Periodica Mathematica Hungarica Vol. 22 (2), (1990), pp. 153–158

A GAME OF D. GALE IN WHICH ONE OF THE PLAYERS HAS LIMITED MEMORY

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0. Introduction

For an infinite set X, let $G(X)$ stand for the following game: at the n-th move Player I chooses a finite non-empty set $X_n \subset X$ and Player II an element x_n from his pile $P_n = X_0 \cup \ldots \cup X_n \setminus \{x_0, \ldots, x_{n-1}\}$. Define $A_n = X_0 \cup \cup \ldots \cup X_n$ and $B_n = \{x_0, \ldots, x_n\}$ and let $A = \bigcup_{n=0}^{\infty} A_n$ and $B = \bigcup_{n=0}^{\infty} B_n$. The game is a win for Player II if $A = B$, otherwise Player I wins.

Obviously if Player II remembers at stage n what the previous plays $X_0, x_0, X_1, x_1, \ldots, x_{n-1}, X_n$ were, then he has an easy winning strategy: at the n-th step he first chooses the smallest $i \leq n$ such that $P_n \cap X_i \neq \emptyset$ and then plays the smallest element (under some fixed linear order of X) from this set. But what if Player II has a bad memory and he remembers at stage n only the position (P_n, B_n) or even only his own pile $P_n = A_n \setminus B_{n-1}$? We will examine these problems and some other related to them. Those questions have been posed by David Gale in [1].

1. Positive results

PROPOSITION 1.1. *If X is countable then Player II has a winning strategy f depending only on his pile P_n.*

PROOF. If $<$ is a well order of X in order type ω then it is enough to put $f(P) = \min(P)$.

Thus we will assume from now on that X is uncountable.

The first theorem tells us that if we allow Player II to remember the last discard element x_{n-1} and his own pile P_n, then he has a winning strategy.

Mathematics Subject Classification, 1980/1985. Primary 03E20; Secondary 03E60.
Key words and phrases: Infinite games, restricted strategies, Axiom of Choice.

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

THEOREM 1.1. *There exists a function $f(P, x) \in P$ which yields a winning strategy for Player II.*

PROOF. Let $<$ be a fixed well order of X. For a finite non-empty set $P \subset X$ and an $x \notin P$ we define $f(P, x) \in P$ by:

$$f(P, x) = \begin{cases} \max \{p \in P\colon\ p < x\} & \text{when } \{p \in P\colon\ p < x\} \neq \emptyset \\ \max (P) & \text{otherwise} \end{cases}$$

To prove that f describes a winning strategy for Player II let the sequence $X_0, x_0, X_1, x_1, \ldots$ represent the game played according to f, i.e., $x_n = f(P_n, x_{n-1})$ for $n > 0$ and $x_0 = \max (P_0)$. Choose $x \in A$ and let i be such that $x \in X_i$. We have to prove that $x = x_j$ for some $j \geq i$.

Firt notice that

(1) for every natural number m there exists $n > m$ such that $x_n < x_{n+1}$.
By the definition of f we have also

(2) for every natural number $n > 0$ if $x \in P_n$ and $x < x_{n-1}$ then $x \leq \leq x_n < x_{n-1}$.

Now let $n > i$ be as in (1), i.e., such that $x_n < x_{n+1}$.

If $x_{n+1} = x$ then put $j = n + 1$.

If $x_{n+1} = \max (P_{n+1}) < x$ then $x \notin P_{n+1} = A_{n+1} \backslash B_n$, i.e., $x \in B_n = \{x_0, \ldots, x_n\}$.

If $x_{n+1} > x$ then we are in situation (2). Moreover the sequence $x_{n+1}, x_{n+2}, \ldots, x_j$ is decreasing as long as $x_{j-1} > x$. So, by (1) and (2), there exists $j > n + 1$ such that $x = x_j$.

The next theorem tells us that Player II also has a winning strategy when he does not remember what his last move was but he is given the position (P_n, B_n). The proof of this theorem generalizes the technique of the proof of Theorem 1.1.

