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Examples of Sierpiński–Zygmund maps in the class of Darboux-like functions

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Abstract

The Darboux-like functions represent a group of maps that are continuous in a generalized sense. The algebra of subsets of $\mathbb{R}^{\mathbb{R}}$ (i.e., maps from $\mathbb{R}$ to $\mathbb{R}$) generated by these classes has nine atoms, that is, the smallest non-empty elements of the algebra. The subject of this work is to study the intersections of these atoms with the class SZ of Sierpiński–Zygmund functions—the maps that have as little of the standard continuity as possible. Specifically, we will show that it is independent of the standard axioms of set theory that each of these atoms has a non-empty intersection with SZ. For seven of the nine atoms this has been unknown, and the constructions of the examples provide answers to the problems stated in a recent survey *A century of Sierpiński–Zygmund functions* of K. C. Ciesielski and J. Seoane-Sepúlveda. Notice that lineability of the main classes of Darboux-like functions, as well as of Sierpiński–Zygmund functions, has been intensively studied. The presented work opens a possibility to study also the lineability of the nine smaller classes we discuss here.

Keywords Sierpiński–Zygmund functions · Darboux-like functions · Connectivity functions · Almost continuous functions

Mathematics Subject Classification 26A15 · 46T99 · 46T20 · 03E75

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1 Introduction

For $X \subset \mathbb{R}$, a map $f : X \to \mathbb{R}$ is a *Sierpiński–Zygmund function* (or just SZ-*function*) provided $f \restriction S$, its restriction to S, is discontinuous for any $S \subset X$ of cardinality $\mathfrak{c}$. Here $\mathfrak{c}$ stands for the *continuum*, that is, the cardinality of $\mathbb{R}$. The first example of such function $f : \mathbb{R} \to \mathbb{R}$ was constructed in a 1923 paper [28] of Wacław Sierpiński and Antoni Zygmund. The SZ-maps have "as little continuity as possible."

On the other hand, an $f : \mathbb{R} \to \mathbb{R}$ is *Darboux* provided it satisfies the *intermediate value property*, that is, for every $a < b$ and y between $f(a)$ and $f(b)$, there is a $c \in [a, b]$ with $f(c) = y$. This is equivalent to the fact that $f[C]$ is connected (i.e., an interval) for every connected $C \subset \mathbb{R}$. The name is used in honor of Jean Gaston Darboux who, in a 1875 paper [11], has shown that all derivatives, including those that are discontinuous, have the intermediate value property. The classes of Sierpiński–Zygmund and Darboux functions from $\mathbb{R}$ to $\mathbb{R}$ are denoted, respectively, by the symbols SZ and $\mathcal{D}$.

By definition, any $f \in \mathcal{D}$ shares the intermediate value property with the class of all continuous maps from $\mathbb{R}$ to $\mathbb{R}$. As such, $\mathcal{D}$ can be considered as a class of generalized continuous functions. Since SZ contains only extremely discontinuous functions, it is not surprising that SZ contains no generalized continuous function in the most typical uses of such term, including, but not limited to, approximately or $\mathcal{I}$-approximately continuous functions and Borel, Baire, or Lebesgue measurable functions. But this is where the class $\mathcal{D}$ stands apart from the other classes of generalized continuous functions: in a 1997 paper [2], M. Balcerzak, K. Ciesielski and T. Natkaniec proved that existence of Darboux SZ-functions is independent of the standard axioms of set theory ZFC.[1] More precisely, the authors proved in [2] that $\mathrm{SZ} \cap \mathcal{D} = \emptyset$ in the iterated perfect set model and constructed an $f \in \mathrm{SZ} \cap \mathcal{D}$ under the assumption that $\mathrm{cov}_\mathrm{M} = \mathfrak{c}$, where

$$\mathrm{cov}_\mathrm{M} := \{\kappa : \mathbb{R} \text{ is not a union of less than} \kappa\text{-many meager sets in } \mathbb{R}\}.$$

Notice that the property $\mathrm{cov}_\mathrm{M} = \mathfrak{c}$ is consistent with ZFC, as it follows from the Continuum Hypothesis (and, more generally, from the Martin's Axiom). For more details concerning this discussion, see [10]. Notice that $\mathrm{SZ} \cap \mathcal{D} \neq \emptyset$ follows also from the fact that $\mathbb{R}$ is not a union of less than $\mathfrak{c}$-many null sets, as it was just proved by the first author. So, $\mathrm{SZ} \cap \mathcal{D} \neq \emptyset$ can also hold when $\mathrm{cov}_\mathrm{M} < \mathfrak{c}$.

All classes of Darboux-like functions we discuss in this paper are contained in $\mathcal{D}$. Therefore, all our constructions require an additional set-theoretical assumption, which we will keep as in [2], that is, we will use the assumption $\mathrm{cov}_\mathrm{M} = \mathfrak{c}$: $\mathbb{R}$ is not a union of less than $\mathfrak{c}$-many meager sets. More specifically, we will use the following well-known and easy-to-see result.

Proposition 1.1 *If* $\mathrm{cov}_\mathrm{M} = \mathfrak{c}$ *holds, then no* G_δ*-subset of* $\mathbb{R}$ *which is dense in some nontrivial interval is a union of less than* $\mathfrak{c}$*-many meager subsets of* $\mathbb{R}$.

[1] This result settled a problem posed in a 1993 paper [12] by U. Darji, who constructed there, in ZFC, a map in $\mathrm{SZ} \cap \mathrm{PR}$ and asked about a function in $\mathrm{SZ} \cap \mathcal{D}$. For more on this subject, see survey [10].

For a set $X \subset \mathbb{R}$ let $\mathbb{R}^X$ denote the class of all functions from X to $\mathbb{R}$ and let

$$\mathcal{G} := \{f \in \mathbb{R}^G : f \text{ is continuous and } G \text{ is a } G_\delta\text{-subset of } \mathbb{R}\}.$$

A standard construction of an SZ-function, including the original one from the 1923 paper, uses the following result of K. Kuratowski, see e.g., [18, theorem 3.8, p. 16].

