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Lineability of the functions that are Sierpiński–Zygmund, Darboux, but not connectivity

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Abstract
Assuming that continuum $\mathfrak{c}$ is a regular cardinal, we show that the class $\mathrm{PES} \setminus \mathrm{Conn}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are perfectly everywhere surjective (so Darboux) but not connectivity is $\mathfrak{c}^+$-lineable, that is, that there exists a linear space of $\mathbb{R}^\mathbb{R}$ of cardinality $\mathfrak{c}^+$ that is contained in $(\mathrm{PES} \setminus \mathrm{Conn}) \cup \{0\}$. Moreover, assuming additionally that $\mathbb{R}$ is not a union of less than $\mathfrak{c}$-many meager sets, we prove $\mathfrak{c}^+$-lineability of the class $\mathrm{SZ} \cap \mathrm{ES} \setminus \mathrm{Conn}$ of Sierpiński-Zygmund everywhere surjective but not connectivity functions.

Keywords Lineability · Darboux-like functions · Sierpiński–Zygmund functions

Mathematics Subject Classification 26A15 · 54C08 · 15A03 · 46J10 · 54A35

1 Introduction

Over the last two decades, a lot of mathematicians have been interested in finding the largest possible vector spaces that are contained in various families of real functions, see e.g. survey [7], monograph [2], and the literature sited there. (More recent work in this direction include [1,9,18].) Specifically, given a (finite or infinite) cardinal number κ, a subset M of a vector space X is said to be *κ-lineable (in X)* provided there exists a linear space $Y \subset M \cup \{0\}$ of dimension κ. This notion was first studied by Vladimir Gurariy [24], even though he did not use the term lineability. He showed that the set of continuous nowhere differentiable functions on [0,1], together with the constant 0 function, contains an infinite dimensional vector space, that is, it is ω-lineable.

In what follows we consider only real-valued functions and no distinction is made between a function and its graph.

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1.1 Lineability of some Darboux-like maps

We are primarily interested in the lineability related to the two classes of Darboux-like functions $f:\mathbb{R}\to\mathbb{R}$ defined as follows.

- *f is a Darboux function*, $f\in\mathcal{D}$, provided $f[K]$ is a connected subset of $\mathbb{R}$ (i.e, an interval) for every connected $K\subset\mathbb{R}$.
- *f is a connectivity function*, $f\in$ Conn, provided the restriction $f\restriction Z$ of f to Z is connected in the product $\mathbb{R}\times\mathbb{R}$ for any connected $Z\subset\mathbb{R}$.

We associate with $\mathcal{D}$ its name in honor of Jean Gaston Darboux (1842–1917) who, in his 1875 paper [19], showed that all derivatives, including those that are discontinuous, satisfy the intermediate value theorem, that is, are in the class $\mathcal{D}$. The notion of connectivity map can be traced to a 1956 problem [27] stated by John Forbes Nash (1928–2015).[1] See also [25,30]. Clearly Conn $\subset\mathcal{D}$. It is also not difficult to see that this inclusion is proper.

It has been shown in [12, thm 3.1], using a result from [20], that the class Conn (so, also $\mathcal{D}$) is $2^{\mathfrak{c}}$-lineable, where $\mathfrak{c}$ denotes the *continuum*, that is, the cardinality of $\mathbb{R}$. The first goal of this paper will be to show that, under a minor set theoretical assumption of regularity of $\mathfrak{c}$,[2] the class $\mathcal{D}\setminus$ Conn is $\mathfrak{c}^+$-lineable. More specifically, this result follows from Theorem 1.1, where we prove $\mathfrak{c}^+$-lineability of the class $\mathcal{F}\setminus$ Conn, where $\mathcal{F}$ is any of the following refinements of the class $\mathcal{D}$.

Given an $f:\mathbb{R}\to\mathbb{R}$, we say (see e.g. [11]) that:

- *f is everywhere surjective*, $f\in$ ES, provided $f[G]$ is equal to $\mathbb{R}$ for every non-empty open set $G\subset\mathbb{R}$;
- *f is strongly everywhere surjective*, $f\in$ SES, provided $f^{-1}(y)\cap G$ has cardinality $\mathfrak{c}$ for every non-empty open set $G\subset\mathbb{R}$;
- *f is perfectly everywhere surjective*, $f\in$ PES, provided $f[P]$ is equal to $\mathbb{R}$ for every non-empty perfect set $P\subset\mathbb{R}$.

It is well known (see [11]) and easy to see that PES $\subsetneq$ SES $\subsetneq$ ES $\subsetneq\mathcal{D}$.

The first result of this paper, proved in the next section, is as follows.