THEOREM 1.2. *There exists a function $g(P, B) = x \in P$ yielding a winning strategy for Player II.*

PROOF. Let ∞ be a symbol such that $\infty \notin X$ and let us fix a well order $<$ of $X \cup \{\infty\}$ such that $x < \infty$ for all $x \in X$. For a natural number $n > 0$, write $n = 2^{i(n)} + k(n)$ with $0 \leq k(n) < 2^{i(n)}$.

If B is an $(n - 1)$-element subset of X then we define

$$t(B) = \begin{cases} b_{2 \cdot k(n)} & \text{for } k(n) > 0 \\ \infty & \text{for } k(n) = 0 \end{cases}$$

where $b_1, \ldots, b_{n-1}$ is a decreasing enumeration of B_n.

Finally for a non-empty finite set $P \subset X$ and finite $B \subset X \setminus P$ we define

$$g(P, B) = f(P, t(B))$$

where f is as in Theorem 1.1.

To prove that g gives a winning strategy for Player II, let a sequence $X_0, x_0, X_1, x_1, \ldots$ represent the game played according to g, i.e., $x_n = g(P_n, B_n)$ for all n. We are supposed to prove that for every element $x \in A = \bigcup_{n=0}^{\infty} A_n$ there exists n such that $x = x_n$.

By way of contradiction let us assume that for some $x \in A$ there is no n such that $x = x_n$. Let j be such that $x \in X_j$. Thus $x \in P_m$ for all $m \geq j$.

First we prove that for every $n \geq j$, $n = 2^{i(n)}$, that

(*) if $b \in B_n$ and $b > x$ then there exists m such that $n \leq m < 2 \cdot n$ and $b > x_m > x$.

So let $b \in B_n$ and $n = 2^{i(n)}$. Then

$$\infty = t(B_n) > t(B_{n+1}) > \ldots > t(B_{2 \cdot n-1}) = \min(B_{2 \cdot n-1}).$$

and hence $t(B_n) > b \geq t(B_{2 \cdot n-1})$.

Let

$$m = \max\{k\colon n \leq k < 2 \cdot n \ \&\ t(B_k) \geq b\}.$$

As $x \in P_m$ and $x < b \leq t(B_m)$ then, by definition of f, we conclude that $x < x_m < t(B_m)$.

To prove (*) it is enough to show that $b > x_m$. But if $b < x_m$ then, by maximality of m, $t(B_{m+1}) < b < x_m < t(B_m)$ and $b, x_m \in B_{m+1}$. This contradicts the fact that there is at most one element y in B_{m+1} such that $t(B_{m+1}) < y < t(B_m)$.

So (*) is proved.

Now to obtain a contradiction with the assumption that $x \in A \setminus B$ we will define by induction an infinite sequence $x_{n_0} > x_{n_1} > x_{n_2} > \ldots$ such that $x_{n_i} > x$ for all i.

Choose $n_0 = 2^k > j$. Thus $x_{n_0} = \max(P_{n_0}) > x$.

If for some $i \geq 0$ we have already defined $x_{n_0} > x_{n_1} > \ldots > x_{n_i} > x$ choose $n = 2^k > n_i$. Then $x_{n_i} \in B_n$ and, by (*), there exists $n_{i+1} \geq n > n_i$ such that $x_{n_i} > x_{n_{i+1}} > x$.

Therefore $A = B$. This finishes the proof of Theorem 1.2.

2. Negative results

In this section we prove that Player II does not have a winning strategy depending only on his pile P_n (according to [1] this result seems to have been first proved by Martin Fürer) and that the situation does not change when we also allow Player II to remember how many moves have been made in the game (a result which was also proved by R. McKenzie).

We start with the following definitions.