Proposition 1.2 *For every continuous function g from an $S \subset \mathbb{R}$ to $\mathbb{R}$, there exist a G_δ-set $G \subset \mathbb{R}$ containing S and a continuous extension $\bar{g} : G \to \mathbb{R}$ of g.*

In what follows, we will repeatedly use the following well-known result that follows immediately from Proposition 1.2. In its statement, and in what follows, we identify any function with its graph. Also, for a set X, the symbol $|X|$ denotes the cardinality of X.

Proposition 1.3 *If f is a map from an $X \subset \mathbb{R}$ into $\mathbb{R}$ such that $|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$, then f is an* SZ*-function.*

Notice that a *partial function* f from (a subset of) $\mathbb{R}$ to $\mathbb{R}$ is an SZ-map, provided $|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$. Other simple and well-known properties of SZ maps that we will use are as follows.

Proposition 1.4 *(i) Any restriction of an* SZ*-map is* SZ*.*
(ii) If $f : \mathbb{R} \to \mathbb{R}$ is a union of countably many SZ *partial maps, then $f \in$* SZ*.*

In the standard setting, the collection of Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$ encompasses, beside of the class $\mathcal{D}$, also seven other classes of maps: PC of *peripherally continuous functions*, PR of *functions with perfect road*, Conn of *connectivity functions*, AC of *almost continuous functions*, Ext of *extendable functions*, CIVP of *functions with Cantor Intermediate Value Property*, and SCIVP of *functions with Strong Cantor Intermediate Value Property*. We will provide their definitions in the following sections on the as needed basis. The inclusion relations among them are presented in Fig. 1.

It is worthy to mention that while Fig. 1 remains unchanged when we restrict Darboux-like classes to Baire class 2 (so Borel) functions, all classes represented there coincide (i.e., are equal) when restricted to the class of Baire 1 functions. (See

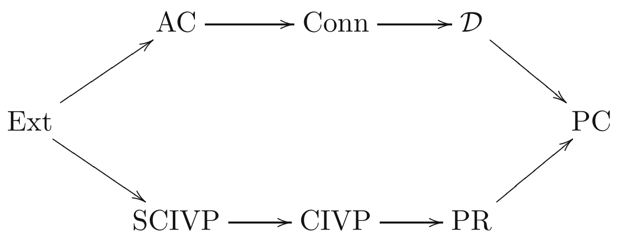

Fig. 1 All inclusions, indicated by arrows, among the Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$. The only inclusions among the intersections of these classes are those that follows trivially from this schema. (See [6,14], or [10].)

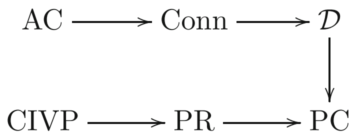

Fig. 2 Six Darboux-like classes of functions that consistently contain SZ-maps. Arrows indicate strict inclusions

the references in [6] or [10].) Also, directly by the definition (not provided here), any SCIVP function has continuous restrictions to many perfect sets. Thus, $\mathrm{SZ} \cap \mathrm{SCIVP} = \emptyset$ and, by Fig. 1, also $\mathrm{SZ} \cap \mathrm{Ext} = \emptyset$. In particular, in what follows we will be interested only in the classes in Fig. 2.

Notice, that the algebra of subsets of $\mathcal{D}$ generated by the classes in Fig. 2 has 9 atoms: $\mathcal{D} \setminus (\mathrm{Conn} \cup \mathrm{PR})$, $\mathcal{D} \cap \mathrm{PR} \setminus (\mathrm{Conn} \cup \mathrm{CIVP})$, $\mathcal{D} \cap \mathrm{CIVP} \setminus \mathrm{Conn}$, $\mathrm{Conn} \setminus (\mathrm{AC} \cup \mathrm{PR})$, $\mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{AC} \cup \mathrm{CIVP})$, $\mathrm{Conn} \cap \mathrm{CIVP} \setminus \mathrm{AC}$, $\mathrm{AC} \setminus \mathrm{PR}$, $\mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}$, and $\mathrm{AC} \cap \mathrm{CIVP}$. (The algebra of subsets of $\mathrm{PC} \setminus \mathcal{D}$ generated by the classes in Fig. 2 has only 3 atoms and it can be proved in ZFC that they have a non-empty intersections with SZ. See e.g. [10].)

The main goal of this paper is to construct, under the additional set-theoretical assumption that $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$, the examples of functions belonging to the following nine classes, the intersections of the above-mentioned atoms with SZ:

(i) $\mathrm{SZ} \cap \mathcal{D} \setminus (\mathrm{Conn} \cup \mathrm{PR})$;
(ii) $\mathrm{SZ} \cap \mathcal{D} \cap \mathrm{PR} \setminus (\mathrm{Conn} \cup \mathrm{CIVP})$;
(iii) $\mathrm{SZ} \cap \mathcal{D} \cap \mathrm{CIVP} \setminus \mathrm{Conn}$.
(iv) $\mathrm{SZ} \cap \mathrm{Conn} \setminus (\mathrm{AC} \cup \mathrm{PR})$;
(v) $\mathrm{SZ} \cap \mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{AC} \cup \mathrm{CIVP})$;
(vi) $\mathrm{SZ} \cap \mathrm{Conn} \cap \mathrm{CIVP} \setminus \mathrm{AC}$;
(vii) $\mathrm{SZ} \cap \mathrm{AC} \setminus \mathrm{PR}$;
(viii) $\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}$;
(ix) $\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{CIVP}$.

Notice that for the first seven of these classes, the existence of such examples was previously unknown, and their construction solves [10, problems 4.10, 4.13, 4.14]. We also show that the machinery we construct here allows also easy constructions of the remaining two previously-known examples.

2 A map in $\mathrm{SZ} \cap \mathrm{AC} \setminus \mathrm{PR}$

The goal of this section is to prove the following theorem.

Theorem 2.1 *If* $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{AC} \setminus \mathrm{PR} \neq \emptyset$.

