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LINEARLY CONTINUOUS MAPS DISCONTINUOUS ON THE GRAPHS OF TWICE DIFFERENTIABLE FUNCTIONS

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ABSTRACT. A function $g\colon \mathbb{R}^n \to \mathbb{R}$ is *linearly continuous* provided its restriction $g \restriction \ell$ to every straight line $\ell \subset \mathbb{R}^n$ is continuous. It is known that the set $D(g)$ of points of discontinuity of any linearly continuous $g\colon \mathbb{R}^n \to \mathbb{R}$ is a countable union of isometric copies of (the graphs of) $f \restriction P$, where $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz and $P \subset \mathbb{R}^{n-1}$ is compact nowhere dense. On the other hand, for every twice continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ and every nowhere dense perfect $P \subset \mathbb{R}$ there is a linearly continuous $g\colon \mathbb{R}^2 \to \mathbb{R}$ with $D(g) = f \restriction P$. The goal of this paper is to show that this last statement fails, if we do not assume that f'' is continuous. More specifically, we show that this failure occurs for every continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ with nowhere monotone derivative, which includes twice differentiable functions f with such property. This generalizes a recent result of professor Luděk Zajíček [On sets of discontinuities of functions continuous on all lines, arxiv.org/abs/2201.00772v1, 2022] and fully solves a problem from a 2013 paper of the first author and Timothy Glatzer [Real Anal. Exchange 38 (2012/13), pp. 377–389].

1. Introduction

A function $g\colon \mathbb{R}^n \to \mathbb{R}$ is *separately continuous* if the restriction $g \restriction \ell$ is continuous for any line ℓ parallel to one of the coordinate axes. A prehistory of this notion can be traced back at least to 1821 textbook [3, 4] of Cauchy , as discussed in a survey [8]. The first example of discontinuous separately continuous function, attributed to Heine, can be found in the 1870 calculus text of J. Thomae [15, pp. 13–16] and was defined as $g(y,z) = \sin\left(4 \arctan \frac{y}{z}\right)$ for $z \neq 0$ and $g(y,0) = 0$. The best known examples of such maps come from the 1884 treatise on calculus by Genocchi and Peano [10]: one defined as $g(x,y) = \frac{xy}{x^2+y^2}$ for $\langle x,y\rangle \neq \langle 0,0\rangle$ and $g(0,0) = 0$, the other, which is also linearly continuous, is given as

$$g(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } \langle x,y\rangle \neq \langle 0,0\rangle, \\ 0 & \text{for } \langle x,y\rangle = \langle 0,0\rangle. \end{cases} \tag{1}$$

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The first serious studies of separately continuous functions come from the work of Lebesgue and Baire. Thus, independently Lebesgue, in his very first paper [12] of 1898, and Baire, in his Ph.D. thesis [1] defended in 1899, proved that every separately continuous function $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a pointwise limit of a sequence of continuous functions or, in the contemporary terminology, it is of *Baire class 1*. This result was generalized to functions $g\colon \mathbb{R}^n \to \mathbb{R}$ of $n \geq 2$ variables in 1905 by Lebesgue [13] where it is shown that such map is of Baire class $n-1$, but need not be of the lower class. On the other hand, answering a question posed in [8], it was recently proved in papers [16] and [2], that a linearly continuous function $g\colon \mathbb{R}^n \to \mathbb{R}$ is Baire class 1 for every n.

The characterization of the sets $D(g)$ of points of discontinuity of separately continuous functions $g\colon \mathbb{R}^n \to \mathbb{R}$ was settled in 1943 [11] by Richard B. Kershner and reads as follows.

