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$\mathfrak{c}$-lineability: A general method and its application to Darboux-like maps

Gbrel M. Albkwre, Krzysztof Chris Ciesielski *

Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA

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ABSTRACT

We describe a simple class of $\mathfrak{c}$-dimensional linear subspaces of $\mathbb{R}^{\mathbb{R}}$ and show that many natural classes of real functions contain subspaces of this form, proving their $\mathfrak{c}$-lineability. The examples include all non-empty classes in the algebra of all Darboux-like functions and of their restrictions to the Baire class 2 maps.

1. Lineability and a canonical linear subspace $W_{\mathcal{F}}$ of $\mathbb{R}^X$

Over the last two decades a lot of mathematicians have been interested in finding the largest possible vector spaces that are contained in various families of real functions, see e.g. survey [8], monograph [6], and the literature cited there. (More recent work in this direction includes [5,10,16].) Specifically, given a cardinal number κ, a subset M of a vector space X is said to be *κ-lineable (in X)* provided there exists a linear space $Y \subset M \cup \{0\}$ of dimension κ. This notion was first studied by Vladimir Gurariy [20], even though he did not use the term lineability. He showed that the set of continuous nowhere differentiable functions on [0,1], together with the constant 0 function, contains an infinite-dimensional vector space, that is, it is ω-lineable.

In what follows $\mathfrak{c}$ denotes $|\mathbb{R}|$, the cardinality of $\mathbb{R}$. The goal of this article is to apply the following simple proposition to show $\mathfrak{c}$-lineability of several subclasses of Darboux-like functions.

For a non-empty set X let $\mathbb{R}^X$ be the class of all maps from X into the real line $\mathbb{R}$. We consider $\mathbb{R}^X$ as a linear space over $\mathbb{R}$. For an $f \in \mathbb{R}^X$ its *support* is defined as

$$\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

* Corresponding author.
E-mail addresses: gmalbkwre@mix.wvu.edu (G.M. Albkwre), KCies@math.wvu.edu (K.C. Ciesielski).

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Notice that we do not take the closure of the set above, even when X is a topological space. For a non-empty family $\mathcal{F} \subseteq \mathbb{R}^X$ of non-zero functions with pairwise disjoint supports consider the following vector subspace of $\mathbb{R}^X$:

$$\mathcal{L}_{\mathcal{F}} = \left\{ \sum_{f \in \mathcal{F}} s(f) \cdot f \colon s \in \mathbb{R}^{\mathcal{F}} \right\}.$$

The maps $\sum_{f \in \mathcal{F}} s(f) \cdot f$ are well defined, since the functions in $\mathcal{F}$ have disjoint supports. The following proposition is obvious, unless $2^{|\mathcal{F}|} = \mathfrak{c}$. (In the case $2^{|\mathcal{F}|} > \mathfrak{c}$ it can be found, for example, in [12]. Compare also proof of [14, theorem 2.14].) The cases with $|\mathcal{F}|$ were also considered, but in considerable less general format. (See for example [7, theorem 3.2], where the linear space associated with $\mathcal{F}$ is defined as the closure of the space spanned by $\mathcal{F}$.)

Proposition 1.1. *If $\mathcal{F} \subseteq \mathbb{R}^X$ is an infinite family with pairwise disjoint supports, then $\mathcal{L}_{\mathcal{F}}$ has dimension $2^{|\mathcal{F}|}$.*

Proof. Let $\kappa = |\mathcal{F}|$ and B be a basis for $\mathcal{L}_{\mathcal{F}}$. If $2^\kappa > \mathfrak{c}$, then the conclusion is obvious, since otherwise $|B| < 2^\kappa$ and $2^\kappa = |\mathcal{L}_{\mathcal{F}}| = |\mathbb{R}| \cdot |B| = \mathfrak{c} \cdot |B| < 2^\kappa$, a contradiction. So, assume that $2^\kappa = \mathfrak{c}$. To finish the proof it is enough to find $\mathfrak{c}$-many linearly independent functions in $\mathcal{L}_{\mathcal{F}}$.