The fact that, under the Continuum Hypothesis, $\mathrm{SZ} \cap \mathrm{AC} \neq \emptyset$ was first noticed in a 1982 paper [19] of K. R. Kellum, who pointed out that a function in $\mathrm{SZ} \cap \mathrm{Conn}$ constructed by J. Ceder in his 1981 paper [4] is also AC. An example of a function in $\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{PR}$ was constructed, under $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$, in a 1997 paper [2] of M. Balcerzak, K. Ciesielski, and T. Natkaniec.

Recall, that an $f : \mathbb{R} \to \mathbb{R}$ has a *perfect road* (at every $x \in \mathbb{R}$), denoted as $f \in \mathrm{PR}$, provided for every $x \in \mathbb{R}$ there exists a perfect $P \subseteq \mathbb{R}$ having x as a bilateral limit point (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) such that $f \restriction P$ is continuous at x. This class was first introduced in a 1936 paper [20] of I. Maximoff, where he proved that a Baire class 1 function is Darboux if, and only if it has a perfect road.

In what follows, we will use the following simple fact.

Proposition 2.2 *If $f: \mathbb{R} \to \mathbb{R}$ is unbounded on every perfect subset of $\mathbb{R}$, then f has perfect road at no point.*

Proof Suppose f has a perfect road at $x \in \mathbb{R}$. Then, there exists a perfect P having x as a bilateral limit point such that $f \restriction P$ is continuous at x. By continuity, there exists an $\delta > 0$ such that $f \restriction P$ is bounded on $P \cap [x-\delta, x+\delta]$. However, $P \cap [x-\delta, x+\delta]$ contains a perfect set, a contradiction. □

A function $f: \mathbb{R} \to \mathbb{R}$ is *almost continuous* (in the sense of Stallings), denoted as $f \in \mathrm{AC}$, provided every open set in $\mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. This class was first seriously studied in a 1959 paper [29] of J. Stallings. However it appeared already in an earlier paper [16] of O. H. Hamilton.

The above definition emphasizes the similarities between continuous and almost continuous functions. However, a more useful characterization of maps in AC relies on the notion of blocking sets. Specifically, a $B \subset \mathbb{R}^2$ is a *blocking set* provided it is closed, meets the graph of every continuous function, and is disjoint with some (arbitrary) function $h \in \mathbb{R}^{\mathbb{R}}$. In what follows, the family of all blocking sets will be denoted by $\mathbb{B}$. It is an easy exercise to see that

$$\textit{a map } \bar{f}: \mathbb{R} \to \mathbb{R} \textit{ is } \mathrm{AC} \textit{ if, and only if, } \bar{f} \cap B \neq \emptyset \textit{ for every } B \in \mathbb{B}. \tag{1}$$

(See e.g., [19, lemma 1].) More interestingly, every $\hat{B} \in \mathbb{B}$ contains another blocking set B such that $\pi[B]$ is a non-trivial interval (see e.g. [24]), where $\pi[B]$ is a projection of B onto the first coordinate (which, in case when B is a function f, is also denoted as $\mathrm{dom}(f)$ and referred to as the *domain* of f). In particular, if

$$\mathbb{K} := \{K \subset \mathbb{R}^2 \colon K \text{ is compact and } \pi[K] = [a, b] \text{ for some } a < b\},$$

then we have the following easy and well-known result.

Proposition 2.3 *If $f \in \mathbb{R}^{\mathbb{R}}$ is such that $f \cap K \neq \emptyset$ for every $K \in \mathbb{K}$, then $f \in \mathrm{AC}$.*

Proof By (1) it is enough to show that every $B \in \mathbb{B}$ contains some $K \in \mathbb{K}$. But, by the above discussion, $\pi[B]$ has a non-empty interior. Since B is a countable union of compact sets B_n, $n < \omega$, and $\pi[B]$ is a union of compact sets $\pi[B_n]$, by Baire Category Theorem there exists an $n < \omega$ such that $\pi[B_n]$ has a non-empty interior. Clearly, such B_n contains a $K \in \mathbb{K}$. □

Our constructions of almost continuous functions will rely on the following result, that implicitly is already in [2]. (Compare also [10, lemma 4.6].)

Lemma 2.4 *For every $K \in \mathbb{K}$, the following holds.*

(i) There exists a $\hat{g} \in \mathcal{G}$ contained in K and with $\mathrm{dom}(\hat{g})$ *dense in* $\pi[K]$.
(ii) If $\hat{g}$ is as (i) and $g \in \mathcal{G}$ is such that $\mathrm{dom}(g \cap \hat{g})$ *is dense in some non-trivial interval J, then $g \restriction J \subset K$.*

Proof (i) is a well known fact. The map $h: \pi[K] \to \mathbb{R}$, $h(x) = \inf\{y : \langle x, y\rangle \in K\}$, is of Baire class one. (See e.g. [19, lemma 1].) So, the set G of its points of continuity is a dense G_δ-subset of $\pi[K]$. (See e.g., [21, theorem 48.5].) In particular, $\hat{g} := h \restriction G$ is as needed. (See also [19, lemma 1] or [8, p. 117].)

(ii) First notice that

$$g \restriction J \subset \mathrm{cl}(\hat{g} \cap g).$$

Indeed, the function $\gamma : J \cap \mathrm{dom}(g) \to g \subset \mathbb{R}^2$, given as $\gamma(x) := \langle x, g(x)\rangle$, is continuous and the set $D := \mathrm{dom}(\hat{g} \cap g)$ is dense in J. Therefore, we have $g \restriction J = \gamma[J \cap \mathrm{dom}(g)] \subset \gamma[\mathrm{cl}(D)] \subset \mathrm{cl}(\gamma[D]) \subset \mathrm{cl}(\hat{g} \cap g)$.

Hence, $g \restriction J \subset \mathrm{cl}(g \restriction J) \subset \mathrm{cl}(\hat{g} \cap g) \subset \mathrm{cl}(\hat{g}) \subset \mathrm{cl}(K) = K$, as needed. □

The most important component in the construction of every example presented in this paper is a result stated as the following lemma.

