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Abstract. One of the most influential examples in analysis is a Weierstrass function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous but differentiable at no point. However this map, as well most of the others among the myriad similar examples, still admits vertical tangent lines. The examples of continuous maps that admit no tangent line in any direction are also known; however, all currently existing presentations of such maps are not easily accessible due to their very complicated descriptions and hard-to-follow proofs of their desired properties. The goal of this article is to present in an accessible way two such examples. The first—a coordinate of the classical Peano space-filling curve—is simpler, but admits one-sided vertical tangent lines at some points. The second is a variation of a function from a 1924 paper of Besicovitch, which is continuous but admits no *one-sided* tangent line in any direction. The proofs of nondifferentiability of these two examples will be facilitated by a simple yet general lemma that also implies nondifferentiability of other similar maps, including those of Takagi and van der Waerden.

1. BACKGROUND. In 1872 Karl Weierstrass (1815–1897) presented to the Prussian Academy of Sciences an example of a function $W\colon \mathbb{R} \to \mathbb{R}$ (compare [20, p. 19] or [49])

$$W(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \text{ with } 0 < a < 1, \text{ odd } b, \text{ and } ab > 1 + \tfrac{3}{2}\pi$$

which, as he showed, is everywhere continuous but differentiable at no point. The printed version of this result appeared for the first time only 14 years later, in an 1886 paper [45]. (For its English translation, see [46].) Although the fact of an existence of this kind of map can be traced back to an 1822 work of Bernard Bolzano (1781–1848), see [21], and was also hinted at by several other prominent mathematical figures before 1872 (see [20, p. 2]), the mathematicians of this period commonly believed that a continuous function must have a derivative at a "significant" set of points. This belief was certainly fueled by a "proof" of this "fact" presented by the French physicist and mathematician André-Marie Ampère (1775–1836) in his 1806 paper [1]. Typical of the standards of mathematical proof of this time, Ampère's argument was not written in the rigorous standards we apply nowadays. In fact, the discovery of the Weierstrass example had a significant impact on the "revolutionary" increase in the rigor of mathematical proofs that was introduced in the late 19th century and is still currently practiced. Not all mathematicians contemporary to Weierstrass were happy to see such drastic changes in the expected standards of mathematical writing. It was probably due to such misgivings that led to the introduction of the term *Weierstrass's monsters*, or shortly *W-monsters*, when referring to the continuous nowhere differentiable maps. It seems that Henri Poincaré (1854–1912) was the first to use term monsters for such functions; see [20, p. 2].

Of course, in the 150 years of mathematical development since the original introduction of W-monster by Weierstrass, a myriad of similar examples were described. In this MONTHLY alone there are 14 published articles concerning such maps; see

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[2, 3, 6, 7, 9, 22, 26, 28, 29, 34, 37, 41, 47, 48]. An extensive bibliography on this subject can be found in a 300-page monograph [20] dedicated to W-monsters published in 2015. In particular, while most of existing proofs of nondifferentiability of W are quite involved, the argument is considerably easier for the W-monster $T := T_{1/2,2}$ described in a 1903 paper [42] of Teiji Takagi (1875–1960), where for $0 < a < 1$ and $ab \geq 1$ the map $T_{a,b}\colon \mathbb{R} \to \mathbb{R}$, known as a *Takagi–van der Waerden function*, is defined:

$$T_{a,b}(x) := \textstyle\sum_{n=0}^{\infty} a^n \operatorname{dist}(b^n x, \mathbb{Z})$$

with $\operatorname{dist}(x, \mathbb{Z})$ being the distance from x to the set $\mathbb{Z} := \{\ldots, -1, 0, 1, 2, \ldots\}$ of all integers. The map $T_{1/10,10}$ was also described in the same context in an influential 1930 paper [43] of Bartel Leendert van der Waerden (1903–1996). Our favorite W-monster is $T_{1/2,8}$, since the proof of its W-monstrosity is particularly short and elegant; see [10] and [36, Theorem 7.18]. Functions $T_{a,b}$ with $b \in \mathbb{N}$ are examples of self-affine maps, which have been extensively studied in the context of differentiability (see, e.g., [15]) and, more generally, rectifiability of their graphs.

Although the examples presented above are nowhere differentiable, they all admit vertical tangent lines. In fact, this is the case for most (but not all) W-monsters described in the literature. For the function $T = T_{1/b,b}$ with $b \in \mathbb{N} := \{1, 2, \ldots\}$, $b \geq 2$, the existence of such tangents is easily seen at any b-nary point $\frac{k}{b^n}$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, since at these points the unilateral infinite derivatives of T exist—the right T'_+ equal $+\infty$, the left T'_- equal $-\infty$—as proved in a 1936 paper [3] of Edward Griffith Begle (1914–1978) and William Leake Ayres (1905–1976); see also [20, Theorem 9.3.1]. The graph of T near such points, which are called *cusps*, resembles the graph of $\pm\sqrt[4]{x^2}$ near 0. Moreover, such T also admits infinite derivatives on a set of Hausdorff dimension one; see, e.g., [17]. For the function W, the existence of vertical tangents is a bit more difficult to see, but it was already known to Arnaud Denjoy (1884–1974) in 1915 as indicated in [14, p. 210]. (See also [33].) Specifically, in a 1916 paper [44] (see also [20, Theorem 3.5.5]) Grace Chisholm Young[1] (1868–1944) proved that W admits cusps on a dense set. At the same time, W does not have infinite derivatives, a fact known already to Weierstrass; see [20, Theorem 3.5.1]. To remediate this "weakness" of existing examples of W-monsters, Abram Samoilovitch Besicovitch (1891–1970) constructed, in a 1924 paper [4] (in Russian, for the German version see [5]), a W-monster $B_a\colon [0, 2a] \to \mathbb{R}$, $a > 0$, which admits neither finite nor infinite unilateral derivative at any point. Of course, such a function admits no tangents in any direction. Such maps are nowadays often called *Besicovitch functions* and, for brevity, we will refer to them in what follows as *B-monsters*. The different constructions of B-monsters were also given in a 1938 paper [31] of Anthony Perry Morse (1911–1984) and in the early 1940s by Avadhesh Narayan Singh [39, 40].[2] (Singh announced the existence of his examples already at the 1936 International Congress of Mathematicians [38].) However, the descriptions of all these B-monsters were quite complicated, often unclear, and the proofs of their desired properties involved and hard to follow. In particular, the constructions of Singh and Morse, described in the monograph [20, Chapter 11]), are each over 10 pages long and give no clear indications what the intuitive reasons are for these examples to have their properties. The actual construction of Besicovitch is actually not explicitly described in [20]. Already in a 1928 paper [33], Echo Dolores Pepper (1897–1979)[3] provided a revised description

[1]The first woman who received a doctorate in any field in Germany, degree granted in 1895.