Let $\{f_k \colon k < \omega\}$ be a family of distinct non-zero functions in $\mathcal{F}$ and $\mathcal{A}$ be a family of $\mathfrak{c}$-many infinite pairwise almost disjoint subsets of ω. (See e.g. [24, proposition 5.26].) For every $A \in \mathcal{A}$ let $F_A = \sum_{k \in A} f_k \in \mathcal{L}_{\mathcal{F}}$ and notice that $\{F_A \colon A \in \mathcal{A}\} \subset \mathcal{L}_{\mathcal{F}}$ is linearly independent. To see this, choose $c_1, \ldots, c_n \in \mathbb{R}$, distinct $A_1, \ldots, A_n \in \mathcal{A}$, and notice that

$$c_1 F_{A_1} + c_2 F_{A_2} + \cdots + c_n F_{A_n} \neq 0,$$

unless $c_1 = c_2 = \cdots = c_n = 0$. Indeed, if $c_i \neq 0$, then there is a $k \in A_i \setminus \bigcup_{j \neq i} A_j$ and an $x \in \operatorname{supp}(f_k)$ for which $(c_1 F_{A_1} + c_2 F_{A_2} + \cdots + c_n F_{A_n})(x) = c_i f_k(x) \neq 0$. Thus, indeed the equation $2^\kappa = \mathfrak{c}$ implies that $\mathcal{L}_{\mathcal{F}}$ has dimension 2^κ. □

2. $\mathfrak{c}$-lineability of the families of Baire 2 Darboux-like maps

The name *Darboux-like functions* usually refers to the eight classes of generalized continuous maps from $\mathbb{R}$ to $\mathbb{R}$, five of which are defined below. (The remaining three classes will be discussed in the next section.) These classes have been extensively studied, see e.g. surveys [11,13,18] and the literature cited there. Compare also the recent paper [15]. For a metric space X, the five main classes of Darboux-like functions from X to $\mathbb{R}$ are defined as follows. (These notions will be used here mainly when X is an interval in $\mathbb{R}$.)

- $\mathscr{D}_X$ of all *Darboux functions* $f \in \mathbb{R}^X$, that is, such that $f[C]$ is connected (i.e., an interval) for every connected $C \subset X$. This class, for $X = \mathbb{R}$ was first systematically investigated by Jean-Gaston Darboux (1842–1917) in his 1875 paper [17].
- PC_X of all *peripherally continuous functions* $f \in \mathbb{R}^X$, that is, such that for every $x \in X$, open interval $J \subset \mathbb{R}$ containing $f(x)$, and $\varepsilon > 0$, there exists an open neighborhood U of x of diameter $< \varepsilon$ such that $f[\operatorname{bd}(U)] \subset J$, where $\operatorname{bd}(U)$ is the boundary (or periphery) of U. Notice that for $X = \mathbb{R}$ this is equivalent to the statement that for every $x \in \mathbb{R}$ there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ with $\lim_{n\to\infty} f(s_n) = f(x) = \lim_{n\to\infty} f(t_n)$. This class was introduced in a 1907 paper [32] of John Wesley Young (1879–1932). The name comes from the papers [21,22,31]. Note that any function in $\mathbb{R}^{\mathbb{R}}$ with a graph dense in $\mathbb{R}^2$ is PC.

Conn_X of all *connectivity functions* $f \in \mathbb{R}^X$, that is, such that the graph of f restricted to any connected $C \subset X$ is a connected subset of $X \times \mathbb{R}$. This notion can be traced to a 1956 problem [27] stated by John Forbes Nash (1928–2015). We also refer to [22,30].

AC_X of all *almost continuous functions* $f \in \mathbb{R}^X$ (in the sense of Stallings), that is, such that every open subset of $X \times \mathbb{R}$ containing the graph of f contains also the graph of a continuous function from X to $\mathbb{R}$. This class was first seriously studied in a 1959 paper [30] of John Robert Stallings (1935–2008); however, it appeared already in a 1957 paper [22] by Olan H. Hamilton (1899–1976). See also 1991 survey [28] by T. Natkaniec.

Ext_X of all *extendable functions* $f \in \mathbb{R}^X$, that is, such that there exists a connectivity function $g\colon X \times [0,1] \to \mathbb{R}$ with $f(x) = g(x,0)$ for all $x \in X$. The notion of extendable functions (without the name) first appeared in a 1959 paper [30] of J. Stallings, where he asks a question whether every connectivity function defined on [0, 1] is extendable.