Theorem 1.1 *Assume that continuum $\mathfrak{c}$ is a regular cardinal. Then the class* PES $\setminus$ Conn *is $\mathfrak{c}^+$-lineable.*

Notice that the $2^{\mathfrak{c}}$-lineability of PES (so, also of the other of these classes) has been proved in [21, thm 2.6]. (Compare also [11, prop. 1.9].) It is also easy to see that the classes SES $\setminus$ PES and $\mathcal{D}\setminus$ ES are $2^{\mathfrak{c}}$-lineable: if V is a witness for $2^{\mathfrak{c}}$-lineability of PES, χ_A denotes the characteristic function of $A\subset\mathbb{R}$, and C is the Cantor set, then the vector spaces $\{f\cdot\chi_C: f\in V\}$ and $\{f\cdot\chi_{(0,\infty)}: f\in V\}$ justify $2^{\mathfrak{c}}$-lineability of SES $\setminus$ PES and $\mathcal{D}\setminus$ ES, respectively. Surprisingly, $2^{\mathfrak{c}}$-lineability of the class ES $\setminus$ SES is not known in ZFC, although it has been proved in [11], with a quite delicate argument, that it is $\mathfrak{c}^+$-lineable. Therefore, our Theorem 1.1 (but not our proof) is similar in flavor to that of $\mathfrak{c}^+$-lineability of the class ES $\setminus$ SES.

[1] Nash shared the 1994 Nobel Memorial Prize in Economic Sciences with game theorists Reinhard Selten and John Harsanyi. In 2015, he also shared the Abel Prize with Louis Nirenberg for his work on nonlinear PDEs.

[2] Recall that $\mathfrak{c}$ is regular when the union of less than $\mathfrak{c}$-many sets, each of cardinality less than $\mathfrak{c}$, has cardinality less than $\mathfrak{c}$.

1.2 Intersection of Darboux-like and SZ functions

The second result of this work concerns the class SZ of functions known as *Sierpiński-Zygmund functions*. For a family $\mathcal{G}$ of real valued partial functions on $\mathbb{R}$ (i.e., functions from different subsets X of $\mathbb{R}$ into $\mathbb{R}$) let $\mathrm{SZ}(\mathcal{G})$ be the family of all $f\colon \mathbb{R} \to \mathbb{R}$ such that $|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$, where $|X|$ denotes the cardinality of X. The family SZ of Sierpiński-Zygmund functions is defined as $\mathrm{SZ}(\mathcal{C})$, where $\mathcal{C}$ is the family of all continuous real valued partial functions on $\mathbb{R}$. In other words, $f \in \mathrm{SZ}$ provided $f \restriction X$ is discontinuous for every $X \subset \mathbb{R}$ of cardinality $\mathfrak{c}$. The first example of an SZ-function was constructed by Sierpiński and Zygmund in their 1923 paper [29]. Compare also the recent survey [17] on the SZ-maps.

The functions in SZ are as far from being continuous as possible: by a 1922 theorem of Blumberg [8] for every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a (countable) dense subset D of $\mathbb{R}$ with $f \restriction D$ being continuous.[3] Thus, one might expect that no SZ-map can be continuous in a generalized sense, e.g., to be Darboux. Surprisingly, it has been proved in [4], coauthored by the second author, that this last statement is independent of the usual axioms ZFC of set theory:

Proposition 1.2 *The statement* $\mathrm{SZ} \cap \mathcal{D} \neq \emptyset$ *is independent of ZFC. More specifically:*

(i) *it follows from the* CPA *axiom, which holds in the iterated perfect set (Sacks) model, that* $\mathrm{SZ} \cap \mathcal{D} = \emptyset$*;*
(ii) *it follows from* $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ *that* $\mathrm{SZ} \cap \mathcal{D} \neq \emptyset$

The symbol $\mathscr{M}$ in Proposition 1.2 denotes the ideal of all meager subsets of $\mathbb{R}$ and $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ is the statement, consistent with ZFC, that $\mathbb{R}$ is not a union of less than $\mathfrak{c}$-many meager sets. The exposition of the CPA axiom can be found in the monograph [16].

The second main result of this paper, which addresses a problem of κ-lineability of $\mathrm{SZ} \cap \mathcal{D} \setminus \mathrm{Conn}$ indicated in Table 3 of the survey [17], is as follows.

Theorem 1.3 *Assume that* $\mathfrak{c}$ *is a regular cardinal and* $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$*. Then the family* $\mathrm{SZ} \cap \mathrm{ES} \setminus \mathrm{Conn}$ *is* $\mathfrak{c}^+$*-lineable.*

Actually, we will prove in Sect. 3 a slightly stronger result that, under the assumptions of the theorem, the class $\mathrm{SZ}(\mathcal{B}) \cap \mathrm{ES} \setminus \mathrm{Conn}$ is $\mathfrak{c}^+$-lineable, where $\mathcal{B}$ denotes the family of all Borel maps from $\mathbb{R}$ into $\mathbb{R}$. It is well known that $\mathrm{SZ}(\mathcal{B}) \subset \mathrm{SZ}$,[4] so this last result implies the conclusion of Theorem 1.3. However, it has been recently noticed (see [5,17], or [14]) that $\mathrm{SZ}(\mathcal{B}) \neq \mathrm{SZ}$ is independent of ZFC.