Theorem 1.1 (Kershner 1943)**.** *$S = D(g)$ for some separately continuous $g\colon \mathbb{R}^n \to \mathbb{R}$ if, and only if, S is an F_σ-set and every orthogonal projection of D onto a coordinate hyperplane is of first category.*

The natural question concerning characterization of the family

$$\mathcal{D}^n := \{D(g)\colon g\colon \mathbb{R}^n \to \mathbb{R} \text{ is linearly continuous}\}$$

of the sets of points of discontinuity of linearly continuous maps of n variables was initiated in 1945 by Alexander S. Kronrod and was partially solved in 1976 by Semen G. Slobodnik [14], who gave the following necessary condition for sets in $\mathcal{D}^n$. (See also [8].)

Theorem 1.2 (Slobodnik 1976)**.** *For every $n \geq 2$, if $D \in \mathcal{D}^n$, then D admits a representation $D = \bigcup_{i=1}^{\infty} D_i$, where each D_i is isometric to the graph of a Lipschitz function $f_i\colon K_i \to \mathbb{R}$ where $K_i \subset \mathbb{R}^{n-1}$ is compact and nowhere dense.*

Let $\mathrm{Lip}(\mathbb{R}^n)$ be the family of all Lipschitz functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{K}_n$ be the collection of all compact nowhere dense sets $K \subset \mathbb{R}^n$. Recall also that every Lipschitz function $f\colon K \to \mathbb{R}$ with $K \subset \mathbb{R}^n$ can be extended to an $\bar{f} \in \mathrm{Lip}(\mathbb{R}^n)$, see e.g. [9, p. 80]. Using this notation, the above theorem says that every $D \in \mathcal{D}^n$ is a countable union of isometric copies of the sets of the form $f \restriction K$, where $f \in \mathrm{Lip}(\mathbb{R}^{n-1})$ and $K \in \mathcal{K}_{n-1}$. It is also clear that the family $\mathcal{D}^n$ is closed under isometric images and under countable unions, see e.g. [6]. Therefore, to turn Theorem 1.2 into full characterization of the family $\mathcal{D}^n$, we must answer the following question:

(Q) for which pairs $\langle f, K\rangle \in \mathrm{Lip}(\mathbb{R}^{n-1}) \times \mathcal{K}_{n-1}$ is $f \restriction K \in \mathcal{D}^n$?

Question (Q) was studied in a 2013 paper of the first author and T. Glatzer [6] where the following results have been proved.

Theorem 1.3 (Ciesielski and Glatzer 2013)**.**

(i) *If $n \geq 2$, $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is convex, and $K \in \mathcal{K}_{n-1}$, then $f \restriction K \in \mathcal{D}^n$.*

(ii) *If function $f\colon \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $K \in \mathcal{K}_1$, then $f \restriction K \in \mathcal{D}^2$.*

(iii) *There exist differentiable $f\colon \mathbb{R} \to \mathbb{R}$ and $K \in \mathcal{K}_1$ with $f \restriction K \notin \mathcal{D}^2$.*

The authors also ask [6, problem 5.3] if the property (ii) of Theorem 1.3 holds either for every continuously differentiable f or for every twice differentiable f (with no continuity assumption on f''). The goal of this paper is to give a negative answer to both versions of this question, see Theorem 2.1 and Corollary 2.2.

The proof of Theorem 2.1 borrows from a very recent manuscript [17] of Luděk Zajíček who constructed (in a very complicated and technical way) a continuously differentiable function f and nowhere dense compact $K \subset \mathbb{R}$ with $f \restriction K \notin \mathcal{D}^2$, thus solving the first part of [6, problem 5.3]. However, the result from [17] does not (in any clear way) imply either our Theorem 2.1 or Corollary 2.2.

Notice that the results we list above do not provide a complete characterization of the sets of points of discontinuity of linearly continuous functions, even just in the case of functions of two variables. (This would require full answer of the question (Q).) Nevertheless, there exist two characterizations of such sets.