Let $\mathcal{P}_X = \{\mathrm{Ext}_X, \mathrm{AC}_X, \mathrm{Conn}_X, \mathscr{D}_X, \mathrm{PC}_X\}$ be the collection of all these classes. We will drop the subscript X in this notation when $X = \mathbb{R}$ or when X is clear from the context.

The topological nature of these definitions and their format easily justifies the following properties of these classes.

Remark 2.1. Let $P \in \mathcal{P}$ and X and Y be metric spaces.

(i) If $g \in P_Y$ and f is a homeomorphism from X to Y, then $g \circ f \in P_X$.
(ii) If $g \in P_Y$ and $h\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism, then $h \circ g \in P_Y$.
(iii) If $g \in P_Y$, $P \neq \mathrm{AC}$, and $B \subset Y$, then $g \restriction B \in P_B$.

The format of the definition of the class AC does not allow to immediately conclude for it the above property (iii). In fact, in such generality the statement is false, see [28, example 2.1]. However, the following result for this family can be found in [28, corollary 2.2].

Proposition 2.2. *Let* $f \in \mathbb{R}^{\mathbb{R}}$. *Then*

$$f \in \mathrm{AC} \text{ if, and only if, } f \restriction [k, k+1] \in \mathrm{AC}_{[k,k+1]} \text{ for every } k \in \mathbb{Z}.$$

The sufficiency condition in Proposition 2.2 will be also needed for the other three classes:

Lemma 2.3. *If* $f \in \mathbb{R}^{\mathbb{R}}$, $P \in \{\mathrm{AC}, \mathrm{Conn}, \mathscr{D}, \mathrm{PC}\}$, *and* $f \restriction [k, k+1] \in P_{[k,k+1]}$ *for every* $k \in \mathbb{Z}$, *then* $f \in P$.

Proof. For $P = \mathrm{AC}$ this is implied by Proposition 2.2.

For $P = \mathrm{PC}$ the result is obvious from the definition of the class PC.

To see this for $P = \mathrm{Conn}$ choose a connected $C \subset \mathbb{R}$, that is, an interval, and notice that the non-empty consecutive sets in the family $\{f \restriction ([k, k+1] \cap C)\colon k \in \mathbb{Z}\}$ are connected (since $f \restriction [k, k+1] \in \mathrm{Conn}_{[k,k+1]}$ and $[k, k+1] \cap C$ is connected) and have non-empty intersections. So, by a well known result (see e.g. [26, theorem 23.3]) their union $f \restriction C$ is connected, as needed, that is, indeed $f \in \mathrm{Conn}$.

The argument for $P = \mathscr{D}$ is similar, where for a connected $C \subset \mathbb{R}$ we just notice that $f[C]$, the union of the family $\{f[[k, k+1] \cap C]\colon k \in \mathbb{Z}\}$, is connected. □

It is well known (see e.g. one of the papers [11,13,15,18] or Example 2.4 below) that these classes are related as follows:

$$\mathrm{Ext} \subsetneq \mathrm{AC} \subsetneq \mathrm{Conn} \subsetneq \mathscr{D} \subsetneq \mathrm{PC}\,. \tag{1}$$

It is also known that within the family B_1 of Baire class 1 functions (i.e., pointwise limits of continuous functions) all these classes coincide. On the other hand, within the family B_2 of Baire class 2 functions (i.e., pointwise limits of B_1 functions) all inclusions presented in (1) remain strict. This means that neither of the classes

$$B_2 \cap (\mathrm{PC} \setminus \mathscr{D}),\ B_2 \cap (\mathscr{D} \setminus \mathrm{Conn}),\ B_2 \cap (\mathrm{Conn} \setminus \mathrm{AC}),\ \text{and}\ B_2 \cap (\mathrm{AC} \setminus \mathrm{Ext}) \tag{2}$$

is empty. The goal of this section is to show that each of these classes is $\mathfrak{c}$-lineable, which is the best result in this direction, since their cardinalities are bounded by $|B_2| = \mathfrak{c}$. To prove this lineability result, we recall the following fact. (Compare also citations in [13].)

Example 2.4. Each of the classes listed in (2) contains a Baire class 2 function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$.

Proof. For the class $B_2 \cap (\mathrm{AC} \setminus \mathrm{Ext})$ see [13, theorem 3.1].