Notice that, by Proposition 1.2, it is consistent with ZFC that $\mathrm{SZ} \cap \mathrm{ES} \subset \mathrm{SZ} \cap \mathcal{D} = \emptyset$, in which case $\mathrm{SZ} \cap \mathrm{ES} \setminus \mathrm{Conn}$ cannot be even 1-lineable. Therefore, some extra set theoretical assumption is necessary in Theorem 1.3.

Finally, notice that the lineability of the class SZ is well understood, as described in [23] and stated below.

Proposition 1.4 *The family SZ is* $\mathfrak{c}^+$*-lineable. In particular, the Generalized Continuum Hypothesis,* GCH*, implies that* SZ *is* $2^{\mathfrak{c}}$*-lineable. On the other hand,* $2^{\mathfrak{c}}$*-lineability of* SZ *is independent of ZFC.*

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[3] For a new easier proof of this theorem see [13]. Compare also [17].

[4] This follows from the fact that $\mathrm{SZ}(\mathcal{B}) = \mathrm{SZ}(\{f \restriction X : f \in \mathcal{B} \ \&\ X \subset \mathbb{R}\})$ justified by a theorem of Kuratowski (see e.g. [26, p. 73]) that every partial Borel map from $X \subset \mathbb{R}$ into $\mathbb{R}$ can be extended to a Borel function from $\mathbb{R}$ to $\mathbb{R}$.

1.3 Additional remarks and notation

Clearly, when studying lineability of a family M we are interested in a vector subspace of $M \cup \{0\}$ of the largest possible dimension. However, it was shown in [3] that there is an $M \subset \mathbb{R}^{\mathbb{R}}$ with no linear subspace of infinite dimension but having subspaces of arbitrary high finite dimension,[5] making the above-mentioned problem ill-posed. To solve this dilemma, we ask instead for the largest possible cardinal number κ for which $M \cup \{0\}$ admits no vector subspace of dimension κ. This leads to the following definition of the lineability coefficient, that comes from [6]. (See also [11,12].)

Definition 1.5 The lineability coefficient of an $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is defined as

$$\mathscr{L}(\mathcal{F}) = \min\{\kappa : \text{there is no } \kappa\text{-dimensional vector subspace } V \subset \mathcal{F} \cup \{0\}\}.$$

Using this notation, we can state Theorems 1.1 and 1.3 in form of the following corollary.

Corollary 1.6 *Assume that* $\mathfrak{c}$ *is a regular cardinal. Then*

(i) $\mathscr{L}(\text{PES} \setminus \text{Conn}) > \mathfrak{c}^+$;
(ii) *if* $\text{cov}(\mathscr{M}) = \mathfrak{c}$, *then* $\mathscr{L}(\text{SZ} \cap \text{ES} \setminus \text{Conn}) > \mathfrak{c}^+$.

In the reminder of the paper we will use also the following notations. We let $\mathcal{B}_0$ to be the family of all $g \in \mathcal{B}$ such that g is either constant or $g^{-1}(r)$ is meager (i.e.,belongs to $\mathscr{M}$) for every $r \in \mathbb{R}$. The symbol $\mathbb{L}$ will stand for the family of all lines in the plane that are neither horizontal nor vertical. Also, we put $\mathbb{L}_0 := \{\ell \in \mathbb{L} \colon \ell(0) = 0\}$. The symbol $[X]^{<\kappa}$ will denote the family of all $Y \subset X$ with cardinality less than κ.

The paper is organized as follows. In Sects. 2 and 3 we prove, respectively, Theorems 1.1 and 1.3. In the last section we state some open questions.

2 $\mathfrak{c}^+$-lineability of PES \ Conn

The goal of this section is to prove Theorems 1.1, that is, $\mathfrak{c}^+$-lineability of the family PES \ Conn under the assumption of regularity of $\mathfrak{c}$. The main step in its proof is the following lemma.

It is worth to mention, that the conclusion of the lemma cannot be deduced if we drop the assumption that $\mathcal{F} \subset \text{SZ}(\mathbb{L}) \cup \{0\}$. More specifically, for the family $\mathcal{F} := \mathcal{B}$ of all Borel functions from $\mathbb{R}$ into $\mathbb{R}$ there is no $h \in \mathbb{R}^{\mathbb{R}}$ for which $h + \mathcal{F} \subset \text{ES} \setminus \text{Conn}$. This has been recently noticed in a paper [15], coauthored by the second author, which constitutes a deep study of the value of so called *additivity coefficient* A[6] (introduced by T. Natkaniec in his 1991 paper [28]) for different classes of Darboux-like functions, including the class $\mathcal{D} \setminus \text{Conn}$. The specific result we mention above means that $A(\text{ES} \setminus \text{Conn}) \le A(\mathcal{D} \setminus \text{Conn}) \le \mathfrak{c}$. This is of importance here, as this implies that we cannot deduce $\mathfrak{c}^+$-lineability of the family PES \ Conn from the value of $A(\mathcal{D} \setminus \text{Conn})$. (A simple criterion for such deduction has been given in a 2010 paper [22]. See also [11].)

