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by

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$\mathfrak{c}^+$-LINEABILITY OF THE CLASS OF DARBOUX MAPS WITH THE STRONG CANTOR INTERMEDIATE VALUE PROPERTY WHICH ARE NOT CONNECTIVITY

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We dedicate this work to the memory of Professor Ralph Kopperman, a coauthor and friend of the second author.

Abstract. We prove, under an additional set-theoretic assumption, specifically *continuum is a regular cardinal*, that there exists a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ of dimension $\mathfrak{c}^+$ whose non-zero elements are the functions that are everywhere surjective (ES), have strong Cantor intermediate value property (SCIVP), and are not connectivity (Conn). Since every map in ES is Darboux (D), this means that the class $\text{SCIVP} \cap \text{D} \setminus \text{Conn}$ is $\mathfrak{c}^+$-lineable under our set-theoretic assumption.

1. Introduction

For sets X and Y, let Y^X be the family of all functions from X to Y and let $|X|$ denote the cardinality of X.

Let V be a vector space, let $M \subseteq V$, and let κ be a cardinal number. We say that M is κ-lineable if there exists a subspace W of V contained in $M \cup \{0\}$ such that the dimension of W is κ. This notion was motivated by the result of V. I. Gurariĭ [16], which, in the language of lineability, says that the set of continuous nowhere differentiable functions from $[0,1]$ to $\mathbb{R}$ (treated as a subset of the vector space $[0,1]^{\mathbb{R}}$ over $\mathbb{R}$) is ω-lineable.

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See [5] and [7] for the development in this area and [1], [2], [4], and [9] for recent results of the authors of this note. In what follows, $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

Recall that an infinite cardinal number κ is a *regular cardinal* provided a union of less than κ-many sets of cardinality less than κ has cardinality less than κ. The goal of this paper is to show that if $\mathfrak{c}$ is a regular cardinal, then the class $\mathrm{SCIVP} \cap \mathrm{D} \setminus \mathrm{Conn}$ is $\mathfrak{c}^+$-lineable, where

- ES is the class of all *everywhere surjective functions* $f \in \mathbb{R}^\mathbb{R}$, that is, such that $f^{-1}(r)$ is dense in $\mathbb{R}$ for every $r \in \mathbb{R}$;
- D is the class of all *Darboux functions* $f \in \mathbb{R}^\mathbb{R}$, that is, such that $f[C]$ is connected (i.e., an interval) for every connected $C \subset \mathbb{R}$ (or, equivalently, that f has the intermediate value property); this class was first systematically investigated by Jean-Gaston Darboux (1842–1917) in his 1875 paper [14];
- Conn is the class of all *connectivity functions* $f \in \mathbb{R}^\mathbb{R}$, that is, such that the graph of f is a connected subset of $\mathbb{R}^2$; this notion can be traced to a 1956 problem [19] stated by John Forbes Nash (1928–2015);
- SCIVP is the class of all functions $f \in \mathbb{R}^\mathbb{R}$ with *strong Cantor intermediate value property*, that is, such that for all $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$ and $f \restriction P$ is continuous; this notion is introduced in a 1992 paper [20] of H. Rosen, R. Gibson, and F. Roush.

Clearly, $\mathrm{ES} \subset \mathrm{D}$. It is also well known and easy to see that $\mathrm{Conn} \subset \mathrm{D}$.

The classes D, Conn, and SCIVP are among the eight classes of generalized continuous maps from $\mathbb{R}$ to $\mathbb{R}$ known as Darboux-like functions and extensively studied; see, e.g., surveys [15], [8], and [10] and the literature cited therein. See also a recent paper [12], which initiated a systematic study of the 18 atoms of the algebra of subsets of $\mathbb{R}^\mathbb{R}$ generated by the eight Darboux-like classes of functions. The class $\mathrm{SCIVP} \cap \mathrm{D} \setminus \mathrm{Conn}$ is one of the above-mentioned atoms and the problem of determining its best possible lineability is a part of a broad study of determining such lineabilities for all these atoms—the subject of a Ph.D. dissertation of the first author, written under the supervision of the remaining two authors.