The first one comes from a 2013 paper of the first author and T. Glatzer [7] and describes sets $D(g)$ for two variable functions in terms of the topology on the set of all lines in $\mathbb{R}^2$. The second comes from the 2020 paper [2] of T. Banakh and O. Maslyuchenko and reads as follows,[1] where a set $K \subset \mathbb{R}^n$ is *ℓ-miserable* provided there exists a closed set $L \subset \mathbb{R}^n$ containing K with the properties:

(i) L is an *ℓ-neighborhood* of K: for any line ℓ in $\mathbb{R}^n$ and any point $\bar{p} \in \ell \cap K$ there is an open interval J in ℓ such that $\bar{p} \in J \subset L$;

(ii) $K \subset \mathrm{cl}(\mathbb{R}^2 \setminus L)$.

Theorem 1.4 (Banakh and Maslyuchenko 2020). *$M = D(g)$ for some linearly continuous $g\colon \mathbb{R}^n \to \mathbb{R}$ if, and only if, M is a countable union of closed ℓ-miserable sets $K \subset \mathbb{R}^n$.*

Although the characterization from Theorem 1.4 is simpler and more complete than the one from [7], neither of these characterizations is either simple to apply or easy to grasp. On the other hand, the full answer to question (Q) could provide such simple and natural characterization. The main result of this paper is a considerable step towards this goal.

2. The main result and its proof

In what follows symbol $C^1(\mathbb{R})$ stands for the functions $f\colon \mathbb{R} \to \mathbb{R}$ that are continuously differentiable, while $D^2(\mathbb{R})$ denotes all $f \in C^1(\mathbb{R})$ that are twice differentiable. Recall also that $f\colon \mathbb{R} \to \mathbb{R}$ is *nowhere monotone* provided it is not monotone on any non-empty open interval.

Theorem 2.1. *For every $f \in C^1(\mathbb{R})$ with nowhere monotone derivative f' there exists a nowhere dense perfect $P \subset \mathbb{R}$ such that every linearly continuous map $g\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at some point of $f \restriction P$.*

This theorem immediately implies the following result, which fully solves [6, problem 5.3].

Corollary 2.2. *There exists an $f \in D^2(\mathbb{R})$ and a nowhere dense perfect $P \subset \mathbb{R}$ such that every linearly continuous map $g\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at some point of $f \restriction P$.*

[1]Since in the proof of our main result we use only one direction of Theorem 1.4—the necessity of $D(g)$ being a countable union of closed ℓ-miserable sets—it is perhaps worth to notice that the argument for it is relatively simple: if $g\colon \mathbb{R}^n \to \mathbb{R}$ is linearly continuous, $\{V_n : n \in \mathbb{N}\}$ is a basis for $\mathbb{R}$, and $E_n := f^{-1}(V_n) \setminus \mathrm{int}(\mathrm{cl}(f^{-1}(V_n)))$, then $D(g) = \bigcup_{n\in\mathbb{N}} E_n$, each set E_n is F_σ, and any subset of E_n is ℓ-miserable.

Proof. Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable nowhere monotone, see e.g. [5]. Then $f(x) := \int_0^x h(t)\,dt$ is as needed. □

2.1. **Lemmas.** In what follows we always assume that $f \in C^1(\mathbb{R})$. For an $x \in \mathbb{R}$ let $\tau_{f,x}$ be the equation of the tangent line to f at x, that is,

$$\tau_{f,x}(s) := f(x) + f'(x)(s - x)$$

and let

$$T_{f,x} := \{\langle t, \tau_{f,x}(s)\rangle \colon s \in \mathbb{R}\}$$

be this tangent line. Also, for $X \subset \mathbb{R}$ and $a \in \mathbb{R}$ let

$$T_{f,X} := \bigcup_{x\in X} T_{f,x} \quad \text{and} \quad T^a_{f,X} := T_{f,X} \cap \pi^{-1}((a,\infty)),$$

where $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ is the projection on the first coordinate.

Lemma 2.3. *Let $f \in C^1(\mathbb{R})$ and $x_1 < x_0$ be such that*

$$f'(x_1) = \max f'([x_1, x_0]) > f'(x_0).$$

Then $\tau_{f,x_1}(c) > \tau_{f,x_0}(c)$ for every $c > x_0$.