[2]More specifically, the function from [39] is just a different description of the original Besicovitch function $B_{1/2}$.

[3]One of the pioneering women in American mathematics; see [18].

of the original Besicovitch function B and gave a proof that it indeed has no one-sided (finite of infinite) derivatives as "his proof that the function has no derivative is rather complicated." Unfortunately, the description of the function B given in [33] is still quite hard to uniquely interpret.[4]

The goal of this article is to provide an accessible description of W-monsters that admit tangents at no points: the coordinate of the classical Peano space-filling curve, presented in Section 3, and a version of the original B-monster of Besicovitch, described in Section 4. Final remarks on the subject are presented in Section 5.

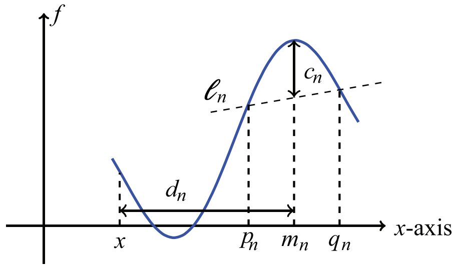

Figure 1. Illustration of the quantities used in the quotient $\frac{c_n}{d_n} = \frac{f(m_n)-\ell_n(m_n)}{m_n-x}$ from Lemma 1. The dashed line ℓ_n is passing through $\langle p_n, f(p_n)\rangle$ and $\langle q_n, f(q_n)\rangle$, $c_n := f(m_n) - \ell_n(m_n)$ represents the displacement of the value of f at m_n with respect to the value $\ell_n(m_n)$ on the chord $\ell \restriction [p_n, q_n]$, while $d_m := m_n - x$. Notice that the distance $q_n - p_n$ has no direct influence on the value of $\frac{c_n}{d_n}$. On the other hand, the proportion $\frac{c_n}{d_n}$ of the "linearity displacement" measure c_n to the "distance to the point of tangency" d_n must approach to 0, as $n \to \infty$, for the right tangent line at x to exist.

2. CONDITION ENSURING NO FINITE DERIVATIVE. The condition, presented in the following lemma, will considerably simplify our proof that the B-monster presented in Section 4 does not admit unilateral finite derivatives. However, it can also be used to deduce similar properties of the other examples of W-monsters, including maps $T_{a,b}$ and f from Section 3, as we show below.

Lemma 1 resembles a bit one from the paper [47], where the existence of the limit $\lim_{n\to\infty} \frac{f(q_n)-f(p_n)}{q_n-p_n}$ is deduced from the fact that the quotients $\frac{q_n-p_n}{q_n-x}$ are bounded away from 0. However, this last assumption is too strong for our purposes, as we use our lemma in the proof of Theorem 4, in which for many points $x \in \mathbb{R}$ we must consider cases with $\frac{q_n-p_n}{q_n-x} \to_n 0$.

Lemma 1 is especially helpful in proving nondifferentiability of functions that have some traces of self-similarity[5] and are representable as limits $f = \lim_{n\to\infty} f_n$ of piecewise linear functions with $f_n^{-1}(0) \subset f_{n+1}^{-1}(0)$ for every $n \in \mathbb{N}$ and with $\bigcup_{k=1}^{\infty} f_k^{-1}(0)$ dense in $\mathbb{R}$. The quantity $f(m_n) - \ell_n(m_n)$ in the lemma measures the deviation of f from its linear approximation ℓ_n: this deviation must be small relative to the distance of m_n to x for the finite right derivative of f at x to exist.

Lemma 1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ has a finite right derivative at $x \in \mathbb{R}$. For every $n \in \mathbb{N}$ let $x \le p_n < m_n < q_n$ and $\ell_n : \mathbb{R} \to \mathbb{R}$ be the line that passes through points $\langle p_n, f(p_n)\rangle$ and $\langle q_n, f(q_n)\rangle$; see Figure 1. If $q_n \to_n x$, then*

$$\lim_{n\to\infty} \frac{f(m_n)-\ell_n(m_n)}{m_n-x} = 0. \tag{1}$$

[4]This seems to be a reason for an incorrect interpretation of the function B_1 in a 2020 paper [16] of Serge Dubuc leading to a conclusion that the Besicovitch function B_1 admits one-sided infinite derivatives (while the claim is true only for the function that was understood as function B_1).

[5]However, the function need not be affine-similar for the lemma to be useful, as we see in its use in Section 4.

Proof. A simple geometric argument shows that $\frac{\ell_n(m_n)-f(x)}{m_n-x}$ is between $\frac{f(p_n)-f(x)}{p_n-x}$ and $\frac{f(q_n)-f(x)}{q_n-x}$. Since the last two difference quotients converge to $f'(x)$, as $n\to\infty$, by the squeeze theorem so does $\frac{\ell_n(m_n)-f(x)}{m_n-x}$. In particular,

$$\frac{f(m_n)-\ell_n(m_n)}{m_n-x}=\frac{f(m_n)-f(x)}{m_n-x}-\frac{\ell_n(m_n)-f(x)}{m_n-x}\underset{n\to\infty}{\longrightarrow} f'(x)-f'(x)=0$$

as promised. ∎

Corollary 2. *Takagi–van der Waerden functions $T_{a,b}$, with $0<a<1$, $ab\geq 1$, and $b\in\mathbb{N}$, admit finite unilateral derivatives at no point.*

Proof. Since $T_{a,b}(-x)=T_{a,b}(x)$, we need to show only that $T_{a,b}$ does not admit a finite right derivative at any $x\in\mathbb{R}$. For this, it is enough to find numbers $p_n<m_n<q_n$ as in Lemma 1 for which (1) fails with $f=T_{a,b}$.

