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NON-DIFFERENTIABILITY OF THE CONVOLUTION OF DIFFERENTIABLE REAL FUNCTIONS

Abstract

We provide an example of two 2-periodic everywhere differentiable functions $f, g \colon \mathbb{R} \to \mathbb{R}$ whose convolution $f * g$ fails to be differentiable at every point of some perfect (thus, uncountable) set $P \subset \mathbb{R}$. This shows that the convolution operator can actually *destroy* the differentiability of these maps, rather than introducing additional smoothness (as it is usually the case). New directions and open problems are also posed.

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1 Introduction

The aim of this note is to present a series of results in the lines of finding "strange examples" ([1, 3]) when studying "differentiability versus non-differentiability" by using the convolution with two 2-periodic everywhere differentiable functions. The outcome, as we will see here, might fail to be differentiable on "large" sets. Therefore the convolution operator can actually *destroy* the differentiability of the original maps (which goes against one's initial intuition when dealing with this concept).

Let $L^1[-1,1]$ denote the class of all functions f from the real line $\mathbb{R}$ into $\mathbb{R}$ that are 2-periodic (i.e., such that $f(x+2) = f(x)$ for all $x \in \mathbb{R}$) and Lebesgue integrable, in the sense that $\|f\|_1 := \int_{-1}^{1} |f(x)|\,\mathrm{d}x < \infty$. Although functions in $L^1[-1,1]$ are formally defined on the entire $\mathbb{R}$, we will consider them only as functions on $[-1,1]$, where periodicity is used only to determine their behavior at the points ± 1 and also to facilitate an arithmetic on $[-1,1]$, especially translation.

For $f, g \in L^1[-1,1]$, the *convolution* $f * g$ of f and g is the function from $\mathbb{R}$ into $\mathbb{R}$ formally defined, for every $x \in \mathbb{R}$, as

$$(f * g)(x) := \int_{-1}^{1} f(s) g(x-s)\,\mathrm{d}s.$$

The maps f and g are referred to as the *parent functions of* $f * g$.

It is well-known (and a simple consequence of Fubini's Theorem) that $f * g$ is defined a.e. and belongs to $L^1[-1,1]$, see e.g. [10, p. 4]. More importantly for us, $f*g$ is defined for all $x \in \mathbb{R}$ when one of the parent functions is bounded, which is the only situation that shall be considered in what follows. Note that $f * g = g * f$, as the above definition clearly implies.

The operator $f * g$ consists basically of the "limit of averaging of the area between f and translations of g," see [11]. It can be also interpreted as "moving weighted average" of one parent function with respect to another, see [4].

One of the most important properties of the convolution operator is that if one of the parent functions, say f, is smooth, then the "averaging" $f * g$ of g smoothens g, in a sense of the following well known result, see e.g. [13, proposition 4, p. 454].

Proposition 1. *Let $f, g \in L^1[-1,1]$ and $k \in \mathbb{N} := \{1,2,3,\ldots\}$. If f has a continuous k^{th} derivative $\frac{d^k f}{dx^k}$, then so has $f * g$. Moreover,*

$$\frac{d^k}{dx^k}(f * g) = \left(\frac{d^k f}{dx^k} * g\right).$$

The proof of Proposition 1 uses the process of differentiation under the integral sign, for which certain control of the derivative is needed. Continuity of the derivatives ensures that this control is satisfied. Actually, for $k = 1$, we have the following slightly stronger result.

