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THE SPACE OF DENSITY CONTINUOUS FUNCTIONS

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We denote by $\mathbf{R}_d$ the set of real numbers, $\mathbf{R}$, endowed with the density topology. A function $f\colon \mathbf{R}_d \to \mathbf{R}_d$ is said to be *density continuous,* if it is continuous with respect to the topology on $\mathbf{R}_d$ in both the domain and range. The set of density continuous functions has been studied in several limited ways. Bruckner [1] and Niewiarowski [3] have studied density continuous functions which are homeomorphisms under the standard topology on $\mathbf{R}$. Ostaszewski has investigated the local behavior of density continuous functions [4] and has investigated their behavior as a semigroup [5].

In this paper, we consider the composition of the set of density continuous functions. The structure of this set seems to be quite complicated. Ostaszewski [5] has noted that it is not closed under uniform convergence. In Example 2 we show that it is not a vector space. Corollary 3 shows that each real-analytic function is density continuous, but Example 1 is a C^∞ function which is not density continuous. It is not difficult to construct a density continuous function which is not continuous. On the other hand, every density continuous function must be approximately continuous.

In what follows, the right (left) unilateral derivatives of a function f are represented as f^+ (f^-). The Lebesgue measure of a set A is denoted by $|A|$ and the Lebesuge density (right, left Lebesgue density) of A at a point x is written as $d(A,x)(d^+(A,x),d^-(A,x))$. The set of functions which are infinitely differentiable on $\mathbf{R}$ is written as C^∞. Finally, if A and B are two sets such that $\sup A \leqq \inf B$, then we write $A \ll B$.

Before stating the main result, we first present the following lemma.

LEMMA 1. *Suppose I is a compact interval and $f\colon I \to \mathbf{R}$. If there exist numbers α and β such that*

$$(1)\qquad 0<\alpha<\frac{f(x)-f(y)}{x-y}<\beta<\infty, \text{ for all } x,y\in I,\ x\neq y,$$

then f is density continuous on I.

PROOF. From (1) it is easy to see that f is strictly increasing and continuous on I. If $g=f^{-1}$, then it follows from (1) that

$$(2)\qquad 0<\frac{1}{\beta}<\frac{g(u)-g(v)}{u-v}<\frac{1}{\alpha}, \text{ for all } u,v\in f(I),\ u\neq v.$$

The right-hand inequality in (2) implies that g is a Lipschitz function on $f(I)$ and hence g is absolutely continuous and g' is bounded above a.e. The left-hand inequality in (2) shows that g' is bounded away from 0 on $f(I)$ a.e. Now a result of Bruckner [1, Corollary 1] shows that g preserves density points. This implies the density continuity of f.

THEOREM 1. *If I is an open interval and $f\colon I \to \mathbf{R}$ is convex, then f is density continuous.*

PROOF. Fix a point $a \in I$. It will be shown that f is right density continuous at a. To do this, we lose no generality in supposing that $f(a) = a = 0$, because the translation of a density continuous function is obviously density continuous.

According to [6, Theorem 10.11], there exists a nondecreasing function $h\colon I \to \mathbf{R}$ such that

$$f(x) = \int_0^x h(t)dt, \text{ for all } x \in I. \tag{3}$$

Because of this, it is easy to see that there must exist a real number $b > 0$ such that f is monotone on $[0, b]$. We may assume that f is strictly monotone on $[0, b]$ because if it is not, f must be constant on some right neighborhood of 0, and right density continuity at 0 follows at once. With this assumption, f is a homeomorphism from $[0, b]$ onto $f([0, b])$. Denote $g = (f|_{[0,b]})^{-1}$.

There are now two cases to consider, depending upon whether f is strictly increasing or strictly decreasing on $[0, b]$.

Assume first that f is strictly decreasing on $[0, b]$. Then by (3), $h < 0$ on $[0, b)$. There is no generality lost in assuming $h(b) < 0$. If $0 \leqq x < y \leqq b$, then considering the average value of h on (x, y) and recalling that h is nondecreasing, it is obvious that

$$0 > h(b) \geqq \frac{\int_x^y h}{y - x} = \frac{f(y) - f(x)}{y - x} \geqq h(0).$$

This implies

$$0 < -h(b) < \frac{(-f(y)) - (-f(x))}{y - x} < -h(0) < \infty, \text{ for all } x, y \in [0, b].$$

($h(0)$ is finite because h is monotone on a neighborhood of 0.) Lemma 1 now shows that $-f$ is density continuous on $[0, b]$. Since density continuity is easily shown to be preserved under constant multiplication, it follows that f is density continuous on $[0, b]$ and therefore right density continuous at 0.