5 Actually, every infinite dimensional vector space V has such subset M: if B is an infinite linearly independent subset of V, sets $\{B_n \subset B \colon n < \omega\}$ are pairwise disjoint with each B_n having n elements, and each V_n is spanned by B_n, then $M = \bigcup_{n<\omega} V_n$ is as needed.

6 $A(\mathcal{F}) := \min\big(\{|F| \colon F \subset \mathbb{R}^{\mathbb{R}} \text{ and } \varphi + F \not\subset \mathcal{F} \text{ for every } \varphi \in \mathbb{R}^{\mathbb{R}}\} \cup \{(2^{\mathfrak{c}})^+\}\big)$, where $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$.

Lemma 2.1 *Assume that $\mathfrak{c}$ is a regular cardinal. If $\mathcal{F} \subset \mathrm{SZ}(\mathbb{L}) \cup \{0\}$ is an additive group of cardinality $\le \mathfrak{c}$, then there is an $h \in \mathbb{R}^{\mathbb{R}}$ such that*

(i) $h + \mathcal{F} \subset \mathrm{PES} \cap \mathrm{SZ}(\mathbb{L})$;
(ii) *for every $f \in \mathcal{F}$ there is an $\ell \in \mathbb{L}_0$ so that $(h + f)(x) \neq \ell(x)$ for every nonzero $x \in \mathbb{R}$.*

In particular, $h + \mathcal{F} \subset \mathrm{PES} \setminus \mathrm{Conn}$

Proof Let $\mathbb{P}$ be the family of all nonempty perfect subsets of $\mathbb{R}$ and let $\{\langle P_\eta, y_\eta, f_\eta, \lambda_\eta\rangle \colon \eta < \mathfrak{c}\}$ be an enumeration of $\mathbb{P} \times \mathbb{R} \times \mathcal{F} \times \mathbb{L}$. By transfinite induction on $\eta < \mathfrak{c}$ we construct a sequence $\langle\langle h_\eta, D_\eta, \ell_\eta\rangle : \eta < \mathfrak{c}\rangle$ satisfying the following inductive conditions for every $\eta < \mathfrak{c}$:

(a) $D_\eta \in [\mathbb{R}]^{<\mathfrak{c}}$, $h_\eta \colon D_\eta \to \mathbb{R}$, and $\ell_\eta \in \mathbb{L}_0$;
(b) $y_\zeta \in D_\eta$ and $h_\zeta \subset h_\eta$ for every $\zeta \le \eta$;
(c) for every $\zeta \le \eta$ there is an $x \in P_\zeta \cap D_\eta$ with $h_\eta(x) + f_\zeta(x) = y_\zeta$;
(d) $h_\eta(x) + f_\zeta(x) \neq \lambda_\zeta(x)$ for every $\zeta < \eta$ and nonzero $x \in D_\eta \setminus D_\zeta$;
(e) $h_\eta(x) + f_\zeta(x) \neq \ell_\zeta(x)$ for every $\zeta \le \eta$ and nonzero $x \in D_\eta$.

First notice that if such a sequence can be found, then $h := \bigcup_{\eta<\mathfrak{c}} h_\eta$ is as desired. Indeed, $h \in \mathbb{R}^{\mathbb{R}}$ by (a) and (b). To see that (i) and (ii) hold, fix an $f \in \mathcal{F}$. Then $h + f \in \mathrm{PES}$ is ensured by (c), while $h + f \in \mathrm{SZ}(\mathbb{L})$ since for every $\lambda \in \mathbb{L}$ there is a $\zeta < \mathfrak{c}$ with $\langle f, \lambda\rangle = \langle f_\zeta, \lambda_\zeta\rangle$ and so (d) implies that $(h + f)(x) \neq \lambda(x)$ for every $x \notin D_\zeta$. Finally, if $f = f_\zeta$ for some $\zeta < \mathfrak{c}$, then (ii) is justified by ℓ_ζ, as ensured by (e).

It remains to construct our sequence. For this assume that, for some $\xi < \mathfrak{c}$, the sequence $\langle\langle h_\zeta, D_\zeta, \ell_\zeta\rangle : \zeta < \xi\rangle$ is already constructed so that the conditions (a)-(e) are satisfied for every $\eta < \xi$. We need to find $\langle h_\xi, D_\xi, \ell_\xi\rangle$ for which (a)-(e) are satisfied also for $\eta = \xi$.

We first construct h_ξ and D_ξ, leaving the construction of ℓ_ξ to the end of the argument. For this, let $D^\xi := \bigcup_{\eta<\xi} D_\eta$ and notice that, by the inductive assumption (a) and the regularity of $\mathfrak{c}$, we have $D^\xi \in [\mathbb{R}]^{<\mathfrak{c}}$. Also, by (a) and (b), $h^\xi := \bigcup_{\eta<\xi} h_\eta$ is a function from D^ξ into $\mathbb{R}$.

