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Density Continuity versus Continuity

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Abstract. Real-valued functions of a real variable which are continuous with respect to the density topology on both the domain and the range are called density continuous. A typical continuous function is nowhere density continuous. The same is true of a typical homeomorphism of the real line. A subset of the real line is the set of points of discontinuity of a density continuous function if and only if it is a nowhere dense F_σ set. The corresponding characterization for the approximately continuous functions is a first category F_σ set. An alternative proof of that result is given. Density continuous functions belong to the class Baire*1, unlike the approximately continuous functions.

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1. Introduction

The density topology is a completely regular refinement of the natural topology on the real line. It consists of all measurable subsets A of $\mathbb{R}$ such that, for every $x \in A$, x is a density point of A. Ostaszewski [7, 8] studied the class of functions $f: \mathbb{R} \to \mathbb{R}$ which are continuous with respect to the density topology on the domain and the range. These are termed *density continuous*. Bijections of the real line whose inverses are density continuous were investigated by Bruckner [2] and Niewiarowski [5]. Ostaszewski [9] considered the class as a semigroup with composition as the operation, and showed that the semigroup, and three of its subsemigroups, have the inner automorphism property. Ciesielski and Larson [4], and Burke [3] showed that real-analytic functions are density continuous, and that the class of density continuous functions is not a linear space. Furthermore, there exist C^∞ functions which are not density continuous.

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In this work we are concerned with the relationship between the classes of continuous and density continuous functions.

We will use the following notation:

$\mathbb{R}$ – the set of real numbers;
$\mathbb{N}$ – the set of natural numbers;
$\mathscr{C}$ – the space of continuous functions $f\colon [0,1] \to \mathbb{R}$;
$\|f\|$ – the norm of an $f \in \mathscr{C}$, $\|f\| = \sup_{x\in[0,1]}|f(x)|$;
$C(f)$ – the set of points at which f is continuous;
$Z(f)$ – the set of points at which f is not continuous;
$\omega(f, x)$ – the oscillation of f at x;
$\mathrm{supp}(f) = \{x : f(x) \neq 0\}$ – the support of f;
$\mathscr{H}$ – the space of all automorphisms of $[0,1]$ equipped with the metric

$$\sigma(g, h) = \|g - h\| + \|g^{-1} - h^{-1}\|$$

for $g, h \in \mathscr{H}$;
$|A|$ – the Lebesgue measure of a measurable set $A \subset \mathbb{R}$;
A^c – the complement of the set A;
$\mathrm{int}(A)$ – the interior of the set A;
$\bar{d}(A, x)$, $\underline{d}(A, x)$, $d(A, x)$ – the upper, lower, and ordinary (respectively) densities of a set $A \subset \mathbb{R}$ at a point $x \in \mathbb{R}$.

2. Typical continuous functions

In this section we prove that a typical continuous function is nowhere density continuous. The same is true of a typical homeomorphism of the real line. To do this, some preliminary definitions and lemmas must be presented.

Let $\{J_n\}$ be a sequence of intervals and let $\{I_n\}$ be a sequence of closed intervals such that I_n and J_n have the same center and $I_n \subset J_n$, for each n. We say that the sequence J_n *captures* the sequence I_n. This relationship between the sequences is denoted $I_n \lhd J_n$.

If $I_n \lhd J_n$, as above, we define

$$J_x = \bigcup_{\{n : x \notin J_n\}} I_n .$$

The properties of captured sequences which are useful in what follows are contained in the following two propositions.

Lemma 1. *If $\{I_n\}$ and $\{J_n\}$ are sequences of intervals such that $I_n \lhd J_n$ and*

$$\sum_{n\in\mathbb{N}} \frac{|I_n|}{|J_n|} < \infty ,$$

then $d(J_x, x) = 0$, $\forall x \in \mathbb{R}$.

Proof. Without loss of generality, we may assume that $x \notin \bigcup_{n\in\mathbb{N}} J_n$. Let $\varepsilon \in (0, 1)$ and choose n_0 and $\delta_0 > 0$ such that

$$(1) \qquad \sum_{n \geq n_0} \frac{|I_n|}{|J_n|} < \varepsilon/3 \quad \text{and} \quad (x - \delta_0, x + \delta_0) \cap I_n = \emptyset, \ \forall n \leq n_0 .$$

Observe that the choice of $\varepsilon \in (0, 1)$ and (1) guarantees that for all $n \geq n_0$, it is true that if $(x - \delta, x + \delta) \cap I_n \neq \emptyset$, then $J_n \subset (x - 3\delta, x + 3\delta)$.

