbf{S}_\theta = \{r(S) : S \in \mathbf{S}_0\}$.

If 2^κ is regular then put $\theta = 2^\kappa$ and $Y = Y_\theta$ satisfies the requirements of Theorem 3 as $\{L(S) : S \in \mathbf{S}_\theta\}$ is a family of rigid pairwise nonisomorphic subsets of Y.

Suppose 2^κ is singular with cofinality θ, and fix a sequence θ_ξ $(\xi < \theta)$ of successor cardinals increasingly converging to 2^κ such that $\theta_0 > \theta$. For each ξ fix a family $\mathbf{S}_{\theta_\xi}$ as above and let $K = \{k_\xi : \xi < \theta\}$ be a suborder of $(\{0, 1\}^\kappa, \leq_{\text{lex}})$. Let Y be an ordering obtained by inserting $L(\lim(\omega) \cap \theta_\xi)$ in place of k_ξ $(\xi < \theta)$ in K. It is not difficult to see that Y is κ-short and has cardinality 2^κ. Moreover, if for each $\xi < \theta$ we choose $S_\xi \in \mathbf{S}_{\theta_\xi}$, and taking only $L(S_\xi)$ in place of k_ξ, we get (compare Proposition 5.5) a rigid linear subordering of Y. Since there are 2^{2^κ} different choices of S_ξ's, and different choices give nonisomorphic orders (compare Proposition 5.4), Theorem 3 is proved.

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