We will define $D_\xi := D^\xi \cup \{y_\xi, z\}$ for some $z \in P_\xi \setminus (D^\xi \cup \{y_\xi\})$ and extend h^ξ to $h_\xi \colon D_\xi \to \mathbb{R}$. This will ensure that (b) and the first two conditions of (a) are still satisfied, for $\eta = \xi$. Also, we will define

$$h_\xi(z) := y_\xi - f_\xi(z),$$

so that $(h_\xi + f_\xi)(z) = y_\xi$. This and the inductive assumption will ensure that also (c) holds for $\eta = \xi$. Notice also that for every $\zeta < \eta = \xi$ and $x \in D^\xi$ the properties (d) and (e) are satisfied by the inductive assumption. In particular, they are also satisfied for $x = y_\xi$ when $y_\xi \in D^\xi$. To ensure that they also hold for $x = y_\xi$ when $y_\xi \notin D^\xi$ it is enough to choose

$$h_\xi(x) \notin \bigcup_{\zeta<\xi} \{(\lambda_\zeta - f_\zeta)(x), (\ell_\zeta - f_\zeta)(x)\}.$$

Similarly, (d) and (e) will hold for $x = z$ when

$$h_\xi(z) \notin \bigcup_{\zeta<\xi} \{(\lambda_\zeta - f_\zeta)(z), (\ell_\zeta - f_\zeta)(z)\}.$$

But we are to have $h_\xi(z) = y_\xi - f_\xi(z)$, so we need to choose z outside the sets

$$S_\zeta := \{x \in \mathbb{R} \colon y_\xi - f_\xi(x) = (\lambda_\zeta - f_\zeta)(x)\} = \{x \in \mathbb{R} \colon (f_\zeta - f_\xi)(x) = \lambda_\zeta(x) - y_\xi\}$$

and

$$T_\zeta := \{x \in \mathbb{R}\colon y_\xi - f_\xi(x) = (\ell_\zeta - f_\zeta)(x)\} = \{x \in \mathbb{R}\colon (f_\zeta - f_\xi)(x) = \ell_\zeta(x) - y_\xi\}.$$

But, by our assumption, we have $f_\zeta - f_\xi \in \mathrm{SZ}(\mathbb{L}) \cup \{0\}$ while the maps $\lambda_\zeta - y_\xi$ and $\ell_\zeta - y_\xi$ belong to $\mathbb{L}$. Hence, the sets S_ζ and T_ζ have cardinality less than $\mathfrak{c}$. In particular, by the regularity of $\mathfrak{c}$, also the set

$$J_\xi := \bigcup_{\zeta<\xi} (S_\zeta \cup T_\zeta)$$

has cardinality less than $\mathfrak{c}$. Therefore, we can choose

$$z \in P_\xi \setminus (J_\xi \cup D^\xi \cup \{y_\xi\}).$$

The above analysis shows that this choice of z and the definition $h_\xi(z) = y_\xi - f_\xi(z)$ ensure that (d) and (e) hold for $x = z$ and every $\zeta < \eta = \xi$.

To finish the construction, choose $\ell_\xi \in \mathbb{L}_0$ disjoint with the set

$$\{\langle x, h_\xi(x) + f_\xi(x)\rangle \colon x \in D_\xi \setminus \{0\}\}.$$

It is possible since $|D_\xi| < \mathfrak{c}$ and the distinct lines in $\mathbb{L}_0$ intersect only in the origin.

This choice of ℓ_ξ ensures that it satisfies (e) and the last part of (a). Thus, this finishes the construction of the sequence $\langle\langle h_\eta, D_\eta, \ell_\eta\rangle : \eta < \mathfrak{c}\rangle$ and the proof of the lemma. □

Proof of Theorem 1.1 Let $X := (\mathrm{PES} \cap \mathrm{SZ}(\mathbb{L}) \setminus \mathrm{Conn}) \cup \{0\}$. A standard Zorn's lemma argument shows that the family of all linear subspaces of X contains a maximal element, say $\mathcal{F}$. But if $\mathcal{F}$ has cardinality $\le \mathfrak{c}$, then, by Lemma 2.1, it is not maximal. Indeed, if h is as in the lemma, then, by (i), $h \notin \mathcal{F}$ and $V := \mathbb{R}h + \mathcal{F}$ is a linear subspace of $(\mathrm{PES} \cap \mathrm{SZ}(\mathbb{L})) \cup \{0\}$ property extending $\mathcal{F}$. Moreover, by (ii), V is also disjoint with $\mathrm{Conn} \setminus \{0\}$. Therefore, $V \subset X$ is a proper extension of $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$.