We concentrate here only on the class $\mathrm{SCIVP} \cap \mathrm{D} \setminus \mathrm{Conn}$, since finding the optimal lineability for it has proved more challenging than for most of other atoms above discussed and the presented construction is different from the other technique involved in this endeavor. Nevertheless, we should point out that the method we use below is a variation of one used in [4].

2. The Main Result: Statement, Discussion, Sketch of Proof

For a family $\mathcal{G}$ of partial functions from $\mathbb{R}$ into $\mathbb{R}$, let

SZ($\mathcal{G}$) be the family of all generalized Sierpiński–Zygmund functions with respect to $\mathcal{G}$, that is, all maps $f \in \mathbb{R}^{\mathbb{R}}$ such that $|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$; see [11].

Of course, in this language, the classical class of Sierpiński–Zygmund (SZ) functions is defined as SZ($\mathcal{C}$), where $\mathcal{C}$ is the family of all partial continuous functions from $\mathbb{R}$ into $\mathbb{R}$. Below we will use the class SZ($\mathbb{L}$), where

$\mathbb{L}$ is the family of all non-constant affine functions $\ell \in \mathbb{R}^{\mathbb{R}}$, defined as $\ell(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$.

As in [4, §2], we prove $\mathfrak{c}^+$-lineability of ES \ Conn by actually showing $\mathfrak{c}^+$-lineability for a smaller class of SZ($\mathbb{L}$)∩ES \ Conn. In addition, to make sure that all constructed maps are also SCIVP, we will find a family $\mathcal{P}$ of pairwise disjoint perfect subsets of $\mathbb{R}$ with the properties described in Lemma 2.2 and make sure that all constructed functions also belong to the class

$[C_{\mathcal{P}}]$ of all $f \in \mathbb{R}^{\mathbb{R}}$ such that f is constant on every $P \in \mathcal{P}$.

In other words, the $\mathfrak{c}^+$-lineability of ES ∩ SCIVP \ Conn will be shown by proving the following theorem.

Theorem 2.1. *If continuum $\mathfrak{c}$ is a regular cardinal number, then there exists a family $\mathcal{P}$ of pairwise disjoint perfect subsets of $\mathbb{R}$ such that the class $C_{\mathcal{P}} \cap$ SZ($\mathbb{L}$) ∩ ES ∩ SCIVP \ Conn is $\mathfrak{c}^+$-lineable.*

Of course, since $\mathcal{E} := C_{\mathcal{P}} \cap$ SZ($\mathbb{L}$) ∩ ES ∩ SCIVP \ Conn is contained in D ∩ SCIVP \ Conn, the theorem immediately implies that the class D ∩ SCIVP \ Conn is also $\mathfrak{c}^+$-lineable as long as $\mathfrak{c}$ is regular.

The perfect sets in $\mathcal{P}$ are chosen small enough so that, from the perspective of the proof of Theorem 2.1, they will behave like singletons. The precise meaning of this last statement is expressed in properties (b) and (c) in Lemma 2.2.

Lemma 2.2. *There exists a family $\mathcal{P}$ of pairwise disjoint perfect subsets of $\mathbb{R}$ such that, for every non-empty open interval I,*

(a) *the collection $\{P \in \mathcal{P} : P \subset I\}$ has cardinality $\mathfrak{c}$;*
(b) *for every $\mathcal{P}_0 \subset \mathcal{P}$ of cardinality less than $\mathfrak{c}$ and for every map $\mathcal{P}_0 \ni P \mapsto \lambda_P \in \mathbb{L}$, the set $\bigcup_{P \in \mathcal{P}_0} \lambda_P[P]$ intersects less than $\mathfrak{c}$-many sets in $\mathcal{P}$;*

(c) *for every $\mathcal{P}_0 \subset \mathcal{P}$ and $S \subset \mathbb{R} \setminus \{0\}$, both of cardinality less than $\mathfrak{c}$, there is a set $A \subset (0, \infty)$ of cardinality $\mathfrak{c}$ such that the sets in the family $\{a\,(S \cup \bigcup \mathcal{P}_0) \colon a \in A\}$ are pairwise disjoint.*