Next, assume that f is strictly increasing on $(0,b)$ and that $I_n = [a_n, b_n]$ is a sequence of disjoint intervals from $(0, f(b))$ such that I_n decreases to 0 and

$$\frac{\left|\bigcup_{n=1}^{\infty} I_n \cap (0,t)\right|}{t} > \varrho > 0, \text{ for all } t \in (0, f(b)). \tag{4}$$

Let $S = \bigcup_{n=1}^{\infty} I_n$, $J_n = g(I_n)$ and $G_n = (b_{n+1}, a_n)$. From (4), it follows that

$$\frac{\left|\bigcup_{k=n}^{\infty} I_k\right|}{\left|\bigcup_{k=n-1}^{\infty} G_k\right|} > \frac{\varrho}{1-\varrho}, \text{ for all } n > 1. \tag{5}$$

Before proceeding with the proof, we make the following useful observations. From (3) and the assumption that f is increasing we see that $h > 0$ on $(0,b)$. Let A and B be intervals contained in $(0,b)$ such that $A \ll B$. Then because h is nondecreasing,

$$\frac{|f(A)|}{|A|} = \frac{\int_A h}{|A|} \leqq \sup_{t\in A} h(t) \leqq \inf_{t\in B} h(t) \leqq \frac{\int_B h}{|B|} = \frac{|f(B)|}{|B|}.$$

This implies the statement

$$|g(C)| \geqq |g(D)| \frac{|C|}{|D|} \tag{6}$$

for all intervals C and D from $(0, f(b))$ such that $C \ll D$, and this estimate immediately extends to the case when C, D are finite unions of disjoint intervals.

We define an infinite partition S_n of S as follows. Let $\alpha_1 = a_1$. By (5), there exists an $\alpha_2' < \alpha_1$ such that

$$\frac{|(\alpha_2', \alpha_1) \cap S|}{|G_1|} = \frac{\varrho}{1-\varrho}.$$

Let $\alpha_2 = \min\{\alpha_2', a_2\}$. Assume that α_k has been chosen for $k = 1, 2, \dots, n-1$ so that either $\alpha_k \geqq a_k$ or $\alpha_k < a_k$ and

$$\frac{|(\alpha_k, \alpha_{k-1}) \cap S|}{|G_{k-1}|} = \frac{\varrho}{1-\varrho},$$

and equality holds if $\alpha_k < a_k$. Choose $\alpha'_n < \alpha_{n-1}$ such that

$$\frac{|(\alpha'_n, \alpha_{n-1}) \cap S|}{|G_{n-1}|} = \frac{\varrho}{1-\varrho}.$$

To see that such a choice is possible, there are two cases to consider, depending on α_{n-1}. If $\alpha_{n-1} = a_{n-1}$, it can be seen immediately from (5). In case $\alpha_{n-1} < a_{n-1}$, let

$$m = \max\{k < n : \alpha_k = a_k\}.$$

Then $|(\alpha_k, \alpha_{k-1}) \cap S| = \varrho |G_{k-1}|/(1-\varrho)$ for $m+1 \leqq k \leqq n-1$ so that

$$|(\alpha_{n-1}, \alpha_m) \cap S| = \frac{\varrho}{1-\varrho} \sum_{k=m}^{n-1} |G_{k-1}|. \tag{7}$$

According to (5), there is a $t < \alpha_{n-1}$ such that

$$|(t, \alpha_m) \cap S| = \frac{\varrho}{1-\varrho} \sum_{k=m}^{n} |G_{k-1}|. \tag{8}$$

Subtracting (7) from (8) gives

$$|(t, \alpha_{n-1}) \cap S| = \frac{\varrho}{1-\varrho} |G_{n-1}|.$$

We set $\alpha'_n = t$ in this case. Then let $\alpha_n = \min\{\alpha'_n, a_n\}$. Define $S_n = [\alpha_{n+1}, \alpha_n) \cap S$. From the choice of $\alpha_n \leqq a_n$, and the fact that $a_n \notin S_n$, we see $\sup S_n \leqq b_{n+1}$. So $S_n \ll G_n = (b_{n+1}, a_n)$ and

$$\frac{|S_n|}{|G_n|} \geqq \frac{\varrho}{1-\varrho}.$$

Finally, we use (6) and the preceding inequality to see

$$\frac{\left| g\left(\bigcup_{n=1}^{\infty} S_n \right) \right|}{\left| g\left(\bigcup_{n=1}^{\infty} G_n \right) \right|} = \frac{\sum_{n=1}^{\infty} |g(S_n)|}{\sum_{n=1}^{\infty} |g(G_n)|} \geqq \frac{\sum_{n=1}^{\infty} |g(G_n)| \frac{|S_n|}{|G_n|}}{\sum_{n=1}^{\infty} |g(G_n)|} \geqq \frac{\varrho}{1-\varrho}.$$