Proof. Indeed, if $\xi \in (x_1, x_0)$ is such that $f'(\xi) = \frac{f(x_0)-f(x_1)}{x_0-x_1}$ then $\tau_{f,x_1}(c) > \tau_{f,x_0}(c)$ is equivalent to each of the following inequalities.

$$\begin{aligned}
f(x_1) + f'(x_1)(c - x_1) &> f(x_0) + f'(x_0)(c - x_0)\\
f'(x_1)(c - x_1) - f'(x_0)(c - x_0) &> f(x_0) - f(x_1)\\
f'(x_1)(c - x_0) - f'(x_0)(c - x_0) &> f(x_0) - f(x_1) - f'(x_1)(x_0 - x_1)\\
(f'(x_1) - f'(x_0))(c - x_0) &> f'(\xi)(x_0 - x_1) - f'(x_1)(x_0 - x_1)\\
(f'(x_1) - f'(x_0))(c - x_0) &> (f'(\xi) - f'(x_1))(x_0 - x_1).
\end{aligned}$$

But the last inequality holds, since $f'(x_1) - f'(x_0) > 0 \geq f'(\xi) - f'(x_1)$. □

The next lemma is proved similarly.

Lemma 2.4. *Let $f \in C^1(\mathbb{R})$ and $x_2 < x_0$ be such that*

$$f'(x_2) = \min f'([x_2, x_0]) < f'(x_0).$$

Then $\tau_{f,x_2}(c) < \tau_{f,x_0}(c)$ for every $c > x_0$.

We will also need the following consequence of the previous two lemmas.

Lemma 2.5. *Let $f \in C^1(\mathbb{R})$ be such that f' is nowhere monotone. If $a \in \mathbb{R}$ and $Z \subset (-\infty, a]$, then for every non-empty $(r, s) \subset Z$ there exists a non-empty $(u, v) \subset (r, s)$ such that*

$$T^a_{f,Z\setminus(u,v)} = T^a_{f,Z}.$$

In particular, if Z is compact, then there exists a non-empty closed nowhere dense $N \subset Z$ such that

$$T^a_{f,N} = T^a_{f,Z}.$$

Proof. To see the first part, choose $x_0, x_1, x_2 \in (r,s)$ such that $x_1, x_2 < x_0$, $f'(x_1) > f'(x_0)$, and $f'(x_2) < f'(x_0)$. The choice is possible since f' is nowhere monotone. We can also ensure by continuity of f' that the assumptions of Lemmas 2.3 and 2.4 are satisfied by replacing each x_i, $i \in \{1,2\}$, by its prime version defined as

$$x_i' = \sup\{x \in [x_i, x_0] \colon f'(x) = f'(x_i)\}.$$

Let $y < z$ be such that $\{y, z\} = \{x_1, x_2\}$.

Since f' is continuous, we can choose $(u,v) \subset (z,s)$ containing x_0 such that $f'(x) \in (f'(x_2), f'(x_1))$ for all $x \in (u,v)$. Then the interval (u,v) is as needed.

To see this, notice that $T^a_{f,Z} = T^a_{f,Z\setminus(u,v)} \cup T^a_{f,(u,v)}$. Thus, it is enough to show that $T^a_{f,(u,v)} \subset T^a_{f,(y,z)}$, as clearly $T^a_{f,(y,z)} \subset T^a_{f,Z\setminus(u,v)}$.

But if $\langle c,d\rangle \in T^a_{f,(u,v)}$, then there is $x \in (u,v)$ with $\langle c,d\rangle \in T^a_{f,x}$, that is, $d = \tau_{f,x}(c)$. So, by Lemmas 2.3 and 2.4, we have $d \in (\tau_{f,x_2}(c), \tau_{f,x_1}(c))$. But the map $t \mapsto \tau_{f,t}(c)$ is continuous. So, by the intermediate value theorem, there is $w \in (y,z)$ with $d = \tau_{f,w}(c)$. This means, that $\langle c,d\rangle \in T^a_{f,w} \subset T^a_{f,(y,z)}$, as needed.