We will prove Theorem 2.1 by recursively constructing a strictly increasing sequence $\langle V_\xi \colon \xi < \mathfrak{c}^+\rangle$ of vector spaces contained in $\mathcal{E} \cup \{0\}$ and of cardinality at most $\mathfrak{c}$. Then we note that the union $\bigcup_{\xi<\mathfrak{c}^+} V_\xi$ justifies $\mathfrak{c}^+$-lineability of $\mathcal{E}$. The construction of the sequence is facilitated by the following lemma.

Lemma 2.3. *If $\mathfrak{c}$ is a regular and $\mathcal{P}$ is as in Lemma 2.2, then for every additive group $V \subset C_{\mathcal{P}} \cap \mathrm{SZ}(\mathbb{L}) \cup \{0\}$ of cardinality at most $\mathfrak{c}$, there exists an $f \in \mathbb{R}^{\mathbb{R}}$ not in V and such that $f + V := \{f + g \colon g \in V\}$ is contained in $C_{\mathcal{P}} \cap \mathrm{SZ}(\mathbb{L}) \cap \mathrm{ES} \cap \mathrm{SCIVP} \setminus \mathrm{Conn}$.*

In Lemma 2.3, the assumption $V \subset \mathrm{SZ}(\mathbb{L})$ is essential since there are groups V of cardinality $\mathfrak{c}$ so that $f + V \not\subset \mathrm{D} \setminus \mathrm{Conn}$ for every $f \in \mathbb{R}^{\mathbb{R}}$; see [12, lemma 4.1] in which this is proved with V being the family of all Borel functions. Also, since Lemma 2.3 will be used to extend each space V_ξ to the next space $V_{\xi+1}$ in the sequence, we will need to ensure that $f + V_\xi \subset \mathrm{SZ}(\mathbb{L})$. It is perhaps also worth mentioning that the class $\mathrm{SZ}(\mathbb{L})$ in this argument is chosen carefully and that the lemma, in its generality, would be false if we state it for the family $\mathrm{SZ} = \mathrm{SZ}(\mathcal{C})$ in place of $\mathrm{SZ}(\mathbb{L})$. This is the case since there are models of ZFC in which $\mathfrak{c} = \omega_2$, so $\mathfrak{c}$ is regular and the class $\mathrm{SZ} \cap \mathrm{D}$ is empty; see [6] or [13, §6.2].

3. The Proofs

For $S \subset \mathbb{R}$, let $\mathbb{Q}(S)$ denote the subfield of $\mathbb{R}$ generated by S (i.e., the smallest subfield of $\mathbb{R}$ containing S), and let $\bar{\mathbb{Q}}(S)$ be the algebraic closure of $\mathbb{Q}(S)$ in $\mathbb{R}$. Recall that S is *algebraically independent* provided that $s \notin \bar{\mathbb{Q}}(S \setminus \{s\})$ for every $s \in S$ and that S is a *transcendental basis* provided it is a maximal algebraically independent subset of $\mathbb{R}$. Every algebraically independent set can be extended to a transcendental basis; see, e.g., [17]. If $T \subset \mathbb{R}$ is a transcendental basis, then, for every $x \in \mathbb{R}$, there exists finite $T_x \subset T$ such that $x \in \bar{\mathbb{Q}}(T_x)$.

Proof of Lemma 2.2. Let $\mathcal{B}$ be the family of all non-empty open intervals with rational endpoints. First, note that

★ there exists a family $\{P_I \subset I \colon I \in \mathcal{B}\}$ of pairwise disjoint perfect sets such that $\bigcup_{I\in\mathcal{B}} P_I$ is algebraically independent.