So, $\mathcal{F}$ has cardinality and dimension at least $\mathfrak{c}^+$, as needed. □

Theorem 1.1 and the monotonicity of the operator $\mathscr{L}$ immediately imply the following

Corollary 2.2 *If continuum $\mathfrak{c}$ is a regular cardinal, then*

$$\begin{aligned}\mathscr{L}(\mathcal{D} \setminus \mathrm{Conn}) &\ge \mathscr{L}(\mathrm{ES} \setminus \mathrm{Conn}) \ge \mathscr{L}(\mathrm{SES} \setminus \mathrm{Conn}) \\ &\ge \mathscr{L}(\mathrm{PES} \setminus \mathrm{Conn}) \ge \mathscr{L}(\mathrm{PES} \cap \mathrm{SZ}(\mathbb{L}) \setminus \mathrm{Conn}) > \mathfrak{c}^+.\end{aligned}$$

3 $\mathfrak{c}^+$-lineability of SZ ∩ ES \ Conn

In this section, we are going to prove Theorem 1.3 that the class $\mathrm{SZ} \cap \mathrm{ES} \setminus \mathrm{Conn}$ is $\mathfrak{c}^+$-lineable under the assumption that $\mathfrak{c}$ is a regular cardinal and $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$. Similarly as in the previous section, the theorem will be deduced from the following analog of Lemma 2.1.

Lemma 3.1 *Assume that $\mathfrak{c}$ is a regular cardinal and $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$. If $\mathcal{F} \subset \mathrm{SZ}(\mathcal{B}) \cup \{0\}$ is an additive group of cardinality $\le \mathfrak{c}$, then there exists an $h \in \mathbb{R}^\mathbb{R}$ such that*

(i) $h + \mathcal{F} \subset \mathrm{ES} \cap \mathrm{SZ}(\mathcal{B})$;
(ii) for every $f \in \mathcal{F}$ there is an $\ell \in \mathbb{L}$ so that $(h + f)(x) \ne \ell(x)$ for every nonzero $x \in \mathbb{R}$.

Before we provide the proof of Lemma 3.1 we need also the following two simple results.

Lemma 3.2 *Let $g \in \mathbb{R}^{\mathbb{R}}$ be a Borel function and*

$$A = \{r \in \mathbb{R} : g^{-1}(r) \textit{ is not meager}\}.$$

Then $|A| \leq \omega$.

Proof For each $r \in A$, let U_r be a maximal open subset of $\mathbb{R}$ such that $g^{-1}(r)$ is residual in U_r. Then $\{U_r : r \in A\}$ is a family of disjoint open subsets of $\mathbb{R}$, hence it is countable. (See e.g. [10, theorem 6.2.1].) Thus, A is indeed countable. □

Lemma 3.3 $\mathrm{SZ}(\mathcal{B}) = \mathrm{SZ}(\mathcal{B}_0)$.

Proof The inclusion $\mathrm{SZ}(\mathcal{B}) \subset \mathrm{SZ}(\mathcal{B}_0)$ is obvious. To see the other inclusion, let $f \in \mathrm{SZ}(\mathcal{B}_0)$ and choose a $g \in \mathcal{B}$. We need to show that $|f \cap g| < \mathfrak{c}$. Let

$$A = \{r \in \mathbb{R} : g^{-1}(r) \text{ is not meager}\}.$$

By Lemma 1, $|A| \leq \omega$. Choose $\overline{g} \in \mathcal{B}_0$ extending $g \restriction (\mathbb{R} \backslash g^{-1}(A))$. For example, $g \restriction g^{-1}(A)$ can be the identity map. For every $r \in \mathbb{R}$, let c_r be the constant map with value r Then

$$f \cap g \subset (\overline{g} \cap f) \cup \bigcup_{r \in A} (c_r \cap f).$$

The sets $\overline{g} \cap f$ and $c_r \cap f$ have cardinality $< \mathfrak{c}$, since $f \in \mathrm{SZ}(\mathcal{B}_0)$ and $\overline{g}, c_r \in \mathcal{B}_0$. Therefore, also $f \cap g$ has cardinality $< \mathfrak{c}$, since it is contained in a countable family of sets of cardinality $< \mathfrak{c}$ and the cofinality of $\mathfrak{c}$ is greater than ω. So, indeed $f \in \mathrm{SZ}(\mathcal{B})$. □

Proof of Lemma 3.1 Let $\{\langle f_\eta, g_\eta \rangle : \eta < \mathfrak{c}\}$ be an enumeration, with no repetition, of $\mathcal{F} \times \mathcal{B}_0$. By transfinite induction on $\eta < \mathfrak{c}$, we construct a sequence $\langle \langle h_\eta, D_\eta, \ell_\eta \rangle : \eta < \mathfrak{c} \rangle$ satisfying the following inductive conditions for every $\eta < \mathfrak{c}$:

(a) $D_\eta \in [\mathbb{R}]^{<\mathfrak{c}}$, $h_\eta \colon D_\eta \to \mathbb{R}$, and $\ell_\eta \in \mathbb{L}_0$;
(b) $h_\zeta \subset h_\eta$ for every $\zeta \leq \eta$;
(c) if $g_\eta \equiv r$ for some number $r \in \mathbb{R}$, then $r \in D_\eta$ and the set $E_\eta := \{x \in D_\eta : h_\eta(x) + f_\eta(x) = r\}$ is dense in $\mathbb{R}$;
(d) $h_\eta(x) + f_\zeta(x) \notin \{g_\zeta(x), \ell_\zeta(x)\}$ for all $\zeta < \eta$ and $x \in D_\eta \backslash D_\zeta$;
(e) $h_\eta(x) + f_\eta(x) \neq \ell_\eta(x)$ for every nonzero $x \in D_\eta$.