To see the additional part, fix a countable basis $\{(r_n,s_n) \colon n \in \mathbb{N}\}$ of $\mathbb{R}$ and inductively construct a sequence $Z \supset Z_1 \supset Z_2 \supset \cdots$ of compact sets such that $T^a_{f,Z} = T^a_{f,Z_n}$ and $(r_n,s_n) \not\subset Z_n$ for every $n \in \mathbb{N}$. Then $N := \bigcap_{n\in\mathbb{N}} Z_n$ is as needed. Indeed, clearly it is compact, nowhere dense, and $T^a_{f,N} \subset T^a_{f,Z}$. To see the other inclusion, choose $\langle s,t\rangle \in T^a_{f,Z} \subset \bigcap_{n\in\mathbb{N}} T^a_{f,Z_n}$. Then, for every $n \in \mathbb{N}$ there exists a $z_n \in Z_n$ such that $\langle s,t\rangle \in T^a_{f,z_n}$, and so

$$t = \tau_{f,z_n}(s) = f(z_n) + f'(z_n)(s - z_n).$$

Since Z is compact, there is a subsequence $(z_{n_k})_k$ of (z_n) that converges to $z \in Z$. Notice that $z \in N$ and that, by the continuity of f and f',

$$t = f(z_{n_k}) + f'(z_{n_k})(s - z_{n_k}) \to_{k\to\infty} f(z) + f'(z)(s - z) = \tau_{f,z}(s).$$

Hence $\langle s,t\rangle \in T^a_{f,z} \subset T^a_{f,N}$. So, indeed $T^a_{f,N} = T^a_{f,Z}$. □

The next simple lemma is a variation of [17, lemma 2.3].

Lemma 2.6. *If $f \in C^1(\mathbb{R})$ is such that f' is nowhere monotone, then for every non-degenerate closed interval $[a,b]$, there exist $d \in (a,b)$ and perfect nowhere dense $N_d \subset (a,d)$ such that $\langle d, f(d)\rangle \in \operatorname{int}(T_{f,N_d})$.*

Proof. First notice the following simple fact, which comes from [17, lemma 2.3]:

($\star$) there exist $s < d$ both in (a,b) such that $\langle d, f(d)\rangle \in T_{f,s}$.

Indeed, since f' is continuous and nowhere monotone, there exist $x_0, x_1, x_2 \in (a,b)$ that satisfy the assumptions of Lemmas 2.3 and 2.4. Then

$$f(x_0) = f(x_1) + \int_{x_1}^{x_0} f'(t)\, dt < f(x_1) + f'(x_1)(x_0 - x_1) = \tau_{f,x_1}(x_0)$$

and

$$f(x_0) = f(x_2) + \int_{x_2}^{x_0} f'(t)\, dt > f(x_2) + f'(x_2)(x_0 - x_2) = \tau_{f,x_2}(x_0).$$

Since map $x \mapsto \tau_{f,x}(x_0)$ is continuous and $\tau_{f,x_2}(x_0) < f(x_0) < \tau_{f,x_1}(x_0)$, there exists s between x_1 and x_2 such that $\tau_{f,s}(x_0) = f(x_0)$. Hence, s and $d := x_0$ satisfy ($\star$).

Also, continuity of f and f' implies that there exists $x_0' \in (\max\{x_1,x_2\}, x_0)$ so that $f'(x_1) = \max f'([x_1,x_0']) > f'(x_0')$ and $f'(x_2) = \min f'([x_2,x_0']) < f'(x_0')$.