To see this, let $K \subset \mathbb{R}$ be a compact perfect algebraically independent set. (See the original construction of such a set by J. v. Neumann in [21]. Compare also [18, Theorem 1] and [13, Theorem 5.1.9].) Choose a family

$\{T_I : I \in \mathcal{B}\}$ of pairwise disjoint perfect subsets of K, and, for every $I \in \mathcal{B}$, choose non-zero $p_I, q_I \in \mathbb{Q}$ so that $P_I := p_I T_I + q_I$ is contained in I. Note that these sets satisfy $\bigstar$.

Next, for every $I \in \mathcal{B}$, let $\mathcal{P}_I$ be a partition of P_I into $\mathfrak{c}$-many perfect sets. Then the family $\mathcal{P} := \bigcup_{I\in\mathcal{B}} \mathcal{P}_I$ is as needed.

Indeed, (a) is obvious from the construction. In particular, there exists a transcendental basis T containing $\bigcup \mathcal{P}$.

To see (b), fix $\mathcal{P}_0 \subset \mathcal{P}$ and the map $P \mapsto \lambda_P$ as in its assumption. For every $P \in \mathcal{P}_0$, there is a finite $T_P \subset T$ such that the two coefficients of the map λ_P are in $\bar{\mathbb{Q}}(T_P)$. In particular, $\lambda_P[P] \subset \bar{\mathbb{Q}}(P \cup T_P)$ so that $\bigcup_{P\in\mathcal{P}_0} \lambda_P[P] \subset \bar{\mathbb{Q}}\left(\bigcup_{P\in\mathcal{P}_0}(P \cup T_P)\right)$. But, by the algebraic independence, $\bar{\mathbb{Q}}\left(\bigcup_{P\in\mathcal{P}_0}(P \cup T_P)\right)$ intersects only less than $\mathfrak{c}$-many sets $P \in \mathcal{P}$: those that $P \in \mathcal{P}_0$ and those for which $P \cap \bigcup_{P\in\mathcal{P}_0} T_P \neq \emptyset$.

To see (c), choose $\mathcal{P}_0 \subset \mathcal{P}$ and $S \subset \mathbb{R} \setminus \{0\}$ as in its assumption. For every $s \in S$, choose a finite $T_s \subset T$ such that $s \in \bar{\mathbb{Q}}(T_s)$ and let $T_S := \bigcup_{s\in S} T_s$. Then, as in (b), the set $T_S \cup \bigcup \mathcal{P}_0$ intersects only less than $\mathfrak{c}$-many sets $P \in \mathcal{P}$. So there is a $P \in \mathcal{P}$ contained in $(0,\infty)$ and disjoint with $T_S \cup \bigcup \mathcal{P}_0$. Then the set $A := P$ satisfies (c). Otherwise, there are distinct $a, a' \in A$ and $x, y \in T_S \cup \bigcup \mathcal{P}_0$ with $ax = a'y$, contradicting the fact that the set $\{a, a'\} \cup T_S \cup \bigcup \mathcal{P}_0 \subset T$ is algebraically independent. $\square$

Proof of Lemma 2.3. Let $\mathcal{B}$ be the family of all non-empty open intervals and $\mathcal{R} = \mathcal{P} \cup \{\{x\} : x \in \mathbb{R} \setminus \bigcup \mathcal{P}\}$. Fix the following enumerations:

- $\{\langle I_\eta, y_\eta, g_\eta\rangle : \eta < \mathfrak{c}\}$ of $\mathcal{B} \times \mathbb{R} \times V$;
- $\{R_\eta : \eta < \mathfrak{c}\}$ of $\mathcal{R}$; and
- $\{\ell_\eta : \eta < \mathfrak{c}\}$ of $\mathbb{L}$.