Proposition 2 (*"folklore"*). *If $f, g \in L^1[-1,1]$ and f is differentiable with bounded derivative, then $f * g$ is differentiable and $(f * g)' = f' * g$.*

Proof. In $[-1,1]$ fix a sequence $x_n \to_n x$. The map $|g(s)| \|f'\|_\infty$ is integrable and dominates all functions

$$h_n(s) := g(s) \frac{f(x_n - s) - f(x - s)}{x_n - x}.$$

Thus, by Lebesgue's dominated convergence theorem,

$$\frac{g * f(x_n) - g * f(x)}{x_n - x} = \int_{-1}^{1} g(s) \frac{f(x_n - s) - f(x - s)}{x_n - x} \, \mathrm{d}s$$

converges, as $n \to \infty$, to $\displaystyle\int_{-1}^{1} g(s) f'(x - s) \, \mathrm{d}s = g * f'(x)$, as needed. $\square$

The goal of this paper is to construct, in Theorem 7, an example for which the convolution $f * g$ fails to be differentiable on a perfect (i.e., closed with no isolated points) subset of $[-1,1]$. This generalizes the following recent result [8]. (See, also, [9].)

Theorem 3. *There exist differentiable functions $f, g \in L^1[-1,1]$ such that $f * g$ is not differentiable on $x = 0$.*

It should be also mentioned here that there exist continuous nowhere differentiable functions whose convolution is differentiable, see e.g. [7]. On the other hand, V. Jarník constructed in [6] two continuous functions whose convolution is nowhere differentiable. In fact, there are many of such functions, as proved in [7]. However, we still do not know, if $f * g$ can be nowhere differentiable, if both f and g are differentiable.

The constructions used in Theorems 3 and 7 rely on the following simple lemma. In what follows $\Psi\colon \mathbb{R} \to [0,1]$ shall stand for a C^∞ "bump" function having support contained in $(0,1)$ and being equal to 1 on its middle half, that is, such that $\Psi^{-1}(1) = [1/4, 3/4]$. See Figure 1.

Lemma 4. *For every non-trivial closed interval I of length ℓ and every $n \in \mathbb{N}$ there exist C^∞ functions $\varphi, \psi\colon \mathbb{R} \to [0,1]$ with disjoint supports contained in I and such that $\left\{s \in I\colon \varphi(s) = \psi\left(s - \frac{\ell}{2n}\right) = 1\right\}$ has measure $\ell/4$.*

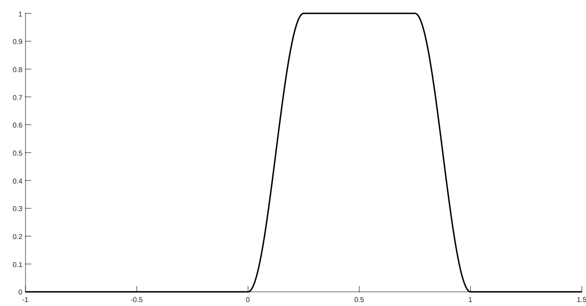

Figure 1: A sketch of the function Ψ.

PROOF. Let $I = [a, b]$ and choose $a = a_0 < \cdots < a_{2n} = b$ equally distributed, that is, with $a_{i+1} - a_i = \frac{\ell}{2n}$ for all $i < 2n$. For every $i < 2n$ let $L_i : \mathbb{R} \to \mathbb{R}$ be the linear map with $L_i(a_i) = 0$ and $L_i(a_{i+1}) = 1$ (i.e., $L_i(x) = \frac{2n}{\ell}(x - a_i)$) and let $\Psi_i := \Psi \circ L_i$. Then, the functions $\varphi = \sum_{i<n} \Psi_{2i}$ and $\psi = \sum_{i<n} \Psi_{2i+1}$, see Figure 2, are as needed. $\square$

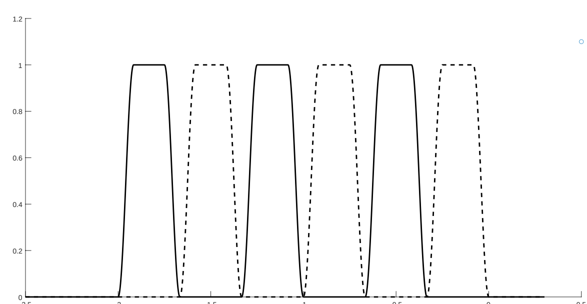

Figure 2: A sketch of the functions φ and ψ, with solid and dotted graphs, respectively. We use $I = [-2, 0]$ and $n = 3$.