For the class $B_2 \cap (\mathrm{Conn} \setminus \mathrm{AC})$ see [23].

For the class $B_2 \cap (\mathscr{D} \setminus \mathrm{Conn})$ see [9, example 2].

For the class $B_2 \cap (\mathrm{PC} \setminus \mathscr{D})$ take a function $F\colon [-1,1] \to \mathbb{R}$ from [13, example 3.5] which is in $B_2 \setminus \mathscr{D}$. It is PC, since it belongs to the class SCIVP $\subset$ PC discussed in the next section. Thus, if ℓ maps linearly $[0,1]$ onto $[-1,1]$, then, by Remark 2.1, the function f defined by $f(x) = F \circ \ell(x) - 1$ is as needed. □

For a non-zero $f\colon [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$ let

$$\mathcal{F}_f := \{f_k \colon k \in \mathbb{Z}\},$$

where $f_0 \in \mathbb{R}^{\mathbb{R}}$ is an extension of f such that $f_0 \equiv 0$ on the complement of $[0,1]$ and, generally, $f_k \in \mathbb{R}^{\mathbb{R}}$ is defined as $f_k(x) := f_0(x-k)$. Notice that $\mathcal{F}_f$ is an infinite countable family of functions that have disjoint supports, since $\mathrm{supp}(f_k) \subset (k, k+1)$ for every $k \in \mathbb{Z}$. So, $\mathcal{L}_{\mathcal{F}_f}$ is well defined and has dimension $2^{|\mathcal{F}_f|} = \mathfrak{c}$.

Theorem 2.5. *Let $f\colon [0,1] \to \mathbb{R}$ be such that $f(0) = f(1) = 0$. If $P, Q \in \mathcal{P}$ are such that $f \in P \setminus Q$, then $\mathcal{L}_{\mathcal{F}_f}$ justifies $\mathfrak{c}$-lineability of $P \setminus Q$.*

Proof. By Proposition 1.1, the space $\mathcal{L}_{\mathcal{F}_f}$ is well defined and has dimension $\mathfrak{c}$. Also, (1) and $P \setminus Q \neq \emptyset$ imply that $P \in \{\mathrm{AC}, \mathrm{Conn}, \mathscr{D}, \mathrm{PC}\}$ and $Q \in \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathscr{D}\}$.

Next, take a non-zero $g \in \mathcal{L}_{\mathcal{F}_f}$. We need to show that $g \in P \setminus Q$. Indeed, $g = \sum_{k\in\mathbb{Z}} c_k f_k$ for some constants c_k not all zero. Moreover, $g \restriction [k, k+1] = c_k f \circ t_k$, where $t_k\colon [k, k+1] \to [0,1]$ is a translation given by $t_k(x) = x - k$. In particular, by Remark 2.1(i)&(ii), $g \restriction [k, k+1] \in P_{[k,k+1]}$ for every $k \in \mathbb{Z}$. Thus, by Lemma 2.3, indeed $g \in P$.

Next, fix a $k \in \mathbb{Z}$ such that $c_k \neq 0$ and notice that, by Lemma 2.3 and the fact that $f_0 \restriction [j, j+1] \in Q$ for every non-zero $j \in \mathbb{Z}$, we have $f_0 \notin Q$. So, by Remark 2.1(i)&(ii), also $c_k f_k \notin Q$. In particular, the contrapositive version of Remark 2.1(iii) implies that $g \restriction [k, k+1] = c_k f_k \restriction [k, k+1] \notin Q_{[k,k+1]}$. Thus, by Proposition 2.2 and contrapositive of Remark 2.1(iii), $g \notin Q$, finishing the proof. □

Now, we are ready for the main result of this section.

Corollary 2.6. *Each of the classes listed in (2) is $\mathfrak{c}$-lineable.*

Proof. Choose $P, Q \in \mathcal{P}$ so that $B_2 \cap (P \setminus Q)$ is one of the classes listed in (2). By Example 2.4, there exists an $f\colon [0,1] \to \mathbb{R}$ in $B_2 \cap (P \setminus Q)$ such that $f(0) = f(1) = 0$. By Theorem 2.5, the family $\mathcal{L}_{\mathcal{F}_f}$ justifies $\mathfrak{c}$-lineability of $P \setminus Q$. To finish the proof, it is enough to notice that if $f \in B_2$, then any map $g = \sum_{k\in\mathbb{Z}} c_k f_k$ from $\mathcal{L}_{\mathcal{F}_f}$ is also Baire class 2. □