Let Fin (X) be the collection of finite non-empty subsets of X. For a function f: Fin $(X) \to X$ such that $f(P) \in P$ and an $a \in X$ let

$$f^{<}(a) = \{x \in X \setminus \{a\}: f(\{a, x\}) = x\}.$$

Lemma 2.1. *The set $F = \{a \in X: f^{<}(a)$ is a finite set$\}$ is at most countable.*

Proof. By way of contradiction assume that F is uncountable, and let $F' \subset F$ be countably infinite. Put $F'' = F' \cup \bigcup \{f^{<}(a): a \in F'\}$. The set F'' is countable; choose $x \in F \setminus F''$. But then $x \notin f^{<}(a)$ for all $a \in F'$, i.e., $F' \subset f^{<}(x)$. Thus the set $f^{<}(x)$ is not finite contradicting the definition of F.

Theorem 2.1. *There does not exist a function $f(P_n, n) = x_n \in P_n$ giving a winning strategy for Player II.*

Proof. Let f be a function defined for all pairs (P, n) where $P \in$ Fin (X) and $n \geq 0$ such that $f(P, n) \in P$. It is enough to show that there exists a game $X_0, x_0, X_1, x_1, \ldots$ played according to f (i.e., with $x_n = g(P_n, n)$ for all n) such that $A \setminus B \neq \emptyset$.

Define f_n: Fin $(X) \to X$ by $f_n(P) = f(P, n)$. Then, by Lemma 2.1, the sets $F_n = \{a \in X: f_n^{<}(a)$ is a finite set$\}$ are at most countable.

Choose $y \in X \setminus \bigcup_{n=0}^{\infty} F_n$. So the sets $Y_n = f_n^{<}(y)$ are infinite.

Let us define $X_0 = \{y, y_0\}$ where $y_0 \in Y_0$ and, for $n > 0$, $X_n = \{y_n\}$ where $y_n \in Y_n \setminus \{x_0, \ldots, x_{n-1}\}$. It is easy to see that $P_n = \{y, y_n\}$ and $x_n = f_n(P_n) = y_n$. Hence $A \setminus B = \{y\} \neq \emptyset$.

3. Without the Axiom of Choice

In Theorem 1.1 and 1.2 we used the Axiom of Choice to well order X. Can these results be proved without the Axiom of Choice? We give a negative answer in the case $X = \mathbf{R}$. This follows from the next theorem, by invoking the consistency ([5], [4]) of $ZF + DC +$ *"Every subset of* **R** *has the Baire Property"*, (where DC stands for the Axiom of Dependent Choice) relative to the consistency of ZF.

Theorem 3.1. ($ZF + DC +$ *"Every subset of* **R** *has the Baire Property"*) *Player II does not have a winning strategy.*

PROOF. Let us assume that Player II plays according to some strategy $f(P, B) \in P$ where $P \neq \emptyset$ and B are finite disjoint subsets of $\mathbf{R}$. We will find a sequence $a, x_0, x_1, x_2, x_3, \ldots$ of different real numbers such that $f(\{a, x_0\}, \emptyset) = x_0$, $f(\{a, x_1\}, \{x_0\}) = x_1$, and, in general, $f(\{a, x_n\}, \{x_0, x_1, \ldots, x_{n-1}\}) = x_n$. This means that the game $\langle X_0, x_0, X_1, x_1, X_2, x_2, \ldots\rangle = \langle\{a, x_0\}, x_0, \{x_1\}, x_1, \{x_2\}, x_2, \ldots\rangle$ is played according to the strategy f, however Player I wins as $A \setminus B = \{a\} \neq \emptyset$. This will show that Player II does not have a winning strategy.

We will need the following lemma.

LEMMA 3.1. *Let $T = \{x_0, x_1, \ldots, x_{n-1}\} \subset \mathbf{R}$ and let us define $f_T(y, x) = f(\{y, x\}, \{x_0, x_1, \ldots, x_{n-1}\}) \in \{y, x\}$ for all $y, x \in \mathbf{R} \setminus T$. Then there exists a first category set M and a countable subset B of $\mathbf{R}$ such that if $U_x = \{y \in \mathbf{R}\setminus T\colon y \neq x \ \& \ f_T(y, x) = x\}$ then*

(1) $\mathbf{R}\setminus M \subset \bigcup\{U_x\colon x \in B\}$,

(2) $U_x\setminus M$ is non-empty and open in $\mathbf{R}\setminus M$ for every x from B.