Lemma 2.5 *Assume that* $\mathrm{cov}_M = \mathfrak{c}$ *holds. Let M be an F_σ meager subset of $\mathbb{R}$. Then there is a partial function f from $\mathbb{R}\backslash M$ to $\mathbb{R}$ such that*

(a) $|f \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$, that is, f is an SZ*-function;*
(b) f is unbounded on every perfect $P \subset \mathbb{R}$ with $P \cap M = \emptyset$;
(c) $K \cap f \neq \emptyset$ for every $K \in \mathbb{K}$.

Proof Let $\{g_\xi : \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{G}$. By induction on $\xi < \mathfrak{c}$, define the sequence of quadruples $\langle C_\xi, Z_\xi, D_\xi, f_\xi\rangle$ as follows.

(C) If $\mathrm{dom}(g_\xi)\backslash M$ is uncountable, then C_ξ is a countable infinite set, enumerated as $\{x_n : n < \omega\}$ and contained in $(\mathrm{dom}(g_\xi)\backslash M) \setminus \bigcup_{\zeta<\xi}(C_\zeta \cup D_\zeta)$. Otherwise we put $C_\xi := \emptyset$.
(Z) $Z_\xi := \mathrm{dom}(g_\xi)\backslash \bigcup_{\zeta<\xi}(C_\zeta \cup D_\zeta \cup \mathrm{dom}(g_\zeta \cap g_\xi))$.
(D) D_ξ is a dense at most countable subset of $Z_\xi\backslash(C_\xi \cup M)$.
(F) $f_\xi : C_\xi \cup D_\xi \to \mathbb{R}$ is defied as:

(i) $f_\xi(x_n) \in (n, \infty)\backslash\{g_\zeta(x_n) : \zeta < \xi\}$ for every $x_n \in C_\xi$;
(ii) $f_\xi(x) = g_\xi(x)$ on D_ξ.

The choice of a set C_ξ is possible, since an uncountable G_δ-set $\mathrm{dom}(g_\xi)\backslash M$ has cardinality $\mathfrak{c}$. It is also clear that $f := \bigcup_{\xi<\mathfrak{c}} f_\xi$ is a partial function from $\mathbb{R}\backslash M$ to $\mathbb{R}$.

To see (a), notice that for every $g \in \mathcal{G}$ there exists a $\zeta < \mathfrak{c}$ such that $g = g_\zeta$ and that for every $\xi > \zeta$ we have $g_\zeta \cap f_\xi = \emptyset$: on C_ξ it is ensured by (i), while on D_ξ by (ii) and the fact that $\mathrm{dom}(g_\zeta \cap g_\xi)$ is disjoint with $Z_\xi \supset D_\xi$. Hence, $f \cap g = f \cap g_\xi$ is a subset of $\bigcup_{\eta\le\xi} f_\eta$ which has cardinality $< \mathfrak{c}$ as a union of $< \mathfrak{c}$-many countable sets. So, indeed, $|f \cap g| < \mathfrak{c}$. Hence, by a remark just before Proposition 1.4, f is an SZ-function.

To see (b), notice that for every perfect $P \subset \mathbb{R}$ with $P \cap M = \emptyset$ there exists a $\xi < \mathfrak{c}$ such that $\mathrm{dom}(g_\xi) = P$ (as every perfect set is a G_δ set). Then, by (C), the set $C_\xi \subset P \cap \mathrm{dom}(f)$ is infinite and, by (i), f is unbounded on it.

Finally, we will argue for (c) using Lemma 2.4. So, fix a $K \in \mathbb{K}$, let $\hat{g} \in \mathcal{G}$ be contained in K and with $\mathrm{dom}(\hat{g})$ dense in $\pi[K]$, and choose a $\xi < \mathfrak{c}$ with $g_\xi = \hat{g}$. Define

$$\alpha := \min\{\zeta \le \xi \colon \mathrm{dom}(g_\zeta \cap g_\xi) \text{ is somewhere dense in } \mathbb{R}\}.$$

It is well-defined, since ξ is in the minimized set. Let J denote a non-trivial interval in which $\mathrm{dom}(g_\alpha \cap g_\xi) = \mathrm{dom}(g_\alpha \cap \hat{g})$ is dense. Then, by Lemma 2.4(ii), we have $g_\alpha \restriction J \subset K$. This and (ii) imply that $f \restriction (D_\alpha \cap J) = f_\alpha \restriction (D_\alpha \cap J) = g_\alpha \restriction (D_\alpha \cap J) \subset K$. Also, $D_\alpha \cap M = \emptyset$. Thus, to finish the proof of (c) it is enough to show that $D_\alpha \cap J \neq \emptyset$.

To see this, first notice that

$$\mathrm{dom}(g_\beta \cap g_\alpha) \cap J \text{ is nowhere dense for every } \beta < \alpha. \tag{2}$$

Indeed, otherwise there exists a $\beta < \alpha$ such that $\mathrm{dom}(g_\beta \cap g_\alpha) \cap J$ is a G_δ-set dense in some non-trivial interval $I \subset J$. In particular, $\mathrm{dom}(g_\beta \cap g_\xi)$ contains the set $\mathrm{dom}(g_\beta \cap g_\alpha) \cap \mathrm{dom}(g_\alpha \cap g_\xi)$, which is a G_δ-set dense in I, contradicting the minimality of α.

Now, by (2) and Proposition 1.1, the set $Z_\alpha \setminus (C_\alpha \cup M)$ is dense in J. Therefore, by (D), also D_α is dense in J, so that $D_\alpha \cap J \neq \emptyset$, as needed. □

Proof of Theorem 2.1 Let $\sigma \in \mathrm{SZ}$ and f be an SZ-function from Lemma 2.5 used with $M = \emptyset$. We claim that $\bar{f} = f \cup \sigma \restriction (\mathbb{R} \setminus \mathrm{dom}(f))$ is as needed, that is, in $\mathrm{SZ} \cap \mathrm{AC} \setminus \mathrm{PR}$.