For every $n\in\omega:=\{0,1,2,\ldots\}$ let $T_n(y):=a^n\operatorname{dist}(b^ny,\mathbb{Z})$, so that $T_{a,b}=\sum_{n=0}^{\infty}T_n$. Notice that $T_n^{-1}(0)=b^{-n}\mathbb{Z}$ and that $T_n^{-1}(0)\subset T_{n+1}^{-1}(0)$, since $b\in\mathbb{N}$. Let p_n be the smallest $r\in b^{-n}\mathbb{Z}$ with $r\geq x$ and notice that $q_n:=p_n+b^{-n}$ and $m_n:=p_n+b^{-(n+1)}$ are the smallest elements of $b^{-n}\mathbb{Z}$ and $b^{-(n+1)}\mathbb{Z}$, respectively, that are greater than p_n. These are our desired numbers.

Indeed, clearly $p_n-b^{-n}<x\leq p_n<m_n<q_n=p_n+b^{-n}$, so $q_n\to_n x$. Moreover, $f(m_n)=\sum_{k=0}^{n}T_k(m_n)=T_n(m_n)+\sum_{k<n}T_k(m_n)=b^{-(n+1)}+\ell_n(m_n)$, since both ℓ_n and $\sum_{k<n}T_k$ are linear on $[p_n,q_n]$ and equal to $T_{a,b}$ at the endpoints. In particular, $\frac{f(m_n)-\ell_n(m_n)}{m_n-x}=\frac{b^{-(n+1)}}{m_n-x}>\frac{b^{-(n+1)}}{2b^{-n}}=\frac{1}{2b}$, that is, (1) indeed fails. ∎

3. W-MONSTER ADMITTING NO VERTICAL TANGENT LINES.

We will start by describing a W-monster $f\colon\mathbb{R}\to\mathbb{R}$ as in the section's title. Then, we will show that $f\restriction[0,1]$ is the second coordinate of the classical Peano curve. This fact, that the graph of the coordinate $f\restriction[0,1]$ of the Peano curve admits no tangent lines on $(0,1)$, was first proved by Eliakim Hastings Moore (1862–1932) in a 1900 paper [30]. However, the proof in [30] is considerably more involved than one presented below.

Define $\psi\colon\mathbb{R}\to\mathbb{R}$ by a formula

$$\psi(k+x):=k+2\operatorname{dist}\left(\tfrac{3}{2}x,\mathbb{Z}\right)\quad\text{for every } k\in\mathbb{Z} \text{ and } x\in[0,1)$$

(see Figure 2) and, by induction on $n\in\omega$, define the functions $f_n\colon\mathbb{R}\to\mathbb{R}$ so that f_0 is the identity function and

$$f_{n+1}(x):=f_n\left(9^{-n}\psi\left(9^nx\right)\right)\quad\text{for every } x\in\mathbb{R};$$

see Figure 3.

Theorem 3. *The function $f\colon\mathbb{R}\to\mathbb{R}$ given as $f=\lim_{n\to\infty}f_n$ is a well-defined continuous map that admits finite unilateral derivatives at no point. Also, at no point does f admit two one-sided infinite derivatives.*

Proof. It is easy to see that ψ is continuous and that for every $k\in\mathbb{Z}$, $i\in\{0,1,2\}$, $j\in\{0,2\}$, and $n\in\omega$,

(a) $f_{n+1}\left(\frac{3k+j}{3.9^n}\right)=f_n\left(\frac{k}{9^n}\right)$, and $f_{n+1}\left(\frac{3k+j+1}{3.9^n}\right)=f_n\left(\frac{k+1}{9^n}\right)$;
(b) f_n maps $\left[\frac{3k+i}{3.9^{n-1}},\frac{3k+i+1}{3.9^{n-1}}\right]$ linearly onto an interval of length 3^{1-n}.

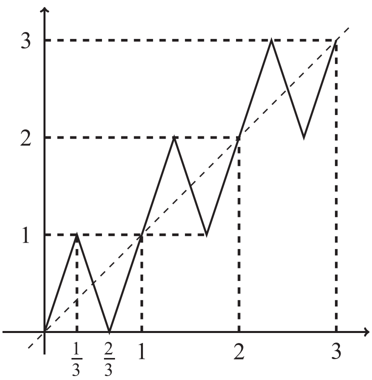

Figure 2. The graph of $f_1 = \psi$, with dashed line representing f_0.

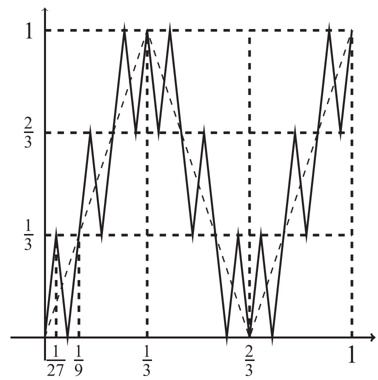

Figure 3. The graph of f_2, with dashed graph representing f_1.