If such a sequence can be found, then $h := \bigcup_{\eta < \mathfrak{c}} h_\eta$ is as desired. Indeed, $h \in \mathbb{R}^{\mathbb{R}}$ by (a), (b), and the first part of (c). To see (i), fix an $f \in \mathcal{F}$. To argue for $h + f \in \mathrm{ES}$, fix an $r \in \mathbb{R}$. We need to show that $(h + f)^{-1}(r)$ is a dense subsets of $\mathbb{R}$. To see this, choose an $\eta < \mathfrak{c}$ such that $\langle f, g \rangle = \langle f_\eta, g_\eta \rangle$ and $g_\eta \equiv r$. Then, by (c), $(h_\eta + f_\eta)^{-1}(r)$ is dense and contained in $(h + f)^{-1}(r)$ as needed. Also, $h + f \in \mathrm{SZ}(\mathcal{B}_0) = \mathrm{SZ}(\mathcal{B})$, since for every $g \in \mathcal{B}_0$ there is a $\zeta < \mathfrak{c}$ with $\langle f, g \rangle = \langle f_\zeta, g_\zeta \rangle$ and, by (d), we have

$$\mathrm{dom}((h + f) \cap g) \subset D_\zeta.$$

So, (i) is indeed satisfied. Also, if $\zeta < \mathfrak{c}$ is such that $f_\zeta = f$, then (ii) holds for ℓ_ζ. Indeed, if $x \in D_\zeta$, then $(h + f)(x) = h_\zeta(x) + f_\zeta(x) \neq \ell_\zeta(x)$ is ensured by (e) used with $\eta = \zeta$. At the same time, if $x \in \mathbb{R} \backslash D_\zeta$, then there exists an $\eta < \mathfrak{c}$ such that $x \in D_\eta \backslash D_\zeta$ and the condition (d) implies that $(h + f)(x) = h_\eta(x) + f_\zeta(x) \neq \ell_\zeta(x)$, as needed.

To define such a sequence, assume that for some $\eta < \mathfrak{c}$ the sequence $\langle \langle h_\zeta, D_\zeta, \ell_\zeta \rangle : \zeta < \eta \rangle$ is already constructed so that the conditions (a)-(e) are satisfied for its every initial segment. We need to find $\langle h_\eta, D_\eta, \ell_\eta \rangle$ for which (a)-(e) are still satisfied. We first construct h_η and D_η, ensuring that the properties (a)-(d) are satisfied.

For this, let $D^\eta := \bigcup_{\zeta<\eta} D_\zeta$ and notice that, by the inductive assumption (a) and the regularity of $\mathfrak{c}$, we have $D^\eta \in [\mathbb{R}]^{<\mathfrak{c}}$. Also, by (a) and (b), $h^\eta := \bigcup_{\zeta<\eta} h_\zeta$ is a function from D^η into $\mathbb{R}$. We define h_η and D_η by considering the following two cases.

Case 1 g_η *is not constant.* Then we just put $h_\eta := h^\eta$ and $D_\eta := D^\eta$. It is easy to see that such definition ensures that the properties (a)-(d) are satisfied.

Case 2 g_η *is constant.* Assume that $g_\eta \equiv r$ for some $r \in \mathbb{R}$. We will find a countable dense set $E \subset \mathbb{R}\setminus D^\eta$ and extend h^η to h_η defined on $D_\eta := D^\eta \cup E \cup \{r\}$ so that h_η and D_η satisfy the properties (a)-(d). Of course, they will satisfy (a) and (b). A delicate issue is to ensure (c) while preserving (d).

The construction will ensure that $E \subset E_\eta$. For this, we will need to define, for every $x \in E$,

$$h_\eta(x) := r - f_\eta(x). \tag{1}$$

On the other hand, to have (d), for every $\zeta < \eta$ our set E cannot contain an $x \in \mathbb{R}$ for which $h_\eta(x) = r - f_\eta(x) = (g_\eta - f_\eta)(x)$ equals to either $(g_\zeta - f_\zeta)(x)$ or $(\ell_\zeta - f_\zeta)(x)$. Thus, it is enough to choose E disjoint with the sets

$$Z_\zeta := \{x \in \mathbb{R}\colon (f_\zeta - f_\eta)(x) = g_\zeta(x) - r\} = \{x \in \mathbb{R}\colon (f_\zeta - f_\eta)(x) = (g_\zeta - g_\eta)(x)\}$$

and

$$T_\zeta := \{x \in \mathbb{R}\colon (f_\zeta - f_\eta)(x) = \ell_\zeta(x) - r\}.$$

We claim that

$$\text{each set } Z_\zeta \text{ and } T_\zeta \text{ is either meager or of cardinality } < \mathfrak{c}. \tag{2}$$