Hence,

$$\frac{\left| g\left(\bigcup_{n=1}^{\infty} S_n \right) \right|}{|g((0, a_1))|} \geqq \varrho.$$

Because ϱ can be made as close to 1 as desired, we see that f is right density continuous at 0.

Similar arguments show that f is left density continuous at every point of I. This completes the proof of the theorem.

COROLLARY 1. *If* $g\colon [a,b] \to \mathbf{R}$ *is convex on* (a,b) *and* $\{g^+(a), g^-(b)\} \subset \subset \mathbf{R}$, *then* g *is density continuous.*

PROOF. Define

$$f(x) = \begin{cases} g^+(a)(x-a) + g(a) & \text{if } x < a, \\ g(x) & \text{if } a \leqq x \leqq b, \\ g^-(b)(x-b) + g(b) & \text{if } x > b \end{cases}$$

and apply Theorem 1.

By using $g = -f$ in Theorem 1 and Corollary 1 we arrive at the following corollary.

COROLLARY 2. *If* g *is concave downward on an open interval* I, *then* g *is density continuous on* I. *Further, if* g *is concave downward on the interval* $[a,b]$ *with both* $g^+(a)$ *and* $g^-(b)$ *finite, then* g *is density continuous on* $[a,b]$.

Ostaszewski [5, Question 4] asked whether polynomials are density continuous. The following corollary provides an affirmative answer to this question.

COROLLARY 3. *Real analytic functions are density continuous.*

PROOF. If f is real analytic, then f' is finite everywhere and f'' has only a finite number of zeroes in every interval, so applications of Corollaries 1 and 2 suffice to establish this corollary.

COROLLARY 4. *If* $f(x) = x^\alpha$ *for* $\alpha \in \mathbf{R}$, *then* f *is density continuous on its domain.*

PROOF. If $\alpha \leqq 0$, then this follows directly from Theorem 1. If $\alpha \geqq 1$, then this corollary is a consequence of Corollary 1.

Suppose $0 < \alpha < 1$. It is clear that Theorem 1 implies f is density continuous on $\mathrm{Dom}(f) \setminus \{0\}$. So, it must be shown that f is density continuous at 0.

Let $h > 0$ and suppose $A \subset (0,h)$. Then, we use the fact that $(f^{-1})'$ is an increasing function to see

$$\frac{|f^{-1}(A)|}{f^{-1}(h)} = \frac{1}{h^{1/\alpha}} \int_A \frac{x^{1/\alpha)-1}}{\alpha} \geqq \frac{1}{h^{1/\alpha}} \int_0^{|A|} \frac{x^{(1/\alpha)-1}}{\alpha} = \frac{|A|^{1/\alpha}}{h^{1/\alpha}} = (|A|/h)^{1/\alpha}.$$

It follows from this inequality that f is right density continuous at 0. A similar argument holds from the left.

EXAMPLE 1. There is a function $f \in C^\infty$ which is not density continuous.

Choose any sequence of disjoint intervals $J_n = [a_n, b_n] \subset [0,1]$ decreasing to 0 such that

$$d^+\left(\bigcup_{n=1}^{\infty} J_n, 0\right) = 0 \tag{9}$$

and let h be a C^∞ function satisfying

(10) $$h(0)=0,\ h(1)=1, \text{ and } h^{(n)}(0)=h^{(n)}(1)=0, \text{ for all } n\in\mathbf{N}.$$

(An example of such a function is

$$h(x)=\varrho\int_0^x \exp(-1/t^2-1/(t-1)^2)\,dt,$$

for suitable ϱ.) Let

(11) $$\alpha_n=\max\{|h^{(k)}(x)|: 0\leqq k\leqq n \text{ and } 0\leqq x\leqq 1\}\geqq 1,$$