Indeed, (a) holds since we have $f_{n+1}\left(\frac{3k+j}{3\cdot 9^n}\right) = f_n\left(9^{-n}\psi\left(\frac{3k+j}{3}\right)\right) = f_n\left(\frac{k}{9^n}\right)$ and $f_{n+1}\left(\frac{3k+j}{3\cdot 9^n}\right) = f_n\left(9^{-n}\psi\left(\frac{3k+j+1}{3}\right)\right) = f_n\left(\frac{k+1}{9^n}\right)$. Part (b) holds for $n = 0$ since f_0 is the identity function and $\left[\frac{3k+i}{3\cdot 9^{0-1}}, \frac{3k+i+1}{3\cdot 9^{0-1}}\right]$ has length $3 = 3^{1-0}$. The inductive step preserves (b), since $9^{-n}\psi(9^n x)$ maps $\left[\frac{3k+i}{3\cdot 9^n}, \frac{3k+i+1}{3\cdot 9^n}\right] = 9^{-n}\left(k + \left[\frac{i}{3}, \frac{i+1}{3}\right]\right)$ linearly onto $J := 9^{-n}[k, k+1] = \left[\frac{k/3}{3\cdot 9^{n-1}}, \frac{(k+1)/3}{3\cdot 9^{n-1}}\right]$ on which f_n is linear, as $J \subset I := \left[\frac{\lfloor k/3 \rfloor}{3\cdot 9^{n-1}}, \frac{\lfloor k/3 \rfloor + 1}{3\cdot 9^{n-1}}\right]$. Also, by the inductive assumption, $f_n[I]$ has length 3^{1-n} so, J being three times shorter than I, $f_{n+1}\left[\frac{3k+i}{3\cdot 9^n}, \frac{3k+i+1}{3\cdot 9^n}\right] = f_n[J]$ has length $\frac{1}{3} \cdot 3^{1-n} = 3^{1-(n+1)}$, as needed.

By (a) and (b), f_n and f_{n+1} map each interval $\left[\frac{k}{9^{n-1}}, \frac{k+1}{9^{n-1}}\right]$ onto the same interval with the endpoints $f_n\left(\frac{k}{9^n}\right)$ and $f_n\left(\frac{k+1}{9^n}\right)$ and of length 3^{1-n}. So the sup norm $\|f_{n+1} - f_n\|$ of $f_{n+1} - f_n$ is bounded by $2 \cdot 3^{1-n}$ and f is continuous, as a uniform limit of continuous functions.

The argument that f admits finite right derivative at no $x \in \mathbb{R}$ uses Lemma 1 similarly as in the proof of Corollary 2. So, for every $n \in \omega$, let $p_n := \frac{k}{9^n} = \frac{3k}{3\cdot 9^n}$, where $k \in \mathbb{Z}$ is the smallest for which $\frac{k}{9^n} \geq x$. Also, let $m_n := \frac{3k+1}{3\cdot 9^n}$ and $q_n := \frac{3k+2}{3\cdot 9^n}$. Then $p_n - \frac{1}{9^n} < x \leq p_n < m_n < q_n < p_n + \frac{1}{9^n}$, so $q_n \to_n x$. So, it is enough to show that (1) fails for this choice of p_n, m_n, and q_n. Indeed, by (a), we have $f(m_n) = f_{n+1}(m_n)$ and $f(q_n) = f(p_n) = f_n(p_n)$, so $\ell_n(m_n) = f_n(p_n) = f_{n+1}(p_n)$. Hence, by (b), $\frac{|f(m_n) - \ell_n(m_n)|}{m_n - x} = \frac{|f_{n+1}(m_n) - f_{n+1}(p_n)|}{m_n - x} = \frac{3^{-n}}{m_n - x} > \frac{3^{-n}}{2/9^n} = 3^n/2$, that is, (1) indeed fails.

f admits finite left derivative at no point, since $f(x) = -f(-x)$ and it admits finite right derivative at no point as we proved above.

Finally, to show that at no point does f admit two one-sided infinite derivatives, it is enough to show that for every $x \in \mathbb{R}$ there is $y \neq x$ with $f(y) = f(x)$ and $|x - y|$ arbitrarily small. Indeed, there is arbitrarily large $n \in \mathbb{N}$ and a $k \in \mathbb{Z}$ so that $x \in \left[\frac{k}{9^n}, \frac{k+1}{9^n}\right]$. But every f_m with $m > n$, so also f, maps each of the three intervals $\left[\frac{3k+i}{3\cdot 9^n}, \frac{3k+i+1}{3\cdot 9^n}\right]$, $i \in \{0, 1, 2\}$, onto the same interval. So, by the intermediate value theorem, there is a $y \in \left[\frac{k}{9^n}, \frac{k+1}{9^n}\right]$ not equal x for which $f(y) = f(x)$, finishing the argument. ■

It is perhaps worth noticing that there exist points $x \in \mathbb{R}$ for which the right-hand side derivative of f at x is ∞. For example, this is the case for $x = \sum_{n=0}^{\infty} \frac{2/3}{9^n} = \frac{3}{4}$, which is the largest point x with $f(x) = 0$.

$f \restriction [0, 1]$ is the second coordinate of the classical Peano curve. We have here in mind the Peano curve $P\colon [0, 1] \to [0, 1] \times [0, 1]$ which is the uniform limit of the piecewise linear paths $P_n\colon [0,1] \to [0,1] \times [0,1]$, $n \in \omega$, defined inductively as follows: $P_0(x) := \langle x, x\rangle$ and P_{n+1} is obtained from P_n by replacing each of its linear parts with the 9-piece linear path using *the basic construction Peano* BCP represented in Figure 4. The graph of P_2 is shown in Figure 5. We assume that the parametrization of curves P_n is uniform with respect to the path length.

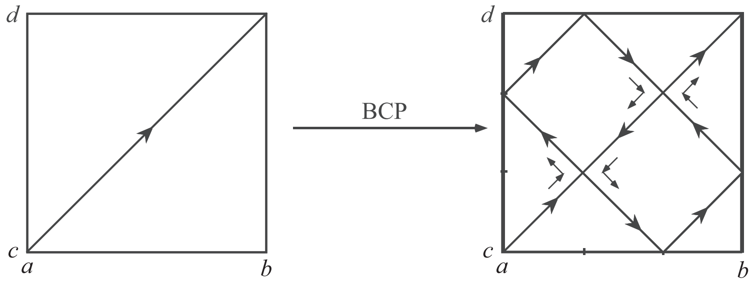

Figure 4. The basic construction BCP. With $a = c = 0$ and $b = d = 1$, the left curve also represents P_0, while the right curve represents P_1.