For the sake of completeness we present below a new, shorter, and considerably more elegant proof of Theorem 3. This will also help the reader to follow the construction used in Theorem 7, which is an elaboration on the one presented below.

PROOF OF THEOREM 3. For every $i \in \mathbb{N}$ let φ_i and ψ_i be the functions from Lemma 4 with $I = I_i = \left[2^{-i}, 2^{-(i-1)}\right]$ and $n = n_i = 2^{7i-1}$. So, they have disjoint supports contained in I_i. Define f (see Figure 3) and g by letting, for each $x \in [-1,1]$,

$$f(x) := x^2 \sum_{i=1}^{\infty} \varphi_i(x) \quad \text{and} \quad g(x) := x^2 \sum_{i=1}^{\infty} \psi_i(-x). \tag{1}$$

Notice that the graph of g is almost a reflection of the graph of f with respect to the y-axis.

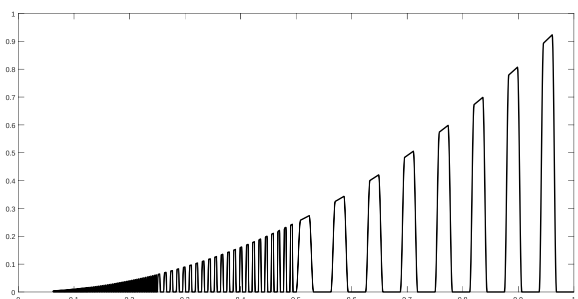

Figure 3: A sketch of the function f. Notice that as the dyadic intervals become smaller, the number of oscillations increase in an exponential way.

They are as needed. To see that f and g are differentiable, first notice that they are C^∞ on $U = (-1,0) \cup (0,1)$, since so is each map $\varphi_i(x)$ and $\psi_i(-x)$ and their supports form a locally finite family on U. They are also 2-periodic and C^∞ at ± 1 by the same reasoning combined with noticing that $f^{(j)}(\pm 1) = g^{(j)}(\pm 1) = 0$ for all $j < \omega$.[1] Finally, $f'(0) = 0$ follows from the squeeze theorem, since $\left|\frac{f(x)-f(0)}{x-0}\right| \leq \left|\frac{x^2-f(0)}{x-0}\right| = |x|$. Similarly $g'(0) = 0$.

To see that $f * g$ is not differentiable at $x = 0$, first notice that $(f*g)(0) = \int_{-1}^{1} f(s)g(0-s)\,\mathrm{d}s = 0$ since, by (1) and Lemma 4, the supports of $f(s)$ and $g(-s)$ are disjoint. Also, if $x_i = \frac{m(I_i)}{2n_i} = 2^{-8i}$, then, by

[1] Here ω stands for the first infinite ordinal. Thus, $j < \omega$ is equivalent to $j \in \{0,1,2,\ldots\}$.

Lemma 4, $E_i := \{s \in I_i \colon \varphi_i(s) = \psi_i(s - x_i) = 1\}$ has measure 2^{-i-2} and

$$\begin{aligned}(f * g)(x_i) &= \int_{-1}^{1} f(s) g(x_i - s)\,\mathrm{d}s \\ &\geq \int_{I_i} s^2\ \varphi_i(s)\ (x_i - s)^2 \psi_i(-(x_i - s))\,\mathrm{d}s \\ &\geq \int_{I_i} 2^{-2i} \varphi_i(s)\ \left(2^{-8i} - 2^{-i}\right)^2 \psi_i(s - x_i)\,\mathrm{d}s \qquad (2) \\ &\geq \int_{I_i} 2^{-4i-1} \chi_{E_i}\,\mathrm{d}s = 2^{-4i-1} m(E_i) = 2^{-5i-3}.\end{aligned}$$