Indeed, $\bar{f} \in \mathrm{SZ}$ by Propositions 1.3 and 1.4. Also, $\bar{f} \notin \mathrm{PR}$, since, by part (b) of Lemma 2.5, $\bar{f} \supset f$ is unbounded on any perfect $P \subset \mathbb{R}$, and so, by Propositions 2.2, is perfect-road free. Finally, by part (c) of Lemma 2.5, $\bar{f} \supset f$ intersects every $K \in \mathbb{K}$. So, by Proposition 2.3, $\bar{f} \in \mathrm{AC}$. □

3 A map in $\mathrm{SZ} \cap \mathcal{D} \setminus (\mathrm{PR} \cup \mathrm{Conn})$

Recall that a map $f \colon \mathbb{R} \to \mathbb{R}$ is *connectivity*, denoted as $f \in \mathrm{Conn}$, provided $f \restriction S$ is a connected subset of $\mathbb{R}^2$ for every connected subset S (i.e., an interval) of $\mathbb{R}$. This concept was first defined in a research problem [22] proposed by J. Nash in 1956. He inquired if an endomorphic map on a cell, that preserves connectedness of any connected subset of its domain to its graph, must have a fixed point or not. This problem was studied and given an affirmative answer by O. H. Hamilton and J. Stallings in their papers [16] and [29], respectively. In addition, J. Stallings proved also in [29] that $\mathrm{AC} \subset \mathrm{Conn}$. Notice, that the inclusion $\mathrm{Conn} \subset \mathcal{D}$ is obvious.

A function $f \colon \mathbb{R} \to \mathbb{R}$ is *everywhere surjective*, denoted as $f \in \mathrm{ES}$, provided $f[(a, b)] = \mathbb{R}$ for all $a < b$. Equivalently, $f \in \mathrm{ES}$ if, and only if, $f^{-1}(y)$ is dense in $\mathbb{R}$ for every $y \in \mathbb{R}$. This class, under different names, has been studied by many authors, see e.g. [7,15,23], or [5, section 7.2]. The name *everywhere surjective* comes from a 2005 paper [1] of R. Aron, V. I. Gurariy, and J. B. Seoane and the consecutive work of these authors. Clearly $\mathrm{ES} \subset \mathcal{D}$.

The goal of this section is to prove the following theorem.

Theorem 3.1 *If* $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{ES} \setminus (\mathrm{PR} \cup \mathrm{Conn}) \neq \emptyset$ *and so also* $\mathrm{SZ} \cap \mathcal{D} \setminus (\mathrm{PR} \cup \mathrm{Conn}) \neq \emptyset$.

An example, under $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$, of an additive function in $\mathrm{SZ} \cap \mathcal{D} \setminus \mathrm{Conn}$ can be found in [26]. In what follows Δ will be defined as the diagonal, that is,

$$\Delta := \{\langle x, x\rangle : x \in \mathbb{R}\}.$$

Proof of Theorem 3.1 Let function $\bar{f}$ be as in the proof of Theorem 2.1, that is, defined as $\bar{f} = f \cup \sigma \restriction (\mathbb{R} \setminus \mathrm{dom}(f))$, where $\sigma \in \mathrm{SZ}$ and f is from Lemma 2.5 used with $M = \emptyset$. Notice that $\bar{f} \in \mathrm{ES}$, since by (c) of Lemma 2.5, f intersects any constant map g defined on any non-trivial interval. Also, by Propositions 1.3 and 1.4, $\bar{f} \in \mathrm{SZ}$.

Since $\Delta \in \mathcal{G}$, the set $D := \mathrm{dom}(\Delta \cap \bar{f})$ has cardinality $< \mathfrak{c}$, as $\bar{f} \in \mathrm{SZ}$. Let $\hat{f} := \bar{f} + \chi_D$, where χ_D is the characteristic function of the set D. We claim that $\hat{f} \in \mathrm{SZ} \cap \mathrm{ES} \setminus (\mathrm{PR} \cup \mathrm{Conn})$.

Indeed, $\hat{f} \notin \mathrm{PR}$ by Proposition 1.4, since for any perfect $P \subset \mathbb{R}$ the map $\hat{f}$ is unbounded. (This is the case, since $\hat{f} := \bar{f} + \chi_D$, where χ_D is bounded, while $f \subset \bar{f}$ is unbounded on P.) Also, $\hat{f} \in \mathrm{ES}$ since, for every $y \in \mathbb{R}$, the set $\bar{f}^{-1}(y)$ is dense in $\mathbb{R}$ (as $\bar{f} \in \mathrm{ES}$), and so is $\hat{f}^{-1}(y)$, as it contains

$$\bar{f}^{-1}(y) \setminus \{x \in D : \hat{f}(x) = y\} \supset \bar{f}^{-1}(y) \setminus \{y\}.$$

We have $\hat{f} \notin \mathrm{Conn}$, since $\hat{f}$ has a graph dense in $\mathbb{R}^2$ (as $\hat{f} \in \mathrm{ES}$) and its graph does not intersect Δ. Finally, $\hat{f} \in \mathrm{SZ}$ by Proposition 1.4, as both $\hat{f} \restriction \mathbb{R} \setminus D = \bar{f} \restriction \mathbb{R} \setminus D$ and $\hat{f} \restriction D$ are SZ. □

4 Maps in SZ ∩ 𝒟 ∩ CIVP \ Conn and in SZ ∩ 𝒟 ∩ PR \ (Conn ∪ CIVP)

A function $f : \mathbb{R} \to \mathbb{R}$ has *Cantor Intermediate Value Property*, what is denoted as $f \in \mathrm{CIVP}$, provided for every $a, b \in \mathbb{R}$ with $f(a) \neq f(b)$ and for every perfect set K between $f(a)$ and $f(b)$, there exists a perfect set C between a and b such that $f[C] \subseteq K$.