Let

$$S_\delta = \{n : (x - \delta, x + \delta) \cap I_n \neq \emptyset\} \quad \text{and} \quad M_\delta = \sup_{n \in S_\delta} |J_n| .$$

If $\delta \in (0, \delta_0)$, the observation and (1) show

$$\begin{aligned} \frac{|(x - \delta, x + \delta) \cap J_x|}{2\delta} &\leq \frac{|\bigcup_{n \in S_\delta} I_n|}{2\delta} \\ &\leq 3 \frac{\sum_{n \in S_\delta} |I_n|}{|(x - 3\delta, x + 3\delta)|} \\ &\leq 3 \frac{\sum_{n \in S_\delta} |J_n| |I_n| / |J_n|}{\bigcup_{n \in S_\delta} |J_n|} \\ &\leq 3 \frac{M_\delta \sum_{n \in S_\delta} |I_n| / |J_n|}{M_\delta} \\ &< \varepsilon . \end{aligned}$$

From this, Lemma 1 follows at once.

It is interesting to note that the following can be proved in much the same way as Lemma 1.

Corollary 1. *If I_n and J_n are sequences of intervals such that $I_n \lhd J_n$, $J_i \cap J_j = \emptyset$ when $i \neq j$ and $|I_n|/|J_n| \to 0$, then $\bigcup_{n=1}^{\infty} I_n$ is density closed.*

Lemma 2. *Let $x \in \mathbb{R}$ and let $\{L_n\}$ be a sequence of intervals such that $x \in \bigcap_{n \geq 1} L_n$ and $\lim_{n \to \infty} |L_n| = 0$. If K_n is a subinterval of L_n for every n and*

$$\limsup_{n \to \infty} |K_n| / |L_n| > 0 ,$$

then $\bar{d}(\bigcup_{n \geq 1} K_n, x) > 0$.

Proof. Let

$$\limsup_{n \to \infty} |K_n| / |L_n| = a > 0 .$$

It will be shown that

$$\bar{d}(\bigcup_{n \in \mathbb{N}} K_n, x) = \limsup_{\delta \to 0+} \frac{|(x - \delta, x + \delta) \cap \bigcup_{n \in \mathbb{N}} K_n|}{2\delta} \geq a/4 .$$

To do this, it is enough to show that for every $\delta_0 > 0$ there is a $\delta \in (0, \delta_0]$ and a number

$n \geq 1$ such that

$$|K_n \cap (x-\delta, x+\delta)|/2\delta \geq a/4.$$

Let n be such that $|L_n| < \delta_0$ and $|K_n|/|L_n| > a/2$. Set $\delta = \inf\{t : L_n \subset (x-t, x+t)\}$. Then $0 < \delta < \delta_0$, $L_n \subset [x-\delta, x+\delta]$ and $|L_n|/2\delta \geq 1/2$. Hence,

$$\frac{|K_n \cap (x-\delta, x+\delta)|}{2\delta} = \frac{|K_n|}{2\delta} = \frac{|K_n|}{|L_n|}\frac{|L_n|}{2\delta} \geq \frac{a}{2}\frac{1}{2} = \frac{a}{4}$$

and the lemma is proved.

Here is the main result of this section.

Theorem 1. *If $\mathscr{C}_{\mathscr{D}}$ denotes the subset of $\mathscr{C}$ consisting of all functions which have at least one point of density continuity, then $\mathscr{C}_{\mathscr{D}}$ is a first category subset of $\mathscr{C}$.*

Proof. We will show that there exists a dense G_δ subset E of $\mathscr{C}$ such that every $f \in E$ is nowhere density continuous.

For every $n \in \mathbb{N}$ denote by D_n the set of all $f \in \mathscr{C}$ such that for every $i = 1, 2, \ldots, 2^n$, f is linear and nonconstant on each interval $[(i-1)2^{-n}, i2^{-n}]$. Notice that $D_{n+1} \subset D_n$ for every $n \in \mathbb{N}$ and $D = \bigcup_{n \in \mathbb{N}} D_n$ is a dense subset of $\mathscr{C}$.