Hence,

$$\frac{(f * g)(x_i) - (f * g)(0)}{x_i - 0} \geq \frac{2^{-5i-3}}{2^{-8i}} \overset{i\to\infty}{\longrightarrow} \infty,$$

so $(f * g)'(0)$ does not exist. □

With the notation of Theorem 3 and given $I = (a, b) \subset (-1, 1)$, let us define two 2-periodic functions $f_I, g_I \colon \mathbb{R} \to \mathbb{R}$, modifications of those from (1), as follows:

Definition 5. *Given $x \in [-1, 1]$, we let*

$$f_I(x) := (x - a)^2 \sum_{i=n_I}^{\infty} \varphi_i(x - a) \qquad \textit{and} \qquad g_I(x) := x^2 \sum_{i=n_I}^{\infty} \psi_i(-x),$$

where n_I is the smallest number such that $2^{-n_I} < m(I)/2$. Notice that this ensures that n_I only depends on $m(I)$ and that, for every $i \geq n_I$, the supports of $\varphi_i(x - a)$ and $\psi_i(x - a)$ are contained in the left half of I. We then extend such functions to the whole $\mathbb{R}$ via 2-periodicity.

2 The example

We will start by defining a nowhere dense perfect set that will serve as a guideline for our construction.

Definition 6. *Construct the families $\{\mathcal{I}_k \colon k < \omega\}$ of intervals such that $\mathcal{I}_0 = \{[-1/2, 1/2]\}$ and $\mathcal{I}_{k+1}$ is formed from $\mathcal{I}_k$ by replacing each $I \in \mathcal{I}_k$ by two closed intervals obtained from I by removing from it its middle half J; that is, with $m(J) = m(I)/2$. Note that each $I \in \mathcal{I}_k$ has length 2^{-2k}. Our perfect set is defined as $\Delta = \bigcap_{k<\omega} \bigcup \mathcal{I}_k$.*

Let $\mathcal{J}$ be the family of all connected components of $(-1,1)\setminus\Delta$. Also, let g be the function constructed in Theorem 3 and $\bar{f}=\sum_{J\in\mathcal{J}} f_J$, where f_J are the corresponding 2-periodic functions obtained in Definition 5. See Figures 4 and 5.

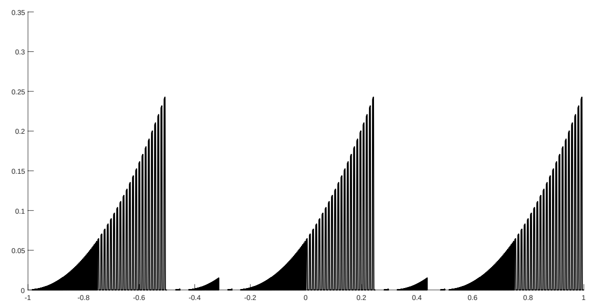

Figure 4: A sketch of the function $\bar{f}$.

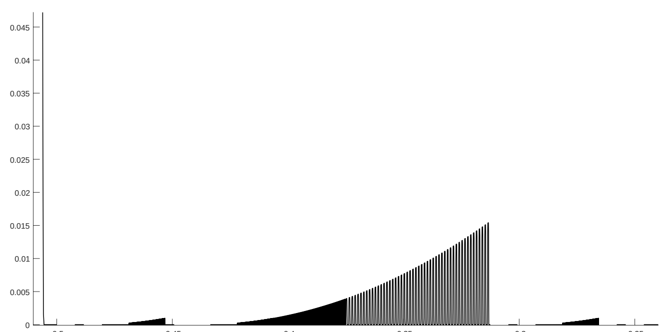

Figure 5: A detail of the sketch of the function $\bar{f}$ on the interval $[-0.53,-0.23]$.

Theorem 7. *If $\bar{f}$ and g are as above, then they are 2-periodic, differentiable, and there is a non-empty perfect $P\subset\Delta$ such that $\bar{f}*g$ is not differentiable at any $x\in P$.*

We will need the following lemma in the proof of Theorem 7.