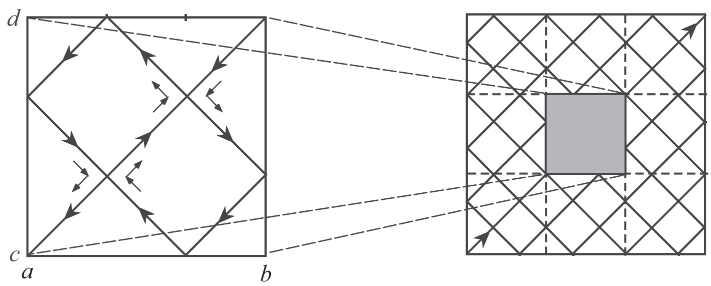

Figure 5. Construction of P_2.

Now, if $P_n = \langle p_1^n, p_2^n\rangle$, so that p_1^n and p_2^n are, respectively, the first and second coordinate function of P_n, then it is easy to see that the second coordinate p_2^n is equal to the restrictions $f_n \restriction [0,1]$ of the maps f_n defined above. In particular, $f \restriction [0,1]$ equals p_2, the second coordinate of the Peano curve $P := \lim_{n\to\infty} P_n$. It is also not difficult to see that $p_2(x) = p_1(x/3)$ for every $x \in [0,1]$. So, $f \restriction [0,1]$ is also a rescaled version of p_1.

This observation, besides showing a curious connection, also allows us to establish several other properties of f, since p_1 has been studied by several authors. Thus, Jan Malý showed in a 1980 paper [27] that p_1 is density continuous (i.e., continuous when both domain and range are equipped with the density topology) but it maps some sets of Lebesgue measure 0 onto sets with positive measure (i.e., p_1 does not satisfy Lusin's condition (N)). In 1989 [12] the author, Lee Larson, and Krzysztof Ostaszewki showed that p_1 is nowhere approximately differentiable, while in a 1992 article [11] the author and Lee Larson proved the category analog of the above-mentioned two results: p_1 is I-density continuous and is nowhere I-approximately differentiable. (See also monograph [13, Section 4.3].) It is also known that p_1 is nowhere approximately differentiable, I-density continuous, and nowhere I-approximately differentiable.

4. BESICOVITCH'S B-MONSTER IN ACCESSIBLE FORMAT. The function $C : \mathbb{R} \to \mathbb{R}$ constructed in this section is a modification the original Besicovitch function B_a from the papers [**4, 5, 33**] (see also [**19, 39**]) in a sense that $C(x) = B_1(x+1)$ on $[-1, 1]$. However, as we indicated earlier, the presentation of the function B_a in these earlier papers makes it hard to recognize this fact.

Let $E \subset [-1, 1]$ be the classical *Smith–Volterra–Cantor set* of Lebesgue measure $|E| = 1$ obtained by consecutive removal of middle open intervals of length $4^{-(n+1)}$ starting separately with intervals $[-1, 0]$ and $[0, 1]$. More specifically, for every $n \in \omega$ we define recursively the families $\mathcal{I}_n$ and $\mathcal{J}_n$ of 2^{n+1} intervals each so that: $\mathcal{I}_0 = \{[-1, 0], [0, 1]\}$, $\mathcal{J}_n$ consists of the open intervals of length $4^{-(n+1)}$ each sharing the center with an $I \in \mathcal{I}_n$, and $\mathcal{I}_{n+1}$ is the family of connected components of the sets $[0, 1] \setminus \bigcup_{k \le n} \bigcup \mathcal{J}_k$ and $[-1, 0] \setminus \bigcup_{k \le n} \bigcup \mathcal{J}_k$. The set E is defined as $E := [-1, 1] \setminus \bigcup_{k=0}^{\infty} \bigcup \mathcal{J}_k$. Notice also that E is symmetric, in the sense that $-E = E$, and uniformly distributed: $|E \cap I| = \frac{1}{2^{n+1}}$ for any $I \in \mathcal{I}_n$.

Let $\gamma : [0, 1] \to [0, 1]$ be a version of a Cantor devil's staircase function associated with E and given for each $x \in [0, 1]$ by a formula $\gamma(x) := 2|E \cap [0, x]|$. Define a *step-triangle map* $\Psi : \mathbb{R} \to [0, 1]$ as

$$\Psi(x) := \gamma(\mathrm{dist}(x, [-1, 1]^c)) \quad \text{for every } x \in \mathbb{R},$$

where $A^c := \mathbb{R} \setminus A$ and $\mathrm{dist}(x, A^c)$ is the distance from x to A^c. See Figure 6. Notice that the "steps" of Ψ (i.e., the maximal nonempty open intervals on which Ψ is constant) consists of the elements of the family $\mathcal{K} = \bigcup_{k=0}^{\infty} \mathcal{J}_k$.

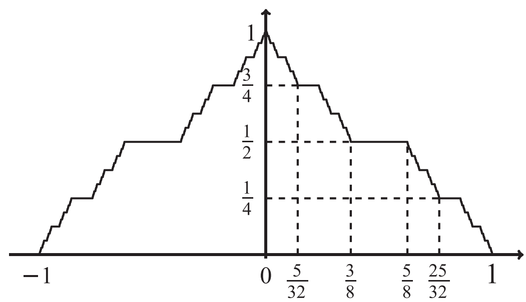

Figure 6. The step-triangle map $\Psi = c_0$. Each of its arms is a version of Cantor devil's staircase function γ.

For every nonempty bounded open interval J in $\mathbb{R}$, let $\ell_J : \mathbb{R} \to \mathbb{R}$ be the increasing linear function mapping J onto $(-1, 1)$. Our B-monster $C : \mathbb{R} \to \mathbb{R}$ is defined as

$$C := \sum_{n=0}^{\infty} (-1)^n c_n \quad \text{where} \quad c_n := \sum_{J \in \mathcal{K}_n} \sqrt{|J|/2}\, \Psi \circ \ell_J; \tag{2}$$

$\mathcal{K}_0 := \{(2k-1, 2k+1) : k \in \mathbb{Z}\}$, and $\mathcal{K}_{n+1} := \{\ell_J^{-1}(K) : J \in \mathcal{K}_n \text{ and } K \in \mathcal{K}\}$ consist of maximal nonempty open intervals on which maps c_n are constant.