By induction on $\eta < \mathfrak{c}$, we define a sequence $\langle\langle \mathcal{D}_\eta, f_\eta, a_\eta\rangle : \eta < \mathfrak{c}\rangle$ with $f_\eta : \bigcup \mathcal{D}_\eta \to \mathbb{R}$ and aiming for $f := \bigcup_{\eta<\mathfrak{c}} f_\eta$ being our desired map. To achieve this, we will ensure that the following inductive conditions are satisfied for each $\eta < \mathfrak{c}$ and every $\xi \leq \eta$:

(i) $a_\eta \in (0,\infty)$, $|\mathcal{D}_\eta| < \mathfrak{c}$; $\mathcal{D}_\xi \subset \mathcal{D}_\eta \subset \mathcal{R}$, and $R_\xi \in \mathcal{D}_\eta$;
(ii) $f_\xi \subset f_\eta$ and f_η is constant on every $P \in \mathcal{D}_\eta$;
(iii) there is a $P \in \mathcal{D}_\eta \cap \mathcal{P}$ contained in I_ξ with $(f_\eta + g_\xi)[P] = \{y_\xi\}$;
(iv) $(f_\eta + g_\xi)(x) \neq a_\xi x$ for every non-zero $x \in \bigcup \mathcal{D}_\xi$;
(v) if $\alpha, \beta < \xi$, then the set $\{x \in \bigcup \mathcal{D}_\eta : (f_\eta + g_\alpha)(x) = \ell_\beta(x)\}$ is contained in $\{0\} \cup \bigcup \mathcal{D}_\xi$.

Before we construct such a sequence, first note that conditions (i)–(v) actually ensure that $f = \bigcup_{\eta<\mathfrak{c}} f_\eta$ has the desired properties. Indeed, (i) and (ii) ensure, in particular, that f is a function from $\mathbb{R}$ into $\mathbb{R}$ constant on every $P \in \mathcal{P}$; that is, $f \in C_{\mathcal{P}}$ and also $f + g \in C_{\mathcal{P}}$ for every $g \in V$. Condition (iii) implies that for such defined f and any $g \in V$, the map

$f+g$ is both ES and SCIVP, where the continuous function $f \restriction P$ in the definition of SCIVP is just a constant map. Condition (iv) justifies that, for every $g = g_\xi \in V$, the map $f+g$ has a disconnected graph, as (by $f+g \in \mathrm{ES}$), there exist $q > p > 0$ such that $(f+g)(p) > a_\xi p$ and $(f+g)(q) < a_\xi q$, and so the three-segment closed set $(\{p\} \times (-\infty, a_\xi p]) \cup \{(x, a_\xi x) \colon x \in [p,q]\} \cup (\{q\} \times [a_\xi q, \infty))$ separates the graph of $f+g$. Finally, to see that $f+g$ is in $\mathrm{SZ}(\mathbb{L})$, choose an $\ell \in \mathbb{L}$ and let $\alpha, \beta < \mathfrak{c}$ be such that $g = g_\alpha$ and $\ell = \ell_\beta$. Choose any $\xi < \mathfrak{c}$ with $\alpha, \beta < \xi$. Then, by (v), $S := \{x \in \mathbb{R} \colon (f+g)(x) = \ell(x)\}$ is contained in $\{0\} \cup \bigcup \mathcal{D}_\xi$. But, by (ii) and the fact that $f+g \in C_{\mathcal{P}}$, we see that $|(f+g)[\bigcup \mathcal{D}_\xi]| \le |\mathcal{D}_\xi| < \mathfrak{c}$. Therefore, since ℓ is injective, $|S| \le |\mathcal{D}_\xi| < \mathfrak{c}$, showing that, indeed, $f+g \in \mathrm{SZ}(\mathbb{L})$.

By the argument in the above paragraph, to finish the proof of the lemma, it is enough to construct a sequence satisfying (i)–(v). So assume that, for some $\eta < \mathfrak{c}$, the sequence $\langle \langle \mathcal{D}_\eta, f_\eta, a_\eta \rangle \colon \eta < \zeta \rangle$ is already constructed and that it satisfies (i)–(v) for every $\eta < \zeta$. We just need to construct $\mathcal{D}_\zeta$, f_ζ, and a_ζ so that (i)–(v) are also satisfied by the sequence $\langle \langle \mathcal{D}_\eta, f_\eta, a_\eta \rangle \colon \eta < \zeta + 1 \rangle$.