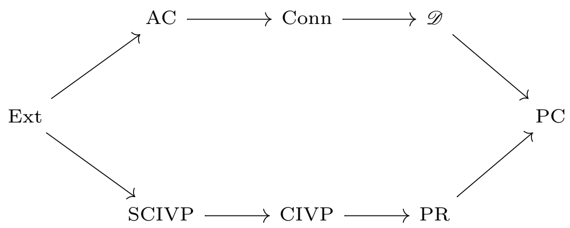

Fig. 1. All inclusions, indicated by arrows, among the Darboux-like classes $\mathbb{D}$. The only inclusions among the intersections of these classes are those that follow trivially from this schema. (See [13,18].)

3. $\mathfrak{c}$-lineability of Darboux-like subclasses of Conn \ AC

For the functions from the interval J into $\mathbb{R}$ there are also three more classes of Darboux-like functions.

PR_J of all functions $f \in \mathbb{R}^J$ *with perfect road*, that is, such that for every $x \in J$ there exists a perfect $P \subset J$ containing x such that: x is a bilateral limit point of P (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) when x is an interior point of J; and that $f \restriction P$ is continuous at x. This class was introduced in a 1936 paper [25] of Isaie Maximoff, where he proved that $\mathscr{D} \cap B_1 = \mathrm{PR} \cap B_1$.

CIVP_J of all functions $f \in \mathbb{R}^J$ *with Cantor Intermediate Value Property*, that is, such that for all distinct $p, q \in J$ with $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$. This class was first introduced in a 1982 paper [19] of Richard G. Gibson and Fred William Roush.

SCIVP_J of all functions $f \in \mathbb{R}^J$ *with Strong Cantor Intermediate Value Property*, that is, such that for all $p, q \in J$ with $p \neq q$ and $f(p) \neq f(q)$ and for every perfect set K between $f(p)$ and $f(q)$, there exists a perfect set P between p and q such that $f[P] \subset K$ and $f \restriction P$ is continuous. This notion was introduced in a 1992 paper [29] of Harvey Rosen, R. Gibson, and F. Roush to help distinguish extendable and connectivity functions on $\mathbb{R}$.

As before, we will drop the subscript J in this notation when $J = \mathbb{R}$ or when J is clear from the context.

Let $\mathbb{D} := \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathscr{D}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}, \mathrm{PC}\}$. The diagram in Fig. 1 shows the relations between the classes in $\mathbb{D}$. The arrows denote strict inclusions.

It is well known (see e.g. [13, theorem 1.2]) and easy to see that

$$\text{every Darboux Borel (so } B_2\text{) map } f \in \mathbb{R}^{\mathbb{R}} \text{ is SCIVP.} \tag{3}$$

This, together with Example 2.4 and Corollary 2.6, implies immediately that

Corollary 3.1. *Each of the following classes (of column 4 of Table 1) is* $\mathfrak{c}$*-lineable:*

$$\mathrm{SCIVP} \setminus \mathscr{D},\ \mathrm{SCIVP} \cap \mathscr{D} \setminus \mathrm{Conn},\ \mathrm{SCIVP} \cap \mathrm{Conn} \setminus \mathrm{AC},\ \mathrm{SCIVP} \cap \mathrm{AC} \setminus \mathrm{Ext}. \tag{4}$$

The Boolean algebra $\mathcal{A}(\mathbb{D})$ of subsets of $\mathbb{R}^{\mathbb{R}}$ generated by $\mathbb{D}$ has 18 atoms: Ext, $\mathbb{R}^{\mathbb{R}} \setminus \mathrm{PC}$, and 16 intersections listed in Table 1, where we use the notation $\neg\mathbb{G} := \mathbb{R}^{\mathbb{R}} \setminus \mathbb{G}$. To see this, notice that the upper row of Fig. 1, that is the inclusions (1), leads to the atoms of $\mathcal{A}(\mathcal{P}) \subset \mathcal{A}(\mathbb{D})$ listed in the left column of the table, while the lower row of Fig. 1

$$\mathrm{Ext} \subsetneq \mathrm{SCIVP} \subsetneq \mathrm{CIVP} \subsetneq \mathrm{PR} \subsetneq \mathrm{PC} \tag{5}$$

Table 1
All atoms of $\mathcal{A}(\mathbb{D})$ except for Ext and $\neg$ PC. The entries constitute the intersections of the classes from the left column and top row.