PROOF. Let $M_0 = \{x\colon x \in \mathbf{R}\setminus T \ \& \ U_x \text{ is of first category}\}$. As $\mathbf{R}\setminus M_0$ is a separable metric space there exists a countable subset B of $\mathbf{R}\setminus T$ such that the set $M_1 = (\mathbf{R}\setminus M_0)\setminus \bigcup\{U_x\colon x \in B\}$ is of first category. Let M_2 be a set of first category such that $U_x\setminus M_2$ is open in $\mathbf{R}\setminus M_2$ for every x from B. Put $M = M_0 \cup M_1 \cup M_2$. Obviously M and B satisfy (1) and (2). Thus it is enough to show that M_0 is of first category.

By way of contradiction, assume that there exists a non-empty interval I such that $I\setminus M_0$ is of first category. We may also assume that I and T are disjoint.

Define a function $F\colon I^2 \to \{0, 1\}$ by

$$F(y, x) = 0 \text{ if and only if } f_T(y, x) < \max\{y, x\}.$$

This function has the Baire property, so there exists a comeager set $C \subset I^2$ such that the restriction of F to C is continuous. Without loss of generality we can also assume that $C \subset (I \cap M_0)^2$ and, by the Kuratowski–Ulam theorem (see [3] or [2] p. 246), that every section of C is comeager in I.

Choose (y_0, x_0) from C such that $y_0 \neq x_0$. By symmetry we may assume that $f_T(y_0, x_0) = x_0$. Then the set U_{x_0} contains

$$\{y\colon F(y, x_0) = F(y_0, x_0)\} = \{y\colon (y, x_0) \in F^{-1}(F(y_0, x_0))\}$$

which is an open section of C given by x_0. Thus U_{x_0} is of second category contradicting the fact that $x_0 \in M_0$.

This finishes the proof of the Lemma.

Now we are ready to construct a sequence $a, x_0, x_1, x_2, \ldots$ as described above.

Let us define for every finite sequence s of natural numbers a real number b_s and a set B_s of second category in $\mathbf{R}$ by induction on the length k of s.

For a given sequence s of length k let $T(s) = \{b_{s|_m}: 1 \leq m < k\}$. We define $\{b_{s^\frown n}: n \in \mathbf{N}\}$ as an enumeration, possibly with repetitions, of the set B from Lemma 3.1 used for $T = T(s)$.

Put

$$B_{s^\frown n} = B_s \cap U_{b_{s^\frown n}} = B_s \cap \{y \in \mathbf{R} \setminus T(s^\frown n): y \neq b_{s^\frown n} \ \& \ f_{T(s^\frown n)}(y, b_{s^\frown n}) = b_{s^\frown n}\}.$$

It is easy to see that for every k the set $\bigcup \{B_s: s$ has length $k\}$ is comeager. Thus choose $a \in \bigcap \{\bigcup \{B_s: s$ has length $\mathbf{n}\}: n \in \mathbf{N}\}$. This means that for every n there exists s_n of length n such that $a \in B_{s_n}$, i.e., $a \in \bigcap \{B_{s_n}: n \in \mathbf{N}\}$. Put $x_n = b_{s_n}$. By our construction if $s = s'$ are sequences of the same length then $B_{s'} \cap B_{s'} = \emptyset$ implies that $b_s = b_{s'}$. Hence $T(s_n)$: $\{x_0, x_1, \ldots, x_{n-1}\}$ and $f(\{a, x_n\}, \{x_0, x_1, \ldots, x_{n-1}\}) = f_{T(s_n)}(a, x_n) = x_n$, as desired.

Note that the proof of Theorem 3.1 gives the stronger statement that, assuming DC + "*Every subset of* $\mathbf{R}$ *has the Baire Property*", Player II does not have a winning strategy $f(P, \langle x_0, x_1, \ldots, x_{n-1}\rangle) \in P$ depending of his pile and the *sequence* of his previous moves.

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(Received July 5, 1988)

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