Lemma 8. *For every $k \in \mathbb{N}$ and every $I := [a,b] \in \mathcal{I}_k$ there exists an $i_b \geq k$ (depending only on b) such that for every $i \geq i_b$ we have*

$$|(\bar{f} * g)(b + x_i) - (\bar{f} * g)(b)| > 2^{-5i-4},$$

where $x_i = \dfrac{m(I_i)}{2n_i} = 2^{-8i}$ is as in the proof of Theorem 3.

PROOF. Let $J = (b,c) \in \mathcal{J}$ and let $g_J : \mathbb{R} \to \mathbb{R}$ be the corresponding 2-periodic function constructed in Definition 5. Notice that $\tilde{g} := g - g_J$ is C^∞, as it is given as a finite sum $\tilde{g}(x) := x^2 \sum_{i=1}^{n_J - 1} \psi_i(-x)$ of C^∞ maps. (Recall that n_J is the minimum natural number for which $2^{-n_J} < m(J)/2$.)

Now, as the support of $g_J(b - x) = (x-b)^2 \sum_{i=n_J}^{\infty} \psi_i(x - b)$ is contained in J, we have that

$$\begin{aligned} 0 \leq (\bar{f} * g_J)(b) &= \int_{-1}^{1} \bar{f}(s) g_J(b-s)\,\mathrm{d}s \\ &= \int_{-1}^{1} \chi_J \bar{f}(s) g_J(b-s)\,\mathrm{d}s \\ &= \int_J f_J(s) g_J(b-s)\,\mathrm{d}s \\ &\leq \int_J f(s-b) g(-(s-b))\,\mathrm{d}s = 0 \end{aligned}$$

since the supports of $f(s)$ and $g(-s)$ are disjoint. Also, for any $i \geq n_J$,

$$\begin{aligned} (\bar{f} * g_J)(b + x_i) &\geq \int_{b+I_i} f_J(s) g_J(x_i + b - s)\,\mathrm{d}s \\ &\geq \int_{I_i} s^2 \ \varphi_i(s)\ (x_i - s)^2 \psi_i(-(x_i - s))\,\mathrm{d}s \geq 2^{-5i-3}, \end{aligned}$$

where the last inequality is justified by (2).

Since $\tilde{g}$ is C^∞, the derivative of $\bar{f} * \tilde{g}$ exists. Choose an $M \in \mathbb{R}$ with $M > |[\bar{f} * (g - g_J)]'(b)|$ and notice that for small enough $x_i > 0$ (i.e., big enough i) we have

$$|(\bar{f} * \tilde{g})(b + x_i) - (\bar{f} * \tilde{g})(b)| < M x_i = M 2^{-8i}.$$

In particular, there is an $i_b \geq k$ such that for every $i \geq i_b$ the above two displayed estimates hold and we also have $2^{-5i-3} - M2^{-8i} > 2^{-5i-4}$. This implies that

$$\begin{aligned}|(\bar{f} * g)(b + x_i) - (\bar{f} * g)(b)| \\ &= |\bar{f} * (g_J + \tilde{g})(b + x_i) - \bar{f} * (g_J + \tilde{g})(b)| \\ &\geq |\bar{f} * g_J(b + x_i) - \bar{f} * g_J(b)| - |\bar{f} * \tilde{g}(b + x_i) - \bar{f} * \tilde{g}(b)| \\ &\geq 2^{-5i-3} - M2^{-8i} > 2^{-5i-4},\end{aligned}$$

as needed. □

PROOF OF THEOREM 7. First notice that $\bar{f}$ is differentiable on Δ. In order to see this, let $h\colon \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = (\operatorname{dist}(x, \Delta))^2$. Then $h'(d) = 0$ for every $d \in \Delta$, since for every $x \neq d$,

$$\left|\frac{h(x) - h(d)}{x - d}\right| = \frac{(\operatorname{dist}(x, \Delta))^2}{|x - d|} \leq \frac{(x - d)^2}{|x - d|} = |x - d| \stackrel{x \to d}{\longrightarrow} 0.$$

Using this and the inequality $0 \leq \bar{f} \leq h$, the similar estimates show that $\bar{f}'(d) = 0$ for every $d \in \Delta$. Thus, indeed $\bar{f}$ and g are differentiable.