$\bigcap$	PC $\setminus$ PR	PR $\setminus$ CIVP	CIVP $\setminus$ SCIVP	SCIVP $\setminus$ Ext
PC $\setminus \mathscr{D}$	PC $\cap$ $\neg$(PR $\cup \mathscr{D}$)	PR $\cap$ $\neg$(CIVP $\cup \mathscr{D}$)	CIVP $\cap$ $\neg$(SCIVP $\cup \mathscr{D}$)	SCIVP $\setminus \mathscr{D}$
$\mathscr{D} \cap$ $\neg$ Conn	$\mathscr{D} \cap$ $\neg$(PR $\cup$ Conn)	$\mathscr{D} \cap$ PR $\cap$ $\neg$(CIVP $\cup$ Conn)	$\mathscr{D} \cap$ CIVP $\cap$ $\neg$(SCIVP $\cup$ Conn)	$\mathscr{D} \cap$ SCIVP $\cap$ $\neg$ Conn
Conn $\cap$ $\neg$ AC	Conn $\cap$ $\neg$(PR $\cup$ AC)	Conn $\cap$ PR $\cap$ $\neg$(CIVP $\cup$ AC)	Conn $\cap$ CIVP $\cap$ $\neg$(SCIVP $\cup$ AC)	Conn $\cap$ SCIVP $\cap \neg$ AC
AC $\cap$ $\neg$ Ext	AC $\setminus$ PR	AC $\cap$ PR $\cap$ $\neg$ CIVP	AC $\cap$ CIVP $\cap$ $\neg$ SCIVP	AC $\cap$ SCIVP $\cap$ $\neg$ Ext

leads to the atoms of $\mathcal{A}(\{\text{Ext}, \text{SCIVP}, \text{CIVP}, \text{PR}, \text{PC}\}) \subset \mathcal{A}(\mathbb{D})$ listed in the top row of Table 1.

The classes from Table 1 have been recently intensively studied, see the long paper [15]. Finding maximal lineabilities of these classes is a subject of a Ph.D. dissertation of the first author, written under the supervision of the second author. In particular, article [3] concerns lineability of the class $\mathscr{D} \setminus \text{Conn}$. We also know, see [1,2], that all the classes in the table—with the exceptions of those in the third row and the last class $\text{SCIVP} \cap (\mathscr{D} \setminus \text{Conn})$ in the second row—are (provable in ZFC) $2^{\mathfrak{c}}$-lineable. Concerning this last class, it is proved in [4] that it is $\mathfrak{c}^+$-lineable whenever $\mathfrak{c}$ is a regular cardinal number. Of course, by Corollary 3.1, all classes in the last column of the table are (provably in ZFC) $\mathfrak{c}$-lineable.

The goal of this section is to prove $\mathfrak{c}$-lineability of the first three classes from the third row, that is, the classes

$$\text{Conn} \setminus (\text{PR} \cup \text{AC}),\ \text{Conn} \cap \text{PR} \setminus (\text{CIVP} \cup \text{AC}),\ \text{Conn} \cap \text{CIVP} \setminus (\text{SCIVP} \cup \text{AC}) \tag{6}$$

for which nothing was known so far in this direction. Notice that each class in (6) is the intersection of the class $\text{Conn} \setminus \text{AC}$ with one of the following classes:

$$\text{PC} \setminus \text{PR},\ \text{PR} \setminus \text{CIVP},\ \text{and CIVP} \setminus \text{SCIVP}. \tag{7}$$

We will use the following variant of Theorem 2.5, with a very similar proof.