For $f \in \mathscr{C}$ define

$$\|f\|_n = \max_{i=1,2,\ldots,2^n} |f(i2^{-n}) - f((i-1)2^{-n})|. \tag{2}$$

We claim that for each open set U in $\mathscr{C}$, there exists an $n \in \mathbb{N}$ and a function $f \in D_n$ such that the ball in $\mathscr{C}$ centered at f of radius $\|f\|_n$ is entirely contained in U. To see this, first find an $m \in \mathbb{N}$ and an $f \in D_m$ such that $f \in U$. Since U is open, there is a $\delta > 0$ such that the open ball of radius δ centered at f is contained in U. Using the uniform continuity of f, we can find an $n > m$ such that whenever $|x-y| < 2^{-n}$, then $|f(x) - f(y)| < \delta$. From this it is clear that $f \in D_n$ and $\|f\|_n < \delta$. The claim is evident.

We will now start the construction of the promised G_δ set E as an intersection of dense open sets, W_k.

Let $k \geq 1$ and U be a nonempty open subset of $\mathscr{C}$, and choose f and n as above. For $j = 0, 1, 2, \ldots, 2^{n+1}$, define

$$g\left(\frac{j}{2^{n+1}}\right) = f\left(\frac{j}{2^{n+1}}\right).$$

If $i2^{-n} \leq j2^{-n-1} < (j+1)2^{-n-1} \leq (i+1)2^{-n}$, where $i \in \{0, 1, 2, \ldots, 2^n - 1\}$, put $L_i = (i2^{-n}, (i+1)2^{-n})$, $M_j = (j2^{-n-1}, (j+1)2^{-n-1})$ and let $K_j = [a_j, b_j]$ be centered in M_j such that

$$\frac{|K_j|}{|M_j|} = 1 - \frac{1}{2^n} = \frac{2|K_j|}{|L_i|}.$$

Let us choose $I_j^0 = [c_j, d_j]$ centered in the interval $f(M_j)$ and such that

$$\frac{|I_j^0|}{|f(M_j)|} = \frac{1}{2^n}.$$

Define g to be linear on each of the intervals

$$[j2^{-n-1}, a_j], [a_j, b_j] \quad \text{and} \quad [b_j, (j+1)2^{-n-1}],$$

such that $g([a_j, b_j]) = [c_j, d_j] = I_j^0$. Thus, if $J_j = f(M_j) = g(M_j)$, then

$$\frac{|g(K_j)|}{|g(M_j)|} = \frac{|I_j^0|}{|J_j|} = \frac{1}{2^n}$$

and

$$\frac{|g^{-1}(I_j^0)|}{|g^{-1}(J_j)|} = \frac{|K_j|}{|M_j|} = 1 - \frac{1}{2^n}.$$

Notice that g is contained in the open ball centered at f of radius $\|f\|_n$. Thus, $g \in U$. Let W_U^k be the open ball centered at g of radius

(3) $$\varepsilon_k = 2^{-n-1} \min_{i=1,2,\ldots,2^n} \left| f\left(\frac{i}{2^n}\right) - f\left(\frac{i-1}{2^n}\right) \right| > 0.$$