Indeed, this is true when $f_\zeta \neq f_\eta$, as then $f_\zeta - f_\eta \in \mathcal{F}\setminus\{0\} \subset \mathrm{SZ}(\mathcal{B})$, so $g_\zeta - r + f_\eta - f_\zeta$ and $\ell_\zeta - r + f_\eta - f_\zeta$ belong to $\mathrm{SZ}(\mathcal{B})$, implying $|Z_\zeta \cup T_\zeta| < \mathfrak{c}$. Thus, we can assume that $f_\zeta = f_\eta$. Then $T_\zeta = (\ell_\zeta - r)^{-1}(0)$ is a singleton. Moreover, $Z_\zeta = (g_\zeta - g_\eta)^{-1}(0)$ is empty when g_ζ is constant (since then $\langle f_\zeta, g_\zeta\rangle \neq \langle f_\eta, g_\eta\rangle$ and $f_\zeta = f_\eta$ implies that $g_\zeta \neq g_\eta$, i.e., $g_\zeta - g_\eta$ is a non-zero constant) and meager when g_ζ is not constant, since then $Z_\zeta = g_\zeta^{-1}(r)$ and g_ζ is a non-constant element of $\mathcal{B}_0$. So, indeed (2) holds.

Let $Z := D^\eta \cup \bigcup_{\zeta<\eta}(Z_\zeta \cup T_\zeta)$. Then, by the regularity of $\mathfrak{c}$, Z is a union of a set, call it T, of cardinality $< \mathfrak{c}$ and of less than $\mathfrak{c}$-many meager sets. Thus, treating T as a union of singletons, we see that Z is a union of less than $\mathfrak{c}$-many meager sets. Therefore, using $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$, we conclude that Z has an empty interior. Hence, there exists a countable dense set $E \subset \mathbb{R}\setminus Z$.

With this choice of E we define $D_\eta := D^\eta \cup E \cup \{r\}$ and h_η on E via (1). Moreover, if $r \notin D^\eta \cup E$, then we choose

$$h_\eta(r) \in \mathbb{R}\setminus \bigcup_{\zeta<\eta}\{(g_\zeta - f_\zeta)(r), (\ell_\zeta - f_\zeta)(r)\}.$$

It is easy to see that such definitions ensure satisfaction of the conditions (a)–(d).

Finally, we can choose ℓ_η satisfying (e) since the graphs of distinct functions in $\mathbb{L}_0$ intersects only at the origin and to satisfy (e) the map $\ell_\eta \restriction \mathbb{R}\setminus\{0\}$ just needs to avoid less than continuum many points on the plane. This finishes the construction of the sequence $\langle\langle h_\eta, D_\eta, \ell_\eta\rangle : \eta < \mathfrak{c}\rangle$ and the proof of the lemma. □

Proof of Theorem 1.3 The argument is very similar to that for Theorem 1.1. We let $X := \mathrm{SZ}(\mathcal{B}) \cap \mathrm{ES} \setminus \mathrm{Conn}$, use Zorn's lemma argument to find a maximal linear subspace $\mathcal{F}$ of $X \cup \{0\}$, and notice that, by Lemma 3.1, $\mathcal{F}$ cannot be maximal unless it has cardinality, so dimension, at least $\mathfrak{c}^+$ □

Corollary 3.4 *Assume that* $\mathfrak{c}$ *is a regular cardinal and* $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$*. Then*

(i) $\mathscr{L}((\mathrm{SZ} \cap \mathrm{ES})) > \mathfrak{c}^+$;
(ii) $\mathscr{L}((\mathrm{SZ}(\mathcal{B}) \cap \mathcal{D}) \setminus \mathrm{Conn}) > \mathfrak{c}^+$.

Recall also that $\mathrm{SZ} \cap \mathcal{D} = \emptyset$ is consistent with ZFC. In particular, this implies that $\mathscr{L}(\mathrm{SZ} \cap \mathcal{D}) = 1$. In particular, the lower bound $\mathfrak{c}^+$ of $\mathscr{L}$ for the classes in Theorem 1.3 and Corollary 3.4 cannot be proved in ZFC.

4 Open problems

We do not know if the lower bounds established in Sect. 2 can be proved without extra set-theoretical assumptions. In particular, we do not know the answers for the following questions.

Problem 4.1 Can the following be proved in ZFC:

(i) $\mathscr{L}(\mathrm{ES} \setminus \mathrm{Conn}) > 2^{\mathfrak{c}}$;
(ii) $\mathscr{L}(\mathrm{SES} \setminus \mathrm{Conn}) > 2^{\mathfrak{c}}$;
(iii) $\mathscr{L}(\mathrm{PES} \setminus \mathrm{Conn}) > 2^{\mathfrak{c}}$?

Problem 4.2 Can the following be proved without assuming the regularity of continuum:

(i) $\mathscr{L}(\mathrm{ES} \setminus \mathrm{Conn}) > \mathfrak{c}^+$;
(ii) $\mathscr{L}(\mathrm{SES} \setminus \mathrm{Conn}) > \mathfrak{c}^+$;
(iii) $\mathscr{L}(\mathrm{PES} \setminus \mathrm{Conn}) > \mathfrak{c}^+$?

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