Choose a countable family $\mathcal{F} = \{P_{p,q} \subseteq (p, q) : p < q \;\&\; p, q \in \mathbb{Q}\}$ of pairwise disjoint nowhere dense perfect sets and define

$$\hat{M} := \bigcup \mathcal{F}. \tag{3}$$

Notice that $\hat{M}$ is a meager and an F_σ-subset of $\mathbb{R}$. Also, for every $P_{p,q} \in \mathcal{F}$ let

$$\{C_{p,q,\xi} : \xi < \mathfrak{c}\} \tag{4}$$

be a partition of $P_{p,q}$ consisting of perfect sets. It exists, since $P_{p,q}$ is homeomorphic to $\mathfrak{C}^2$, where $\mathfrak{C}$ is the Cantor set, and the sets $\{\{x\}\times\mathfrak{C}\colon x\in\mathfrak{C}\}$ form its partition. Thus,

$$\{C_{p,q,\xi}\colon \xi<\mathfrak{c}\ \&\ p<q\ \&\ p,q\in\mathbb{Q}\}$$

is a partition of $\hat{M}$.

Lemma 4.1 *Let $\hat{M}$ be as in (3). Then there exists a function $h\colon \hat{M}\to\mathbb{R}$ such that*

(a) $|h\cap g|<\mathfrak{c}$ for every $g\in\mathcal{G}$, that is, h is an SZ-function;
(b) for every perfect $K\subset\mathbb{R}$ and $a<b$ there exists a perfect $P\subset\hat{M}\cap(a,b)$ such that $h[P]\subset K$.

Proof Let $\{g_\xi\colon \xi<\mathfrak{c}\}$, $\{x_\zeta\colon \zeta<\mathfrak{c}\}$, and $\{K_\xi\colon \xi<\mathfrak{c}\}$ be the enumerations of $\mathcal{G}$, $\hat{M}$, and the family of all perfect subsets of $\mathbb{R}$, respectively. Let sets $C_{p,q,\xi}$ be as in (4). By induction on $\zeta<\mathfrak{c}$, define a function $h\colon \hat{M}\to\mathbb{R}$ such that

$$h(x_\zeta)\in K_\xi\setminus\{g_\eta(x_\zeta)\colon \eta<\zeta\ \&\ x_\zeta\in\mathrm{dom}(g_\eta)\},$$

where $\xi<\mathfrak{c}$ is such that $x_\zeta\in C_{p,q,\xi}$ for some $p,q\in\mathbb{Q}$, $p<q$. This completes the construction.

To see (a), notice that for every $g\in\mathcal{G}$, there exists a $\xi<\mathfrak{c}$ such that $g=g_\xi$ and $h(x_\zeta)\neq g_\xi(x_\zeta)$ for every $\zeta>\xi$. Thus, $|h\cap g|\le\xi<\mathfrak{c}$.

To see (b), take perfect $K\subset\mathbb{R}$ and $a<b$. Find $p,q\in\mathbb{Q}$ with $a<p<q<b$ and a $\xi<\mathfrak{c}$ such that $K=K_\xi$. Then $P:=C_{p,q,\xi}\subset\hat{M}\cap(a,b)$ is perfect and, according to the construction, $h[P]=h[C_{p,q,\xi}]\subseteq K_\xi=K$, as needed. □

Theorem 4.2 *If $\mathrm{cov}_{\mathrm{M}}=\mathfrak{c}$ holds, then $\mathrm{SZ}\cap\mathrm{ES}\cap\mathrm{CIVP}\setminus\mathrm{Conn}\neq\emptyset$, and so also $\mathrm{SZ}\cap\mathcal{D}\cap\mathrm{CIVP}\setminus\mathrm{Conn}\neq\emptyset$.*

Proof Take the SZ-functions: $h\colon \hat{M}\to\mathbb{R}$ from Lemma 4.1, f from Lemma 2.5 with $M:=\hat{M}$, and an arbitrary $\sigma\in\mathrm{SZ}$. Let

$$\bar{f}=f\cup h\cup\sigma\restriction\big(\mathbb{R}\setminus\mathrm{dom}(f\cup h)\big).$$

Then $\bar{f}\in\mathrm{ES}$, since by (c) of Lemma 2.5, f intersects any constant map g defined on any non-trivial interval. Also, by Proposition 1.4, we have $\bar{f}\in\mathrm{SZ}$. So, the set $D:=\mathrm{dom}(\Delta\cap\bar{f})$ has cardinality $<\mathfrak{c}$, as $\Delta\in\mathcal{G}$. Let

$$\hat{f}:=\bar{f}+\chi_D,$$

where χ_D is the characteristic function of D. We claim that $\hat{f}$ is as needed, that is, that $\hat{f}\in\mathrm{SZ}\cap\mathrm{ES}\cap\mathrm{CIVP}\setminus\mathrm{Conn}$.

Indeed, $\hat{f}\in\mathrm{SZ}$ by Proposition 1.4, as a union of two SZ-functions: $\hat{f}\restriction D$ and $\hat{f}\restriction(\mathbb{R}\setminus D)=\bar{f}\restriction(\mathbb{R}\setminus D)$. Also, $\hat{f}\in\mathrm{ES}$ since, for every $y\in\mathbb{R}$, the set $\bar{f}^{-1}(y)$ is dense in $\mathbb{R}$ (as $\bar{f}\in\mathrm{ES}$), and so is $\hat{f}^{-1}(y)$, as it contains

$$\bar{f}^{-1}(y)\setminus\{x\in D\colon \hat{f}(x)=y\}\supset\bar{f}^{-1}(y)\setminus\{y\}.$$

We have $\hat{f} \notin \mathrm{Conn}$, since $\hat{f}$ has a graph dense in $\mathbb{R}^2$ (as $\hat{f} \in \mathrm{ES}$) and its graph does not intersect Δ.