Next we construct, by induction on $j < \omega$, the families $\{\mathcal{J}_j : j < \omega\}$, each $\mathcal{J}_j$ consisting of 2^j intervals from $\bigcup_{k<\omega} \mathcal{I}_k$ of the same length. We start with $\mathcal{J}_0 = \mathcal{I}_0$.

If $\mathcal{J}_j$ is already constructed let $k_j < \omega$ be such that $\mathcal{J}_j \subset \mathcal{I}_{k_j}$. Let us define A_j to be the set of all right endpoints of intervals $J \in \mathcal{J}_j$ and define $N_j = \max\{i_b : b \in A_j\}$. We also choose $k \geq k_j$ to be the smallest number such that 2^{-2k}, the length of each $I \in \mathcal{I}_k$, is at most $2^{-(7N_j-1)}$.

Let $\mathcal{J}_j^*$ be the family of all $I \in \mathcal{I}_k$ that share the right endpoint with some $I' \in \mathcal{J}_j$ and let $\mathcal{J}_{j+1}$ be the family of all intervals in $\mathcal{I}_{k+1}$ contained in some $I \in \mathcal{J}_j^*$. This finished the inductive construction.

Define $P = \bigcap_{j<\omega} \bigcup \mathcal{J}_j$. This is clearly a perfect subset of Δ. We claim that it is as needed. So, let $x \in P$. We need to show that $\bar{f} * g$ is not differentiable at x.

For this, first assume that $x = b$ for some point b considered in Lemma 8. Then, for every $j < \omega$, $i \geq N_j$, and $x_i = \frac{m(I_i)}{2n_i} = 2^{-8i}$ we have

$$\left|\frac{(\bar{f} * g)(b + x_i) - (\bar{f} * g)(b)}{x_i}\right| > \frac{2^{-5i-4}}{2^{-8i}} \stackrel{i \to \infty}{\longrightarrow} \infty$$

and indeed $\bar{f} * g$ is not differentiable at $x = b$.

Next, assume that x is not in the above form. For every $j < \omega$ choose an interval $[a_j, b_j] \in \mathcal{J}_j \subset \mathcal{I}_{k_j}$ containing x. Then $b_j \searrow x$. If the sequence

$\left\langle \frac{(\bar{f}*g)(b_j)-(\bar{f}*g)(x)}{b_j-x} \right\rangle_j$ has no bound, then clearly $\bar{f}*g$ is not differentiable at x and we are done. So assume that there exists an $M \in \mathbb{R}$ such that

$$\left| \frac{(\bar{f}*g)(b_j)-(\bar{f}*g)(x)}{b_j-x} \right| < M \quad \text{for all } j.$$

Let N_j be as in the j^{th} step of the construction of the perfect set P and let $x_{N_j} = 2^{-8N_j}$ be as in Lemma 8. Then, clearly, $b_j + x_{N_j} \overset{j\to\infty}{\longrightarrow} x$. Thus, to finish the proof it is enough to show that

$$\left| \frac{(\bar{f}*g)(b_j+x_{N_j})-(\bar{f}*g)(x)}{b_j+x_{N_j}-x} \right| \overset{j\to\infty}{\longrightarrow} \infty.$$