In particular, if Z is the closed interval with endpoints x_1 and x_2, then, by Lemmas 2.3 and 2.4, $\langle d, f(d)\rangle$ is contained in the interior of the set

$$W := \{\langle c, y\rangle : c > x_0' \ \&\ \tau_{f,x_2}(c) < y < \tau_{f,x_1}(c)\}$$

and $W \subset T_{f,Z}$. Hence, by Lemma 2.5, there exists perfect nowhere dense set $N_d \subset Z \subset (a, d)$ with $\langle d, f(d)\rangle \in \operatorname{int}(T_{f,Z}) = \operatorname{int}(T_{f,N_d})$. □

2.2. **Proof of Theorem 2.1.** By induction on the length of binary sequences $s \in 2^{<\omega}$ we construct a sequence $\langle\langle I_s, d_s, N_s\rangle : s \in 2^{<\omega}\rangle$ subject to the following inductive conditions.

(A_n) $\mathcal{I}_n = \{I_s : s \in 2^n\}$ consists of pairwise disjoint non-trivial closed intervals such that for every $s \in 2^n$ the interval I_s has length $|I_s| \le (\frac{2}{3})^n$.

(B_n) If $s, t \in 2^{\le n}$ and $s \subset t$, then $I_t \subset I_s$ and

$$N_t \cup \{d_t\} \subset \bigcup \mathcal{I}_n\,. \tag{2}$$

(C_n) For every $s \in 2^n$, if $I_s = [a_s, b_s]$, then $d_s \in (a_s, b_s)$, $N_s \subset (a_s, d_s)$ is compact nowhere dense, and

$$\langle d_s, f(d_s)\rangle \in \operatorname{int}(T_{f,N_s}).$$

We start the construction by putting $I_\emptyset = [0, 1]$. This ensures that the conditions (A_0) and (B_0) are satisfied, except for (2) when $t \in 2^n$, since in such case N_t and d_t are not defined yet.

Also, if for some $n < \omega$, the objects satisfying conditions (A_n)–(C_n) are already defined, we construct the intervals in $\mathcal{I}_{n+1}$ as follows. For every $s \in 2^n$ let M_s be the middle third open subinterval of I_s, let J_s be a non-empty open interval contained in $M_s \setminus \bigcup_{t\in 2^{\le n}}(N_t \cup \{d_t\})$, and define two intervals in $\{I_u : u \in 2^{n+1} \ \&\ s \subset u\}$ as two connected components of $I_s \setminus J_s$. Notice that, once again, such choice ensures that the conditions (A_{n+1}) and (B_{n+1}) are satisfied, except for (2) when $t \in 2^{n+1}$, since in such case N_t and d_t are not defined yet.

Finally, if for some $n < \omega$ the intervals in $\mathcal{I}_n$ are already constructed, for every $t \in 2^n$ use Lemma 2.6 with $[a_t, b_t] = I_t$ to find $d_t \in (a_t, b_t)$ and a perfect nowhere dense $N_t \subset (a_t, d_t)$ that satisfy (C_n). Notice that this choice preserves (2) of (B_n). This finishes the inductive construction.

The desired nowhere dense perfect set P is defined as

$$P := \bigcap_{n<\omega} \bigcup \mathcal{I}_n\,. \tag{3}$$

Notice also that, by (2), the set $D := \{d_s : s \in 2^{<\omega}\}$ is contained in P.

To see that P is as needed, assume by way of contradiction that there is a linearly continuous $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that g is discontinuous at each point of $f \restriction P$. Then, by Theorem 1.4, there exists a countable family $\mathbb{K}$ of ℓ-miserable closed subsets of $\mathbb{R}^2$ so that $f \restriction P \subset \bigcup \mathbb{K}$. Therefore, by Baire category theorem used in the compact space $f \restriction P$, there exists a $K \in \mathbb{K}$ containing a non-empty open subset of $f \restriction P$. In particular, there is an $s_0 \in 2^{<\omega}$ such that $K_0 := f \restriction (P \cap I_{s_0})$ is contained in K. Since K is ℓ-miserable, so is K_0. Thus, there exists a closed ℓ-neighborhood $L \subset \mathbb{R}^2$ of K_0 such that K_0 is contained in the closure of $U := \mathbb{R}^2 \setminus L$. Our contradiction will be obtained by showing that such set L cannot be an ℓ-neighborhood of K_0. More specifically, we will find a point $p \in P$ such that the property (i) of ℓ-miserable set fails for $\ell := T_{f,p}$ and $\bar{p} := \langle p, f(p)\rangle$.