To finish the proof, it remains to check that $\hat{f} \in \mathrm{CIVP}$. So, fix $a < b$ with $\hat{f}(a) \neq \hat{f}(b)$ and a perfect set K between $\hat{f}(a)$ and $\hat{f}(b)$. We need to find a perfect $C \subset (a, b)$ with $\hat{f}[C] \subset K$. By part (b) of Lemma 4.1, there exists a perfect $P \subset \hat{M} \cap (a, b)$ such that $h[P] \subset K$. Since $|D| < \mathfrak{c}$, there exists a perfect $C \subset P \setminus D$. But $\hat{f} = \bar{f} = h$ on $P \setminus D$. Hence, $\hat{f}[C] = h[C] \subset h[P] \subset K$, as needed. □

To construct a map in $\mathrm{SZ} \cap \mathrm{ES} \cap \mathrm{PR} \setminus (\mathrm{Conn} \cup \mathrm{CIVP})$, we need yet another lemma, where $\mathfrak{C}$ denotes the classic Cantor ternary set in $[0, 1]$.

Lemma 4.3 *Let $\hat{M}$ be as in (3). Then there exists a function $h \colon \hat{M} \to \mathbb{R}$ such that*

(a) $|h \cap g| < \mathfrak{c}$ for every $g \in \mathcal{G}$, that is, h is an SZ*-function;*
(b) $h[\hat{M}] \cap \mathfrak{C} = \emptyset$;
(c) for any $\langle s, t\rangle \in \mathbb{R}^2$ there is a perfect set $P \subset \hat{M} \cup \{s\}$ having s as a bilateral limit point and such that $\lim_{x \to s,\, x \in P} h(x) = t$.

Proof Let $\{\langle s_\xi, t_\xi\rangle : \xi < \mathfrak{c}\}$ be an enumeration of $\mathbb{R}^2$ and let sets $C_{p,q,\xi}$ be as in (4). For every $\xi < \mathfrak{c}$, let $\{p_n^\xi\}_{n\in\mathbb{N}}$ and $\{q_n^\xi\}_{n\in\mathbb{N}}$ be the sequences of rational numbers converging to s_ξ, the first strictly increasing, the second strictly decreasing. Define

$$P_\xi = \bigcup_{n\in\mathbb{N}} (C_{p_n, p_{n+1}, \xi} \cup C_{q_{n+1}, q_n, \xi}).$$

Note that $\{P_\xi : \xi < \mathfrak{c}\}$ is a family of pairwise disjoint subsets of $\hat{M}$ and that, for every $\xi < \mathfrak{c}$, the set $P_\xi \cup \{s_\xi\}$ is perfect with s_ξ being its bilateral limit point.

Let $\{x_\zeta : \zeta < \mathfrak{c}\}$ be an enumeration of $\hat{M}$. By induction on $\zeta < \mathfrak{c}$, define a function $h \colon \hat{M} \to \mathbb{R}$ so that

$$h(x_\zeta) \in (t_\xi, t_\xi + |x_\zeta - s_\xi|) \setminus \left(\mathfrak{C} \cup \{g_\eta(x_\zeta) \colon \eta < \zeta \;\&\; x_\zeta \in \mathrm{dom}(g_\eta)\}\right) \qquad (5)$$

when x_ζ is in some P_ξ and

$$h_\xi(x_\zeta) \in \mathbb{R} \setminus \left(\mathfrak{C} \cup \{g_\eta(x_\zeta) \colon \eta < \zeta \;\&\; x_\zeta \in \mathrm{dom}(g_\eta)\}\right)$$

otherwise. Such choice is possible since $(y_\xi, y_\xi + |x_\zeta - x_\xi|) \setminus \mathfrak{C}$ has cardinality $\mathfrak{c}$, while $\{g_\eta(x_\zeta) \colon \eta < \zeta \;\&\; x_\zeta \in \mathrm{dom}(g_\eta)\}$ has a smaller cardinality. This completes the construction.

Clearly, h satisfies (a) and (b). To see (c), fix an $\langle s, t\rangle \in \mathbb{R}^2$ and let $\xi < \mathfrak{c}$ be such that $\langle s_\xi, t_\xi\rangle = \langle s, t\rangle$. Then $P := P_\xi \cup \{s_\xi\}$ is as needed, since our construction ensures that $|h(x) - t_\xi| < |x - s_\xi|$ for every $x \in P_\xi$. □

Theorem 4.4 *If* $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{ES} \cap \mathrm{PR} \setminus (\mathrm{Conn} \cup \mathrm{CIVP}) \neq \emptyset$*, and so also* $\mathrm{SZ} \cap \mathcal{D} \cap \mathrm{PR} \setminus (\mathrm{Conn} \cup \mathrm{CIVP}) \neq \emptyset$.

Proof Take the SZ-functions: $h\colon \hat{M} \to \mathbb{R}$ from Lemma 4.3, f from Lemma 2.5 with $M := \hat{M}$, and an arbitrary $\sigma \in \text{SZ}$. Let

$$\bar{f} = f \cup h \cup \sigma \restriction \big(\mathbb{R}\setminus \text{dom}(f \cup h)\big).$$

Then $\bar{f} \in \text{ES}$, since by (c) of Lemma 2.5, f intersects any constant map g defined on any non-trivial interval. Also, by Proposition 1.4, we have $\bar{f} \in \text{SZ}$. So, the set $D := \text{dom}(\Delta \cap \bar{f})$ has cardinality $< \mathfrak{c}$, as $\Delta \in \mathcal{G}$. Let $\hat{\chi}\colon D \to \mathbb{R}\setminus\mathfrak{C}$ be one-to-one and such that $\hat{\chi} \cap \Delta = \emptyset$. Define

$$\hat{f} := \hat{\chi} \cup \bar{f} \restriction (\mathbb{R}\setminus D).$$

We claim that $\hat{f}$ is as needed, that is, that $\hat{f} \in \text{SZ} \cap \text{ES} \cap \text{PR} \setminus (\text{Conn} \cup \text{CIVP})$.