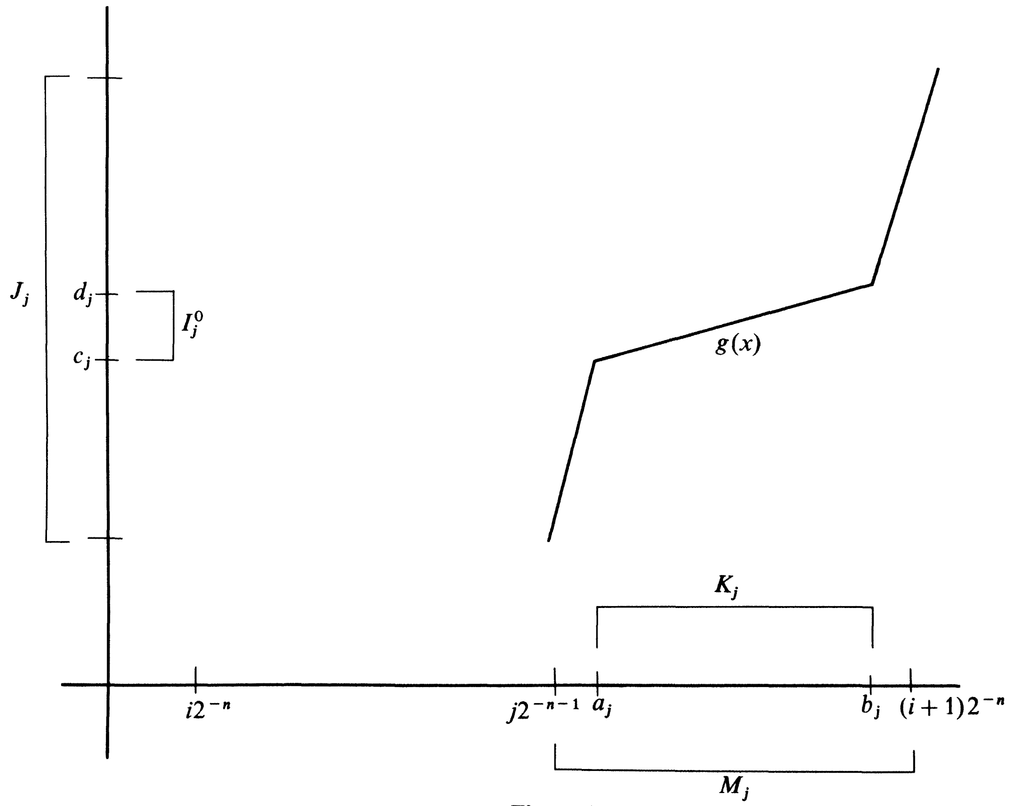

Figure 1

Obviously $W_k = \bigcup\{W_U^k : U$ is open and nonempty in $\mathscr{C}\}$ is open and dense in $\mathscr{C}$, so that $E = \bigcap_{k\in\mathbb{N}} W_k$ is a residual set in $\mathscr{C}$. We will show that if $h \in E$ then h is nowhere density continuous.

Now let x be an arbitrary point of $[0,1]$. We will choose intervals I_m, $m \in \mathbb{N}$ such that

$$d\left(\bigcup_{m\in\mathbb{N}} I_m, h(x)\right) = 0,$$

and

$$\bar{d}\left(h^{-1}\left(\bigcup_{m\in\mathbb{N}} I_m\right), x\right) > 0.$$

This will prove that h is not density continuous at x.

Let $m \in \mathbb{N}$. We have $h \in W_m$ so there exists a set U, open in $\mathscr{C}$, such that $h \in W_U^m$. Let g be the center of W_U^m. Let $n \geq m$ be the number given in the construction of W_U^m. Let $i \in \{0, 1, 2, \dots, 2^n - 1\}$ be such that $x \in [i2^{-n}, (i+1)2^{-n}]$.

Put $L_m = [i2^{-n}, (i+1)2^{-n}]$. Let $M^1 = (2i2^{-n-1}, (2i+1)2^{-n-1})$, $M^2 = ((2i+1)2^{-n-1}, 2(i+1)2^{-n-1})$ and $M_m \in \{M^1, M^2\}$ such that $h(x) \notin g(M_m)$. Put $J_m = g(M_m)$ and let $I_m^0 = [c_j, d_j]$ and $K_m = [a_j, b_j]$ be as in the construction of g. Thus

$$\frac{|I_m^0|}{|J_m|} = 1/2^n \leq 1/2^m \quad \text{and} \quad \frac{|K_m|}{|M_m|} = 1 - 1/2^n \geq 1 - 1/2^m.$$

Put $I_m = [c_j - \varepsilon_m, d_j + \varepsilon_m]$. As $h(x) \notin J_m$ for every m and

$$\sum_{m=1}^{\infty} \frac{|I_m|}{|J_m|} \leq \sum_{m=1}^{\infty} \frac{3|I_m^0|}{|J_m|} \leq 3 \sum_{m=1}^{\infty} \frac{1}{2^m} = 3,$$

Lemma 1 yields $d(\bigcup_{m=1}^{\infty} I_m, h(x)) = 0$.