Indeed, for every $j<\omega$, since $b_j - x < m([a_j,b_j]) \leq 2^{-(7N_j-1)}$, we have

$$\begin{aligned}
&\left| \frac{(\bar{f}*g)(b_j+x_{N_j})-(\bar{f}*g)(x)}{b_j+x_{N_j}-x} \right| \geq \\
&\quad\geq \left| \frac{(\bar{f}*g)(b_j+x_{N_j})-(\bar{f}*g)(b_j)}{b_j+x_{N_j}-x} \right| - \left| \frac{(\bar{f}*g)(b_j)-(\bar{f}*g)(x)}{b_j+x_{N_j}-x} \right| \\
&\quad\geq \left| \frac{(\bar{f}*g)(b_j+x_{N_j})-(\bar{f}*g)(b_j)}{2^{-(7N_j-1)}+2^{-8N_j}} \right| - \left| \frac{(\bar{f}*g)(b_j)-(\bar{f}*g)(x)}{b_j-x} \right| \\
&\quad\geq \frac{2^{-5N_j-4}}{2^{-(7N_j-1)}+2^{-8N_j}} - M \overset{j\to\infty}{\longrightarrow} \infty,
\end{aligned}$$

finishing the proof. □

3 Final remarks and open problems

We first observe that the functions f and $\bar{g}$ from Theorem 7 can be chosen equal. Indeed, we have:

Corollary 9. *There exists a differentiable $h \in L^1[-1,1]$ such that $h*h$ is not differentiable at any point of a perfect subset of $[-1,1]$.*

PROOF. For a map g from $X \subset \mathbb{R}$ to $\mathbb{R}$ let $\mathrm{Dif}(g)$ be the set of all $x \in X$ where g is differentiable and let $\mathrm{nDif}(g) = X \setminus \mathrm{Dif}(g)$ be its complement. It is known, that $\mathrm{Dif}(g)$, and so also $\mathrm{nDif}(g)$, are Borel in X.[2]

[2]For $X=\mathbb{R}$, Zahorski has shown in 1941 (see [12] for the 1946 French edition of his work) that $\mathrm{Dif}(g)$ is an $F_{\sigma\delta}$-set. For an arbitrary set X this was recently proven in [2, lemma 5.1] and (as was recently pointed out to the authors of [2]) earlier by Garg in [5].

Now, let $\bar{f}$ and g be as in Theorem 7. Then

$$(\bar{f}+g)*(\bar{f}+g)=\bar{f}*\bar{f}+g*g+2\bar{f}*g.$$

In particular, $2\bar{f}*g=(\bar{f}+g)*(\bar{f}+g)-\bar{f}*\bar{f}-g*g$. Hence, the uncountable set $\mathrm{nDif}(\bar{f}*g)$ is contained in

$$\mathrm{nDif}((\bar{f}+g)*(\bar{f}+g))\cup\mathrm{nDif}(\bar{f}*\bar{f})\cup\mathrm{nDif}(g*g).$$

Thus, there is an $h\in\{\bar{f},g,\bar{f}+g\}$ with $\mathrm{nDif}(h)$ uncountable. Now, since $\mathrm{nDif}(h)$ is Borel, it must contain a perfect set. □

In the same direction, notice that the continuous functions f and g constructed by Jarník [6], for which $\mathrm{nDif}(f*g)=\varnothing$, are actually equal.

Let us also point out that the above results can be translated to higher order differentiation, as related to Proposition 1. Specifically, if f and $\bar{g}$ are as in Theorem 7, $k=2,3,4,\ldots$, and F is k-times differentiable with $F^{(k-1)}=f$, then $F*\bar{g}$ fails to have k-th derivative on an uncountable set. Thus, the continuity assumption of $f^{(k)}$ in Proposition 1 is essential.

However, for $k>1$, if both f and g are k-times differentiable, then so is $f*g$. Actually, it is at least $2(k-1)$-times differentiable, with

$$(f*g)^{2(k-1)}=(f^{(k-1)}*g)^{(k-1)}=f^{(k-1)}*g^{(k-1)}.$$

We would like to finish this paper with the following interesting open question.

Problem 10. *Assume that $f,g\in L^1[-1,1]$ are differentiable. How small the set of points of differentiability of $f*g$ can be? Could it be empty? What if we, additionally, assume that $f'*g\in L^1[-1,1]$?*

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