The family $\mathcal{D}_\zeta$ is defined as $\{R_\zeta, P\} \cup \bigcup_{\eta<\zeta} \mathcal{D}_\eta$ for appropriately chosen $P \in \mathcal{R}$, so that (iii), (iv), and (v) can be ensured. We define f_ζ as an extension of $\bigcup_{\eta<\zeta} f_\eta$ so that $(f_\zeta + g_\zeta)[P] = \{y_\zeta\}$ and $f_\zeta[R_\zeta] = \{y\}$ for appropriately chosen $y \in \mathbb{R}$. The construction will be finished with an appropriate choice of a_ζ.

The above scheme ensures satisfaction of (i) and (ii), where the property $|\mathcal{D}_\zeta| < \mathfrak{c}$ is implied by the inductive assumption and the regularity of $\mathfrak{c}$. Next, we choose a needed $P \in \mathcal{R} \setminus \bigcup_{\eta<\zeta} \mathcal{D}_\eta$ contained in I_ζ so that the definition of f_ζ on P required for the satisfaction of (iii)

$$f_\zeta(x) := y_\zeta - g_\zeta(x) \quad \text{for every } x \in P \tag{3.1}$$

does not contradict (iv); that is,

$$f_\zeta(x) \neq a_\xi x - g_\xi(x) \quad \text{for every } x \in P \text{ and } \xi < \zeta \tag{3.2}$$

and (v); that is,

$$f_\zeta(x) \neq \ell_\beta(x) - g_\alpha(x) \quad \text{for every } x \in P \text{ and } \alpha, \beta \le \zeta. \tag{3.3}$$

To avoid conflict between (3.1) and (3.2), we need to choose P disjoint with the sets

$$S_\xi := \{x \in \mathbb{R} \colon (g_\xi - g_\zeta)(x) = a_\xi x - y_\zeta\} \quad \text{for every } \xi < \zeta;$$

while, to avoid conflict between (3.1) and (3.3), our P needs be disjoint with

$$T^\alpha_\beta := \{x \in \mathbb{R} \colon (g_\alpha - g_\zeta)(x) = \ell_\beta(x) - y_\zeta\} \quad \text{for every } \alpha, \beta \le \zeta.$$

But each of the sets S_ξ and T^α_β has cardinality less than $\mathfrak{c}$, as maps $a_\xi x - y_\zeta$ and $\ell_\beta(x) - y_\zeta$ are in $\mathbb{L}$, while $g_\xi - g_\zeta, g_\alpha - g_\zeta \in V \subset \mathrm{SZ}(\mathbb{L}) \cup \{0\}$. Therefore, by the regularity of $\mathfrak{c}$, the union $T := \bigcup_{\xi<\zeta} S_\xi \cup \bigcup_{\alpha,\beta\le\zeta} T^\alpha_\beta$ has cardinality less than $\mathfrak{c}$, so we can choose $P \in \mathcal{P} \setminus \bigcup_{\eta<\zeta} \mathcal{D}_\eta$ contained in $I_\zeta \setminus T$. Such a choice ensures that (iv) and (v) are satisfied for every $x \in P$; $\alpha, \beta \le \zeta$; and $\xi < \zeta$. To ensure that the same is true for $x \in R_\zeta$, first note that this follows from the inductive assumption when $R_\zeta \in \{P\} \cup \bigcup_{\eta<\zeta} \mathcal{D}_\eta$. So assume that this is not the case. We need to choose $y \in \mathbb{R}$ so that the definition

$$f_\zeta(x) := y \quad \text{for every } x \in R_\zeta \tag{3.4}$$

does not contradict (3.2) and (3.3) considered with P replaced with R_ζ. Denoting the singleton value of $g_\xi[R_\zeta]$ by $\{z_\xi\}$, this last requirement means that y does not belong to the sets $a_\xi R_\zeta - z_\xi$, $\xi < \zeta$, and $\ell_\beta[R_\zeta] - z_\alpha$. But the existence of such a y is obvious when R_ζ is a singleton and otherwise follows from the property of the family $\mathcal{P}$ expressed in Lemma 2.2(b). Such a choice of y ensures that all conditions (i)–(v) are satisfied, except (iv) with $\xi = \zeta$, as a_ζ is still not defined. This condition that we still need is

$$(f_\zeta + g_\zeta)(x) \neq a_\zeta x \text{ for every non-zero } x \in \bigcup \mathcal{D}_\zeta. \tag{3.5}$$