Notice that the alternating series format of the definition of C and the square root in the coefficient $\sqrt{|J|/2}$ play an essential role in the proof that it does not admit unilateral infinite derivatives at some $x \in \mathbb{R}$; see (B) below. The fact that E used in the construction has a positive measure is crucial in the proof that C does not admit unilateral infinite derivatives at all other points; see (C) below.

As $(-1, 1)$ is the support[6] of Ψ, the supports of the summands $\sqrt{|J|/2}\ \Psi \circ \ell_J$ of each c_n—the intervals $J \in \mathcal{K}_n$—are pairwise disjoint, so the maps c_n are well-defined. It is also easy to see that each c_n is 2-periodic, that is, $c_n(x+2) = c_n(x)$ for all $x \in \mathbb{R}$. Thus, in what follows we will restrict our attention to C on the interval $[-1, 1]$. Notice also that C is symmetric (i.e., $C(-x) = C(x)$ for all $x \in \mathbb{R}$), as the symmetry of E implies that the maps Ψ and c_n are symmetric.

Let $C_n := \sum_{k=0}^{n} (-1)^k c_k$ denote the partial sum of the series (2). The graphs of maps C_0, C_1, C_2, and C, restricted to $[-1, 1]$, are shown in Figure 7. Notice that $C_0 = c_0$ is formed with the copies of Ψ and C_{n+1} is obtained by adding to or subtracting from C_n the rescaled copies of Ψ, one for each $J \in \mathcal{K}_n$.

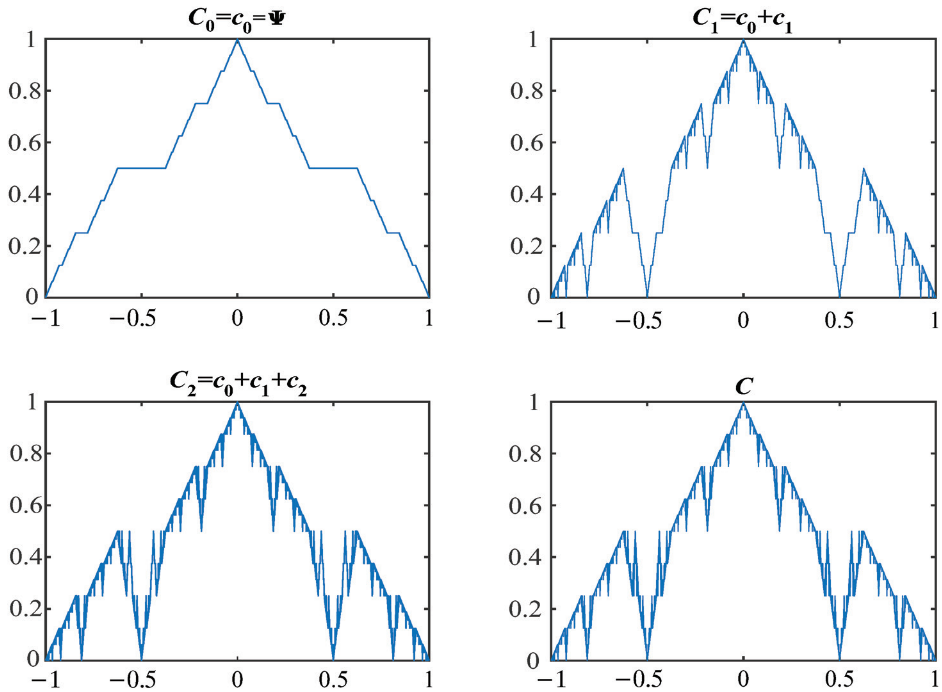

Figure 7. The graphs of maps C_0, C_1, C_2, and C, restricted to $[-1, 1]$ (generating program courtesy of Prof. Serge Dubuc).

Theorem 4. *The map C given by (2) is a B-monster, that is, it is continuous and admits no unilateral, finite or infinite, derivative at any point.*

Since C is symmetric, we shall deduce Theorem 4 by noticing the following relatively simple facts, which constitute the sketch of our proof.

(A) C is continuous, as each c_n is continuous and of sup norm $\|c_n\| \leq \frac{1}{2^n}$.
(B) C admits no right infinite derivative at any $x \in \bigcap_{n=0}^{\infty} \bigcup \mathcal{K}_n$, since the sequence $\langle c_n(x) \rangle_n$ is strictly decreasing and, by the alternating series test, there exists a sequence $\langle r_n > x : n \in \mathbb{N} \rangle$ converging to x with $C(r_n) = C(x)$ for all $n \in \mathbb{N}$. This shows that 0 is the only possible value of the right derivative of C at such x.

[6] We define the support of an $f : X \to \mathbb{R}$ as the set $f^{-1}(\mathbb{R} \setminus \{0\})$ rather than its closure.

(C) C admits no right infinite derivative at any $x \in \mathbb{R} \setminus \bigcup \mathcal{K}_n$, $n \in \omega$, since x is a right limit point of $\mathbb{R} \setminus \bigcup \mathcal{K}_{n+1}$ and C is Lipschitz on $\mathbb{R} \setminus \bigcup \mathcal{K}_{n+1}$. So, the right derivative of C at such x, if it exists, would need to be finite.
(D) The fact that C admits no right finite derivative at any $x \in \mathbb{R}$ is justified by Lemma 1 in an argument similar to that for Corollary 2.

The detailed arguments follow.

Proof of Theorem 4. By the 2-periodicity of C, we can restrict our attention to C on $[-1, 1]$.