On the other hand, by the choice of ε_m, $K_m \subset h^{-1}(I_m)$. Thus, by Lemma 2, the fact that $x \in L_m$ for every m and using

$$\lim_{m\to\infty} \frac{|K_m|}{|L_m|} = \lim_{m\to\infty} \frac{|K_m|}{2|M_m|} = 1/2 > 0$$

we have

$$\bar{d}\left(h^{-1}\left(\bigcup_{m=1}^{\infty} I_m\right), x\right) \geq \bar{d}\left(\bigcup_{m=1}^{\infty} K_m, x\right) > 0.$$

Therefore, h is not density continuous at x.

Theorem 2. *If $\mathscr{H}_{\mathscr{D}}$ denotes the class of all elements of $\mathscr{H}$ which have at least one point of density continuity, then $\mathscr{H}_{\mathscr{D}}$ is a first category subset of $\mathscr{H}$.*

Proof. As discussed in [10, page 50], $\mathscr{H}$ is a G_δ subset of $\mathscr{C}$. It is actually complete with the metric σ defined in the introduction of this work.

Let W be the dense G_δ subset of $\mathscr{C}$ constructed in the proof of Theorem 1. It is obvious that $W \cap \mathscr{H}$ is a G_δ subset of $\mathscr{H}$. Thus, it would be sufficient to show that $W \cap \mathscr{H}$ is dense in $\mathscr{H}$ in order to prove Theorem 2.

Unfortunately, in general, this is not the case. However, the set $D \cap \mathscr{H}$ is dense in $\mathscr{H}$. Thus, if in the choice of f in the proof of Theorem 1, we assume additionally that for nonempty $U \cap \mathscr{H}$ we choose $f \in U \cap \mathscr{H} \cap D$, then the corresponding function g will be also in $\mathscr{H}$. $\mathscr{H} \cap W$ will be dense in W. This proves Theorem 2.

Let us note that the fact that a typical homeomorphism is not density continuous is mentioned in [7], but without a detailed proof.

3. Continuity of density continuous functions

In this section, the set on which a density continuous function can be continuous is characterized as any nowhere dense F_σ set.

A function $f: \mathbb{R} \to \mathbb{R}$ is in the class Baire*1 if for each perfect set P, there is a portion Q of P such that $f|_Q$ is continuous. In other words, f is continuous on a relative subinterval of each closed set. This class was introduced by Richard O'Malley [6], who studied the Baire*1 functions having the Darboux property.

Theorem 3. *If f is a density continuous function, then f is in Baire*1.*

Proof. We assume the theorem is not true. Then there is a nonempty perfect set P such that

$$Z = \{x \in P : f|_P \text{ is not continuous at } x\}$$

is dense in P. We will show that this assumption assures that there is an $x \in P$ such that f is not density continuous at x. The proof uses induction to find a sequence $x_n \in P$ a sequence of open intervals (a_n, b_n) and two sequences of compact intervals, I_n and J_n such that $x_n \in I_n \subset J_n$, $I_n \lhd J_n$ and $x_n \to x$.

To start, let $x_0 \in Z$, $J_0 = I_0 = \emptyset$ and $(a_0, b_0) = (x_0 - 1, x_0 + 1)$. Assume that x_i, closed intervals J_i and I_i, and an open interval (a_i, b_i) have been chosen for $1 \leq i \leq n$ to satisfy the following properties:

(a) $f(x_i) \in I_i \subset J_i$;
(b) $J_{i-1} \cap J_i = \emptyset$;
(c) $0 < |I_i| \leq |J_i|/2^i$ and $|J_i| < \omega(f|_P, x_i)$;
(d) $x_i \in (a_i, b_i) \cap Z \subset [a_i, b_i] \subset (a_{i-1}, b_{i-1})$;
(e) $b_i - a_i < 1/2^i$; and,
(f) $|f^{-1}(I_i) \cap (a_i, b_i)| > (1 - 2^{-i})(b_i - a_i)$.

To continue with the inductive step, we note that from (c), we are able to choose

$$y \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n).$$

If $y \in Z$, then let $x_{n+1} = y$. Otherwise, $f|_P$ is continuous at y. In this case, the fact that

Z is dense in P guarantees the existence of

$$x_{n+1} \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n) \cap Z.$$

Because J_n is closed and $x_{n+1} \in Z$, there is a closed interval J_{n+1} centered at $f(x_{n+1})$ such that $J_{n+1} \cap J_n = \emptyset$ and $0 < |J_{n+1}| < \omega(f|_P, x_{n+1})$. Setting I_{n+1} to be the closed interval centered at $f(x_{n+1})$ with length $|J_{n+1}|/2^{n+1}$, it follows that (a), (b) and (c) are true with $i = n+1$. Next, use the approximate continuity of f at x_{n+1} to find an interval $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$ containing x_{n+1} such that (d), (e) and (f) are satisfied. The induction is complete.