(12) $$h_n(x)=\begin{cases} 0 & \text{if } x<a_n,\\ \frac{a_n(b_n-a_n)^n}{\alpha_n}h\left(\frac{x-a_n}{b_n-a_n}\right) & \text{if } x\in J_n,\\ \frac{a_n(b_n-a_n)^n}{\alpha_n} & \text{if } x>b_n \end{cases}$$

and

$$f(x)=\sum_{n=1}^\infty h_n(x).$$

From the choice of h, we see that $h_n\in C^\infty$ for each n. Obviously, using (9) and (11), it follows that

(13) $$\sum_{n=1}^\infty \frac{a_n(b_n-a_n)^n}{\alpha_n}\leqq\sum_{n=1}^\infty(b_n-a_n)<\infty,$$

so that f exists everywhere. Moreover, because the J_n are pairwise disjoint, it follows that f is infinitely differentiable on $\mathbf{R}\setminus 0$ and continuous on $\mathbf{R}$.

To prove that $f^{(k+1)}(0)$ exists and equals 0, let us assume that $f^{(k)}(0)=0$ and choose $a_n\leqq s<a_{n-1}$ for some $n>k$. Then it follows from (11) and (12) that

$$\frac{f^{(k)}(s)-f^{(k)}(0)}{s-0}=\begin{cases} \frac{1}{s}\sum_{i=n}^\infty h_i(s)\leqq\sum_{i=n}^\infty(b_j-a_j)^j<b_n & \text{if } k=0,\\ \frac{1}{s}h_n^{(k)}(s)\leqq\frac{a_n(b_n-a_n)^n-\alpha_k}{s\alpha_n(b_n-a_n)^k}\leqq b_n-a_n<b_n & \text{if } k>0. \end{cases}$$

Since $s\to 0$ implies $b_n\to 0$, this shows $f^{(k+1)}(0)=0$. Therefore, f is a C^∞ function.

But, f cannot be density continuous because of (9) and the fact that

$$f\left(\mathbf{R}\setminus\bigcup_{n=1}^\infty J_n\right)$$

is countable.

EXAMPLE 2. There is a continuous, density continuous function $f\colon \mathbf{R} \to \mathbf{R}$ such that $f(x) + x$ is not density continuous.

To construct such a function, we first choose two differentiable functions h_1 and h_2 satisfying:

(i) $0 < h_1 < h_2$ on $(0, \infty)$;
(ii) $h_1(x) = h_2(x) = x$ for $x \leqq 0$; and,
(iii) $1/2 < h_1'(x) < 1 < h_2'(x) < 2$ when $x > 0$.

Let a_n and b_n be any two sequences converging to 0 such that $1 = b_1 > a_1 > b_2 > a_2 > \dots$, and both

$$\frac{h_2(b_n) - h_1(a_n)}{b_n - a_n} = 2 \quad \text{and} \quad \frac{h_1(a_n) - h_2(b_{n+1})}{a_n - b_{n+1}} = 1/2. \tag{14}$$

Define a piecewise linear function f_0 by letting $f_0(a_n) = h_1(a_n), f_0(b_n) = h_2(b_n)$ and $f_0(x) = x + f_0(b_1) - b_1$ when $x > 1$ and $f_0(x) = x$ when $x \leqq 0$. The function f_0 is easily seen to be continuous because h_1 and h_2 are continuous and have value 0 at 0. Equation (14) implies

$$\frac{1}{2} \leqq \frac{f_0(b) - f_0(a)}{b - a} \leqq 2, \text{ for all } a, b \in (0, \infty).$$

It follows from Lemma 1 that f must be density continuous.

Denote $A(1/2) = \bigcup_{n=1}^{\infty} [b_{n+1}, a_n]$ and $A(2) = \bigcup_{n=1}^{\infty} [a_n b_n]$. Either

$$(-\infty, 0] \cup A(1/2) \quad \text{or} \quad (-\infty, 0) \cup A(2)$$

has positive upper density at 0. Without loss of generality we assume that it is the former. Then $f_1(x) = f_0(x) - x/2$ is constant on each component of $A(1/2)$. But this implies that $|f_1(A(1/2))| = 0$ and $A(1/2) = f_1^{-1}(f_1(A(1/2)))$ has positive density at 0. Therefore, f_1 is not density continuous at 0. So, it is enough to define $f(x) = -2f_0(x)$ to obtain the desired function.

We note that the f in Example 2 can actually be constructed as a C^∞ function by a method analogous to the construction in Example 1.

This example answers questions posed by Ostaszewski [5, Questions 5 and 6].

We wish to thank Krzysztof Ostaszewski for bringing to our attention several of the questions we have considered here.

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