Theorem 3.2. *Let $f\colon [0,1] \to \mathbb{R}$ be such that $f(0) = f(1) = 0$. If P and Q are among the classes in $\{\text{SCIVP}, \text{CIVP}, \text{PR}, \text{PC}\}$ and such that $f \in P \setminus Q$, then $\mathcal{L}_{\mathcal{F}_f}$ justifies $\mathfrak{c}$-lineability of $P \setminus Q$.*

Proof. First, notice that the analogues of Remark 2.1(i)–(iii) and Lemma 2.3 hold also for the classes PC, PR, CIVP, and SCIVP. By Proposition 1.1, the space $\mathcal{L}_{\mathcal{F}_f}$ is well defined and has dimension $\mathfrak{c}$. Also, (5) and $P \setminus Q \neq \emptyset$ imply that $P \in \{\text{CIVP}, \text{PR}, \text{PC}\}$ and $Q \in \{\text{SCIVP}, \text{CIVP}, \text{PR}\}$.

Take a non-zero $g \in \mathcal{L}_{\mathcal{F}_f}$. We need to show that $g \in P \setminus Q$. Indeed, $g = \sum_{k \in \mathbb{Z}} c_k f_k$ for some constants c_k not all zero. Moreover, $g \restriction [k, k+1] = c_k f \circ t_k$, where $t_k\colon [k, k+1] \to [0,1]$ is a translation given by $t_k(x) = x - k$. In particular, by above mentioned analog of Remark 2.1(i)&(ii), $g \restriction [k, k+1] \in P_{[k,k+1]}$ for every $k \in \mathbb{Z}$. Thus, by an analogue of Lemma 2.3, indeed $g \in P$. Also, if $k \in \mathbb{Z}$ is such that $c_k \neq 0$, then, by an analogue of Remark 2.1(ii), $g \restriction [k, k+1] \notin Q_{[k,k+1]}$. So, by the analogues of Proposition 2.2 and of the contrapositive of Remark 2.1(iii), $g \notin Q$, finishing the proof. □

Now, we are ready for the main result of this section.

Corollary 3.3. *Each of the classes listed in (6) is $\mathfrak{c}$-lineable.*

Proof. First notice that we have an analogue of Example 2.4 for the classes in (6):

(a) each of the classes in (6) contains an $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$.

To see this, first recall the class ES consist of all $f \in \mathbb{R}^{\mathbb{R}}$ for which every level set $f^{-1}(r)$ is dense in $\mathbb{R}$. It follows from the results presented in the paper [15] that

(b) each of the classes in (6) contains an $F\colon \mathbb{R} \to \mathbb{R}$ that belongs also to ES.

Indeed, the additivity coefficient A associated with each class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ (see e.g. [15, definition 1.1]) has the property that $A(\mathcal{F}) \geq 2$ if, and only if, $\mathcal{F} \neq \emptyset$ (see e.g. [15, proposition 1.2(i)]) while we have the following results:

[15, theorem 6.3]: $A(\mathrm{ES} \cap \mathrm{Conn} \setminus (\mathrm{PR} \cup \mathrm{AC})) \geq \omega_1$,
[15, theorem 7.2]: $A(\mathrm{ES} \cap \mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{CIVP} \cup \mathrm{AC})) \geq \omega_1$,
[15, theorem 8.2]: $A(\mathrm{ES} \cap \mathrm{Conn} \cap \mathrm{CIVP} \setminus (\mathrm{SCIVP} \cup \mathrm{AC})) \geq \omega_1$.

This clearly implies (b). To see (a) let $F \in \mathrm{ES}$ belong to one of the classes in (6), choose $a < b$ so that $F(a) = F(b) = 0$, and notice that $f := F \circ \ell_{[a,b]}$ is as needed for (a), where $\ell_{[a,b]}$ maps linearly $[0,1]$ onto $[a,b]$.

To finish the proof, take an f as in (a) and notice that $\mathcal{L}_{\mathcal{F}_f}$ justifies $\mathfrak{c}$-lineability of an appropriate family from (6). Indeed, every non-zero map $g \in \mathcal{L}_{\mathcal{F}_f}$ is in $\mathrm{Conn} \setminus \mathrm{AC}$ by Theorem 2.5, while by Theorem 3.2 it belongs also to an appropriate class listed in (7). But this means that indeed g is in an appropriate class from (6). $\square$

We should also remark here, that technique used in the proof of Corollary 3.3 can be also used to prove that all other classes from Table 1 are $\mathfrak{c}$-lineable. All one needs for such proof is to notice that each class in the table contains a function as in (a) in the proof of Corollary 3.3. However, the $\mathfrak{c}$-lineability of other classes follows from the stronger lineability results we mentioned above, so there is no reason for completing such argument here.

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