But $|\mathcal{D}_\zeta| < \mathfrak{c}$ and $f_\zeta + g_\zeta \in C_\mathcal{P}$ imply that the set $(f_\zeta + g_\zeta)[\bigcup \mathcal{D}_\zeta]$ has cardinality less than $\mathfrak{c}$. On the other hand, by Lemma 2.2(c), used with $\mathcal{P}_0 := \mathcal{P} \cap \mathcal{D}_\zeta$ and $S := \bigcup \mathcal{D}_\zeta \setminus (\{0\} \cup \bigcup \mathcal{P}_0)$, there is a set $A \subset (0, \infty)$ of cardinality $\mathfrak{c}$ such that the sets in $\{a(\bigcup \mathcal{D}_\zeta) \setminus \{0\} \colon a \in A\}$ are pairwise disjoint. Therefore, there exists an $a_\zeta \in A$ with the set $a_\zeta(\bigcup \mathcal{D}_\zeta) \setminus \{0\}$ disjoint with $(f_\zeta + g_\zeta)[\bigcup \mathcal{D}_\zeta]$. But this ensures that (3.5) is satisfied. This choice finishes the construction and the proof of the lemma. □

Proof of Theorem 2.1. Let $\mathcal{P}$ be as in Lemma 2.2. By induction on $\xi < \mathfrak{c}^+$, construct a sequence $\langle V_\xi \colon \xi \le \mathfrak{c}^+ \rangle$ of linear subspaces of $\mathcal{E} \cup \{0\}$, where $\mathcal{E} = C_\mathcal{P} \cap \mathrm{SZ}(\mathbb{L}) \cap \mathrm{ES} \cap \mathrm{SCIVP} \setminus \mathrm{Conn}$, such that $|V_\xi| \le \mathfrak{c}$ for every $\xi < \mathfrak{c}^+$, $V_\lambda = \bigcup_{\eta<\lambda} V_\eta$ for every limit ordinal number $\lambda \le \mathfrak{c}^+$, and $V_{\xi+1} := \bigcup_{r\in\mathbb{R}} (r f_\xi + V_\xi)$ for every $\xi < \mathfrak{c}^+$, where f_ξ is the function f from Lemma 2.3 used with $V = V_\xi$. Then $V_{\mathfrak{c}^+}$ justifies $\mathfrak{c}^+$-lineability of $\mathcal{E}$, as needed. □

4. Corollaries and Open Problems

Of course, Theorem 2.1 immediately implies the following corollary.

Corollary 4.1. *It is consistent with* ZFC*, and follows, for example, from the generalized continuum hypothesis* GCH*, that the class* $\mathrm{D} \cap \mathrm{SCIVP} \setminus \mathrm{Conn}$ *is* $2^{\mathfrak{c}}$*-lineable.*

We have shown in ZFC (see [1] and [2]) that the majority of classes that constitute the atoms of the algebra of Darboux-like classes of functions are $2^{\mathfrak{c}}$-lineable. In this light, the following open problem is natural to state.

Problem 4.2. Can we prove in ZFC that the class $\mathrm{D}\cap\mathrm{SCIVP}\setminus\mathrm{Conn}$ is $\mathfrak{c}^+$-lineable? What about its $2^{\mathfrak{c}}$-lineabilty?

We believe that both these questions have positive answers. Perhaps this can be proved with the technique developed in [1].

Theorem 2.1 does not tell us anything in ZFC about lineabilty of the class $\mathrm{D}\cap\mathrm{SCIVP}\setminus\mathrm{Conn}$. However, a relatively easy proof in ZFC of $\mathfrak{c}$-lineability of this class can be found in [3].

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