(A): Continuity of C. Each c_n is continuous on $[-1, 1]$, since it is a sum of the continuous maps $\sqrt{|J|/2}\ \Psi \circ \ell_J$ with disjoint supports $J \in \mathcal{K}_n$ contained in $[-1, 1]$, so for every $\varepsilon > 0$ there are only finitely many such summands with their sup norms $\sqrt{|J|/2}$ exceeding ε. Also, since $|J| \le \frac{1}{4}$ for every $J \in \mathcal{K}$, an easy induction argument shows that $|J| \le \frac{2}{8^n}$ for every $n \in \omega$ and $J \in \mathcal{K}_n$. In particular, $\|c_n\| = \sup_{J \in \mathcal{K}_n} \sqrt{|J|/2} \le \sqrt{8^{-n}}$ so, by the Weierstrass M test, C is continuous.

In the next steps we will need the following simple property of C, which holds since the support of every c_k with $k \ge n$ is contained in $\bigcup \mathcal{K}_n$.

(i) $C = C_{n-1}$ on $\mathbb{R} \setminus \bigcup \mathcal{K}_n$ for every $n \in \mathbb{N}$.

(B): No infinite right derivative at any $x \in \bigcap_{n=0}^{\infty} \bigcup \mathcal{K}_n$. First notice that, by the uniform distribution of the measure of E, for any $J \in \mathcal{J}_k$ with $k \in \omega$ the constant value of Ψ on J is at least $\frac{1}{2^{k+1}}$, since it belongs to $\left\{\frac{2i-1}{2^{k+1}} : i \in \{1, \ldots, 2^k\}\right\}$. In particular, $\Psi \restriction K \ge \frac{1}{2^{k+1}} = \sqrt{|K|}$ for any $K \in \mathcal{K}$ of length $\frac{1}{4^{k+1}}$.

Next, for every $n \in \omega$ choose a $J_n := (a_n, b_n) \in \mathcal{K}_n$ containing x and notice that $J_{n+1} = \ell_{J_n}^{-1}(K_n)$ for some $K_n \in \mathcal{K}$. Hence, $|J_{n+1}| = \frac{|J_n|}{2}|K_n|$ as $\frac{|J_n|}{2}$ is the slope of $\ell_{J_n}^{-1}$. Using $\ell_{J_n}(x) \in K_n$, which holds as $x \in J_{n+1} = \ell_{J_n}^{-1}(K_n)$, we get

$$c_n(x) = \sqrt{\tfrac{|J_n|}{2}}\ \Psi(\ell_{J_n}(x)) \ge \sqrt{\tfrac{|J_n|}{2}}\ \sqrt{|K_n|} = \sqrt{|J_{n+1}|} > \sqrt{|J_{n+1}|/2} \ge c_{n+1}(x).$$

In particular, the bounds part of the alternating series test used with the series $C(x) = \sum_{n=0}^{\infty} (-1)^n c_n(x)$ implies that $C_{2n-1}(x) < C(x) < C_{2n}(x)$ for every $n \in \mathbb{N}$. At the same time, for every $k \in \mathbb{N}$ we have $C_{k-1}(x) = C_{k-1}(b_k)$, as C_{k-1} is constant on $[a_k, b_k] \ni x$, and $C_{k-1}(b_k) = C(x)$, which holds by (i), as $b_k \in \mathbb{R} \setminus \bigcup \mathcal{K}_k$. So, for every $n \in \mathbb{N}$,

$$C(b_{2n}) = C_{2n-1}(b_{2n}) = C_{2n-1}(x) < C(x) < C_{2n}(x) = C_{2n}(b_{2n+1}) = C(b_{2n+1}).$$

In particular, by the intermediate value theorem, for every $n \in \mathbb{N}$ there exists an $r_n \in (b_{2n+1}, b_{2n})$ with $C(r_n) = C(x)$. Since $b_{2n} \searrow x$, this implies that the only possible value of the right derivative of C at x is 0.

In the next steps, we will also need the following two simple properties of C.

(ii) For every $n \in \omega$ and $J \in \mathcal{K}_n$ the partial sum C_n of C is Lipschitz on J.

Indeed, (ii) holds, since the step-triangle map Ψ is Lipschitz with constant 2, as $|\gamma(y) - \gamma(x)| = 2|E \cap [x, y]| \le 2|y - x|$ for every $x < y$ from $[0, 1]$, while C_n on J is a sum of a constant and $(-1)^n \sqrt{|J|/2}\ \Psi \circ \ell_J$, which is Lipschitz.

For every $n \in \omega$ let $K_n := \bigcup\{[a, b)\colon (a, b) \in \mathcal{K}_n\}$ and notice that

(iii) every $x \in \mathbb{R} \setminus K_n$ is a right limit point of $\mathbb{R} \setminus K_n \subset \mathbb{R} \setminus \bigcup \mathcal{K}_n$.

(C): No infinite right derivative at any $x \notin \bigcap_{n=0}^{\infty} \bigcup \mathcal{K}_n$. Notice that $K_{n+1} \subset \bigcup \mathcal{K}_n$ for every $n \in \omega$. In particular, $x \notin \bigcap_{n=0}^{\infty} K_n$. Also, $K_0 = \mathbb{R}$, so there exist $n \in \mathbb{N}$ and $(a, b) \in \mathcal{K}_{n-1}$ so that $x \in [a, b) \setminus K_n$. By (ii) the map C_{n-1} on $[a, b)$ is Lipschitz with some constant $L \geq 0$, while (iii) implies that x is a right limit point of the set $[a, b) \setminus K_n \subset \mathbb{R} \setminus \bigcup \mathcal{K}_n$. Since, by (i), $C = C_{n-1}$ on $\mathbb{R} \setminus \bigcup \mathcal{K}_n$, it follows that the only possible value of the right derivative of C at x is in $[-L, L]$.