For this, construct by induction the sequence $\langle\langle s_n, c_n, \varepsilon_n\rangle \in 2^{<\omega} \times U \times \mathbb{R} \colon n < \omega\rangle$ such that the following inductive conditions are satisfied:

(a_n) $c_n \in U \cap \operatorname{int}(T_{f,N_{s_n}})$ and $\|c_n - \langle d_{s_n}, f(d_{s_n})\rangle\| \le 2^{-n}$;
(b_n) $\varepsilon_n \in (0, 2^{-n})$ is such that the open ball $B(c_n, \varepsilon_n) \subset U \cap \operatorname{int}(T_{f,N_{s_n}})$;
(c_n) $s_{n+1} \in 2^{<\omega}$ extends s_n and $T_{f,p} \cap B(c_n, \varepsilon_n) \neq \emptyset$ for every $p \in I_{s_{n+1}}$.

We start with s_0 defined above. If for some $n < \omega$ the sequence s_n is already constructed, we can choose c_n as in (a_n) since $\langle d_{s_n}, f(d_{s_n})\rangle$ belongs to $K_0 \subset \operatorname{cl}(U)$ and to $\operatorname{int}(T_{f,N_{s_n}})$. The number ε_n can be chosen since $U \cap \operatorname{int}(T_{f,N_{s_n}})$ is open. To choose s_{n+1} as in (c_n), first choose $x_n \in N_{s_n} \subset I_{s_n}$ such that $T_{f,x_n} \cap B(c_n, \varepsilon_n) \neq \emptyset$. Since $f \in C^1(\mathbb{R})$, there exists $\delta > 0$ such that $T_{f,p} \cap B(c_n, \varepsilon_n) \neq \emptyset$ for every $p \in (x_n - \delta, x_n + \delta)$. So, it is enough to choose s_{n+1} extending s_n such that $I_{s_{n+1}} \subset (x_n - \delta, x_n + \delta)$. This finishes the inductive construction.

Finally, let p be such that $\{p\} = \bigcap_{n<\omega} I_{s_n}$, put $\ell := T_{f,p}$ and let $\bar{p} := \langle p, f(p)\rangle$. Then for every $n < \omega$ there exists a $p_n \in \ell \cap B(c_n, \varepsilon_n) \subset \ell \cap U$. In particular, since $p_n \to_n \bar{p}$, there is no open interval J in ℓ with $\bar{p} \in J \subset \mathbb{R}^2 \setminus U = L$, that is, indeed the property (i) of ℓ-miserable set is not satisfied.

3. Final remarks

Corollary 2.2 implies that the part (iii) of Theorem 1.3 holds true also for some twice differentiable functions f. In other words, in the natural hierarchy of smoothness of functions $f \colon \mathbb{R} \to \mathbb{R}$, the part (ii) of Theorem 1.3 is the best possible. Nevertheless, we still do not know the full answer for the question (Q):

Problem 3.1. *Find the description of all pairs $\langle f, K\rangle \in \operatorname{Lip}(\mathbb{R}^{n-1}) \times \mathcal{K}_{n-1}$ with $f \restriction K \in \mathcal{D}^n$, that is, such that $D(g) = f \restriction K$ for some linearly continuous $g \colon \mathbb{R}^n \to \mathbb{R}$.*

Of course, the question is of interest only for the functions $f \in \operatorname{Lip}(\mathbb{R}^{n-1})$ that are neither convex nor, for $n = 2$, twice continuously differentiable. Interestingly, we also do not know the answer for the following problem, in spite that we believe that the answer to it is positive.

Problem 3.2. *If $n > 2$, is it true that $f \restriction K \in \mathcal{D}^n$ for every twice continuously differentiable $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$ and $K \in \mathcal{K}_{n-1}$?*

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