From (d) and (e) we see that there is an $x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \cap P$. We claim that there is a subsequence J_{n_m} of J_n such that $f(x) \notin J_{n_m}$ for every m. Otherwise, $f(x)$ is contained in all but a finite number of the J_n, which is easily seen to violate (b). From (c) and the construction, it follows that

$$\sum_{m=1}^{\infty} \frac{|I_{n_m}|}{|J_{n_m}|} < \infty \quad \text{and} \quad I_{n_m} \lhd J_{n_m},$$

so Lemma 1 implies that $d(\bigcup_{m=1}^{\infty} I_{n_m}, f(x)) = 0$. The density continuity of f now implies that

$$d\left(f^{-1}\left(\bigcup_{m=1}^{\infty} I_{n_m}\right), x\right) = 0. \tag{4}$$

On the other hand, $x \in (a_{n_m}, b_{n_m})$ for all m, so (f) implies

$$\bar{d}\left(f^{-1}\left(\bigcup_{m=1}^{\infty} J_{n_m}\right), x\right) \geq \lim_{m \to \infty} \frac{|f^{-1}(I_{n_m}) \cap (a_{n_m}, b_{n_m})|}{|(a_{n_m}, b_{n_m})|} = 1. \tag{5}$$

But, (4) and (5) contradict each other, so we are forced to conclude that Z cannot be dense in P, which finishes the proof.

It is evident from the definition of Baire*1 that if f is in Baire*1, then $C(f)$ contains a dense open set. Because $C(f)$ is a G_δ set, we have proved that a density continuous function can be discontinuous on at most a nowhere dense F_σ set. The converse to this statement is also true.

Theorem 4. *If* $\mathscr{Z} = \{Z(f) : f$ *is density continuous*$\}$, *then*

$$\mathscr{Z} = \{F : F \text{ is a nowhere dense } F_\sigma \text{ set}\}.$$

In order to prove this theorem, it suffices to show that given an arbitrary nowhere dense F_σ set F, a density continuous function f can be constructed such that $Z(f) = F$. In order to do this, two lemmas are needed.

Lemma 3. *If F is a nowhere dense F_σ set, then there exist sequences of pairwise disjoint compact intervals, I_n and J_n with $I_n \lhd J_n$ such that*

$$F \subset \overline{\bigcup_{n \in \mathbb{N}} I_n} \setminus \bigcup_{n \in \mathbb{N}} J_n.$$

Moreover, if $F = \bigcup_{n\in\mathbb{N}} F_n$, *where* F_n *is closed for each* n, *then there are disjoint subsequences* m_k^n *from* $\mathbb{N}$ *such that*

$$F_n = \overline{\bigcup_{k\in\mathbb{N}} I_{m_k^n}} \setminus \bigcup_{k\in\mathbb{N}} J_{m_k^n}.$$

Proof. Let $\{(a_n, b_n) : n \in \mathbb{N}\}$ be the components of $\bar{F}^c$. For each n, choose a decreasing sequence $\{x_i^n\} \subset (a_n, b_n)$ such that $\lim_{i\to\infty} x_i^n = a_n$. The set $\{x_i^n : i, n \in \mathbb{N}\}$ is discrete, so it can be enumerated as a sequence y_i. Let J_i be a sequence of pairwise disjoint closed intervals such that J_i is centered at y_i and let I_i be a closed interval centered in J_i such that $|I_i|/|J_i| = 2^{-i}$. Then $I_n \lhd J_n$ and

$$F \subset \bar{F} = \overline{\{a_i : i \in \mathbb{N}\}} = \overline{\{y_i : i \in \mathbb{N}\}} \setminus \{y_i : i \in \mathbb{N}\} = \overline{\bigcup_{i\in\mathbb{N}} I_i} \setminus \bigcup_{i\in\mathbb{N}} J_i.$$

The second part of the lemma follows easily by choosing appropriate subsequences of y_i.