(D): No finite right derivative at any $x \in \mathbb{R}$. Since $K_0 = \mathbb{R}$, for every $k \in \omega$ we can choose an $n \in \omega$ with $x \in K_n$ such that either $x \notin K_{n+1}$ or $n \geq k$. Let $J := (a, b) \in \mathcal{K}_n$ be such that $x \in [a, b)$. So, $\ell_J(x) \in [-1, 1)$. Choose

(•) $m \in \omega$, $I_m := [c_m, d_m] \in \mathcal{I}_m$, and its middle quarter $J_m := (a_m, b_m) \in \mathcal{J}_m$ such that $\ell_J(x) \in [c_m, b_m)$ and $m + n \geq k$.

Such choice is clearly possible when $x \in K_{n+1}$, since then J_m and I_m are uniquely determined, $\ell_J(x) \in [a_m, b_m) \subset [c_m, b_m)$, and $m + n \geq n \geq k$. On the other hand, if $x \notin K_{n+1}$, then for every $m \in \omega$ there exist $I_m := [c_m, d_m] \in \mathcal{I}_m$ and its middle quarter $J_m := (a_m, b_m) \in \mathcal{J}_m$ such that $\ell_J(x) \in I_m \setminus J_m$. Also, we must have $\ell_J(x) \in [c_m, b_m)$ for infinitely many $m \in \omega$, since otherwise $\ell_J(x)$ would be equal to one of the numbers d_m, which implies that $x = \ell_J^{-1}(d_m) \in K_{n+1}$, a contradiction. So, there is an $m \geq k$ for which the intervals J_m and I_m are as needed.

Now, let $p_k < q_k$ be such that $\hat{J} := \ell_J[(p_k, q_k)] \in \mathcal{J}_{m+1}$ is contained in the interval $[b_m, d_m] \in \mathcal{I}_{m+1}$. Let m_k be the midpoint of (p_k, q_k). Then $x < p_k < m_k < q_k$. To finish the proof it is enough to show that the assumptions of Lemma 1 are satisfied for these numbers and $f = C$, while (1) fails for them.

Indeed, $c_m \leq \ell_J(x) < b_m < \ell_J(p_k) < \ell_J(q_k) < d_m$ and, since $|I_m| \leq 2^{-m}$ and the slope of ℓ_J^{-1} equals $|J|/2$,

$$q_k - x < |\ell_J^{-1}([c_m, d_m])| = \frac{1}{2}|J||I_m| \leq \frac{1}{2}\frac{2}{8^n}2^{-m} \leq 2^{-(m+n)} \leq 2^{-k} \to_k 0, \quad (3)$$

that is, indeed $q_k \to_k x$. Finally, since $(p_k, q_k) = \ell_J^{-1}[\hat{J}] \in \mathcal{K}_{n+1}$ has length $\frac{1}{2}|J||\hat{J}| = \frac{1}{2}|J|4^{-(m+2)}$ and, by (i), $|f(m_k) - \ell_n(m_k)| = |C_{n+1}(m_k) - C_n(m_k)|$, we have

$$|f(m_k) - \ell_n(m_k)| = |c_{n+1}(m_k)| = \sqrt{|(p_k, q_k)|/2} = 2^{-(m+3)}\sqrt{|J|}.$$

Moreover, by (3), $m_k - x \leq \ell_J^{-1}(d_m) - \ell_J^{-1}(c_m) = |\ell_J^{-1}([c_m, d_m])| = \frac{1}{2}|J||I_m|$, so

$$\frac{|f(m_k) - \ell_k(m_k)|}{m_k - x} \geq \frac{2^{-(m+3)}\sqrt{|J|}}{\frac{1}{2}|J||I_m|} = \frac{2^{-(m+3)}\sqrt{|J|}}{\frac{1}{2}|J|2^{-m}} = \frac{1}{4}\frac{1}{\sqrt{|J|}} \geq \frac{1}{8},$$

showing that (1) indeed fails. ∎

5. FINAL REMARKS.

Typical behavior of W-monsters. It should be mentioned here that W-monsters, even those that admit no tangent line, such that the coordinates of Peano curves described in Section 3 are "typical" in the sense that there exists a first category subset M of the space $C([a, b])$ of all continuous functions considered with the supremum distance, such that every $f \in C([a, b])$ that is not in M has such properties. This was first proved in 1931 paper by Stefan Banach (1892–1945); see [20, Theorem 7.2.1]. On the other hand, the B-monsters form a first category subset of $C([a, b])$, as was shown in 1932 by Stanisław Saks (1897–1942); see [20, Theorem 7.5.1].

W-monsters and nowhere monotone maps. The classical theorem of Henri León Lebesgue (1875–1941), known as Lebesgue's differentiation theorem, states that every monotone function defined on an interval J is differentiable at almost all points of J. This clearly implies that any W-monster must be nowhere monotone, that is, monotone on no nontrivial interval. The examples we discussed above show that there are bad (with respect to differentiability) nowhere monotone functions. The related interesting question is how good continuous nowhere monotone functions could be. In particular, can such a map be differentiable at every point?

This question goes back to the late 19th century and was answered in the affirmative in 1887 by Alfred Köpcke (1852–1932) in [23]. (See also [24, 25].) The examples of this kind have recently become known as *differentiable monsters* to stress their curious relation to W-monsters. (See [10].) The long history of the search for differentiable monsters is described in detail in the 1983 paper [8] of Andrew Michael Bruckner (1932–). But unlike in the case of W-monsters, where simple examples were described soon after their original discovery, the short and elementary constructions of differentiable monsters were in short supply until 2018, when the author noticed that for every differentiable auto-homeomorphism h of $\mathbb{R}$ with a dense set of points having zero derivative there is a residual set of points $t \in \mathbb{R}$ for which the map $g(x) := h(x - t) - h(x)$ is a differentiable monster [10]. This makes the construction easy, since the maps h, known as *Pompeiu-like functions*, are easily constructed, as noticed already in the 1907 paper [35] by Dimitrie Pompeiu (1873–1954). Even more interesting, it was just noticed by Chang-Han Pan in [32] that the differentiable monsters that are difference of two Pompeiu-like maps are canonical in the sense that every differentiable monster has a restriction that can be expressed in that way.

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