Lemma 4. *Let F be a closed nowhere dense set, $\lambda > 0$ and suppose that I_n and J_n are sequences of compact intervals such that $I_n \lhd J_n$ and the J_n are pairwise disjoint. If $F = \overline{\bigcup_{n\in\mathbb{N}} I_n} \setminus \bigcup_{n\in\mathbb{N}} J_n$, then there exists a density continuous function $f\colon \mathbb{R} \to [0, \lambda]$ such that*

(a) $Z(f) = F$,
(b) $\omega(f, x) = \lambda$, $\forall x \in F$, *and*
(c) $f^{-1}((0, \lambda]) = \bigcup_{n\in\mathbb{N}} \operatorname{int}(I_n)$.

Proof. Let

$$f_n(x) = \begin{cases} 0 & x \notin I_n \\ 2\lambda \operatorname{dist}(x, I_n^c)/|I_n| & x \in I_n \end{cases}$$

and

$$f(x) = \sum_{n\in\mathbb{N}} f_n.$$

The disjointness of the J_n and the fact that $I_n \subset \operatorname{int}(J_n)$ for all n guarantees that (a), (b) and (c) are true. To see that f is density continuous, there are two cases to consider. First, suppose that $x \in I_n$, for some n. In this case, the definitions of f_n and the fact that the J_n are pairwise disjoint guarantee that f is piecewise linear on some neighborhood of x. So, f is density continuous at x. Second, if x is in no I_n, then (c) implies that $f(x) = 0$. Using (c) again, along with Corollary 1, it follows that $f = 0$ on a density open neighborhood of x. This implies that f is density continuous at x.

We now proceed with the proof of Theorem 4.

Let F be as in the statement of the theorem. Suppose $F = \bigcup_{n\in\mathbb{N}} F_n$, where F_n is closed and $F_n \subset F_{n+1}$ *for* $n \geq 1$. Let $I_n \lhd J_n$ and the sequences m_k^n be as in Lemma 3. For each n, use Lemma 4 with $\lambda = 3^{-n}$ and the pair of intervals $I_{m_k^n} \lhd J_{m_k^n}$ to construct a function f_n. Define

(6) $$f = \sum_{n \in \mathbb{N}} f_n .$$

We see that (6) converges uniformly. Because of this, part (a) of Lemma 4 yields $Z(f) \subset F$. On the other hand, if $x_0 \in F$, then $f(x_0) = 0$ and $x \in F_n$ for some n. It follows that

(7) $$\limsup_{x \to x_0} f(x) \geq \limsup_{x \to x_0} f_n(x) = 3^{-n} > f(x),$$

so $F = Z(f)$.

Since $f = 0$ on $(\bigcup_{n \in \mathbb{N}} I_n)^c$, Corollary 1 implies that f is density continuous on that set. If $x \in I_n$ for some n, then the fact that $\operatorname{supp}(f_n) \cap \operatorname{supp}(f_m) = \emptyset$ whenever $m \neq n$ shows that there is a neighborhood G of x such that $f = f_n$ on G. The density continuity of f_n at x implies the density continuity of f at x. Therefore, f is a density continuous function.

The structure of $Z(f)$ for an approximately continuous function f is well-known. (See, e. g. Bruckner [2, page 48].) But, the proof of Theorem 4 can be used to give an alternative proof of this characterization.

Theorem 5. *If $Z = \{Z(f) : f$ is approximately continuous$\}$ then $Z = \{F : F$ is F_σ and first category$\}$.*

Proof. Since approximately continuous functions are continuous on a dense G_δ set, we see

$$Z \subset \{F : F_\sigma \text{ and first category}\}.$$

Let F be a first category F_σ set and suppose $F = \bigcup_{n \in \mathbb{N}} F_n$, where each F_n is closed and nowhere dense with $F_n \subset F_{n+1}$, $\forall n \in \mathbb{N}$. The functions f_n and f can be defined as in the proof of Theorem 4. Density continuous functions are approximately continuous and the uniform limit of approximately continuous functions is approximately continuous. Therefore, f is approximately continuous.

As before, it is clear that $Z(f) \subset F$. To establish the opposite containment, we note that if $x \in F_{n+1} \setminus F_n$, then

$$\omega(f_{n+1}, x) = \omega\left(\sum_{i \leq n+1} f_i, x\right) = 1/3^{n+1} > \sum_{i > n+1} f_i ,$$

so $x \in Z(f)$ and the theorem follows.

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