Indeed, $\hat{f} \in \text{SZ}$ by Proposition 1.4, as a union of two SZ-functions: $\hat{f} \restriction D$ and $\hat{f} \restriction (\mathbb{R}\setminus D) = \bar{f} \restriction (\mathbb{R}\setminus D)$. Also, $\hat{f} \in \text{ES}$ since, for every $y \in \mathbb{R}$, the set $\bar{f}^{-1}(y)$ is dense in $\mathbb{R}$ (as $\bar{f} \in \text{ES}$), and so is $\hat{f}^{-1}(y)$, as it contains

$$\bar{f}^{-1}(y)\setminus\{x \in D\colon \hat{f}(x) = y\} \supset \bar{f}^{-1}(y)\setminus\hat{\chi}^{-1}(y).$$

We have $\hat{f} \notin \text{Conn}$, since $\hat{f}$ has a graph dense in $\mathbb{R}^2$ (as $\hat{f} \in \text{ES}$) and its graph does not intersect Δ.

For $\hat{f} \notin \text{CIVP}$, notice that $\hat{f} \in \text{ES}$ implies the existence of $a < b$ for which $\mathfrak{C}$ is between $\hat{f}(a)$ and $\hat{f}(b)$. Thus, it is enough to show, that $\hat{f}[C] \not\subset \mathfrak{C}$ for every perfect $C \subset \mathbb{R}$. So, by way of contradiction, assume that there is a perfect $C \subset \mathbb{R}$ with $\hat{f}[C] \subset \mathfrak{C}$. Since $|D| < \mathfrak{c}$, there is a perfect $P \subset C\setminus D$, for which of course $\hat{f}[P] \subset \mathfrak{C}$. Then, $P \cap \hat{M} = \emptyset$, since, by part (b) of Lemma 4.3, for every $x \in \hat{M}\setminus D$ we have $\hat{f}(x) = h(x) \notin \mathfrak{C}$. So, $\hat{f} = f$ on P and, by part (b) of Lemma 2.5, $\hat{f}[P] = f[P]$ is unbounded, contradicting $\hat{f}[P] \subset \mathfrak{C}$.

To finish the proof, we need to show that $\hat{f} \in \text{PR}$. To see this, fix an $s \in \mathbb{R}$. We need to find a perfect $P \subseteq \mathbb{R}$ having s as a bilateral limit point such that $\hat{f} \restriction P$ is continuous at s. For this, let $t = \hat{f}(s)$. By part (c) of Lemma 4.3, there is a perfect set $P \subset \hat{M} \cup \{s\}$ having s as a bilateral limit point and such that $\lim_{x\to s,\, x\in P} h(x) = t$. Since $|D| < \mathfrak{c}$, we can decrease P so that $P\setminus\{s\}$ is disjoint with D. But then, $\lim_{x\to s,\, x\in P} \hat{f}(x) = \lim_{x\to s,\, x\in P} h(x) = t = \hat{f}(s)$, that is, $\hat{f} \restriction P$ is continuous at s, as needed. □

5 Three examples within the class SZ ∩ Conn \ AC

It is well known (see e.g. [17, theorem 2] or [13]), that

Proposition 5.1 *If* $f \in \mathbb{R}^{\mathbb{R}}$ *intersects every compact connected subset* H *of* $\mathbb{R}^2$ *with* $|\pi[H]| > 1$, *then* $f \in \text{Conn}$.

In fact, this follows easily from a theorem, that *if two points of the plane are separated by a closed set* F, *then they are separated by a component of* F.

On the other hand, J. H. Roberts constructed in [27] a subset $Z \subset [0, 1]^2$ homeomorphic to the Cantor set $\mathfrak{C}$ (so, zero-dimensional) which is a blocking set for maps from

[0, 1] to [0, 1], that is, such that $Z \cap g \neq \emptyset$ for every continuous $g : [0, 1] \to [0, 1]$. This construction was modified by K. Ciesielski and A. Rosłanowski in [9, lemma 2.1] to obtain a zero-dimensional blocking set $\bar{Z}$ for functions from $\mathbb{R}$ to $\mathbb{R}$. The following proposition describes the properties of this set that we will use in what follows.

Proposition 5.2 *Let $X := (-1, 1) \cap \mathbb{Q}$ and $G := (-1, 1) \setminus \mathbb{Q}$. There exists an embedding $F = \langle F_0, F_1 \rangle : \mathbb{R} \to (-1, 1) \times \mathbb{R}$ such that F_0 is non-decreasing,*

(a) $B := F[\mathbb{R}]$ is a blocking set;
(b) zero-dimensional $\bar{Z} := F[\mathbb{Z} + \mathfrak{C}] \subset B$ is also a blocking set;
(c) $\gamma := \bar{Z} \cap \pi^{-1}(G) = B \cap \pi^{-1}(G)$ is a continuous function on G; and
(d) for every $x \in X$ the vertical section $B \cap \pi^{-1}(\{x\})$ of B is a non-trivial closed interval and $\bar{Z} \cap \pi^{-1}(\{x\})$ consists of the two endpoints of that interval.

Using Robert's set Z is relatively easy to construct a connectivity function $f : [0, 1] \to [0, 1]$ which is not almost continuous. Below, we will use the set $\bar{Z}$ to construct the functions in $\mathrm{SZ} \cap \mathrm{Conn} \setminus \mathrm{AC}$ with the additional properties we examine. An example of additive function in $\mathrm{SZ} \cap \mathrm{Conn} \setminus \mathrm{AC}$ has been constructed, under the Continuum Hypothesis, in [26, example 9]. The key result for this construction is the following lemma.

Lemma 5.3 *Let H be a compact connected subset of $\mathbb{R}^2$ with $|\pi[H]| > 1$. If H contains no vertical section of the set $B \cap \pi^{-1}(X)$ from Proposition 5.2, then there is a $K \in \mathbb{K}$ contained in $H \setminus \bar{Z}$.*

Proof If $\pi[H] \not\subset [-1, 1]$, then there are $a < b$ with $[a, b] \subset \pi[H] \setminus [-1, 1]$ and $K := H \cap \pi^{-1}([a, b])$ is as needed. So, assume that $\pi[H] \subset [-1, 1]$. Since $\pi[H]$

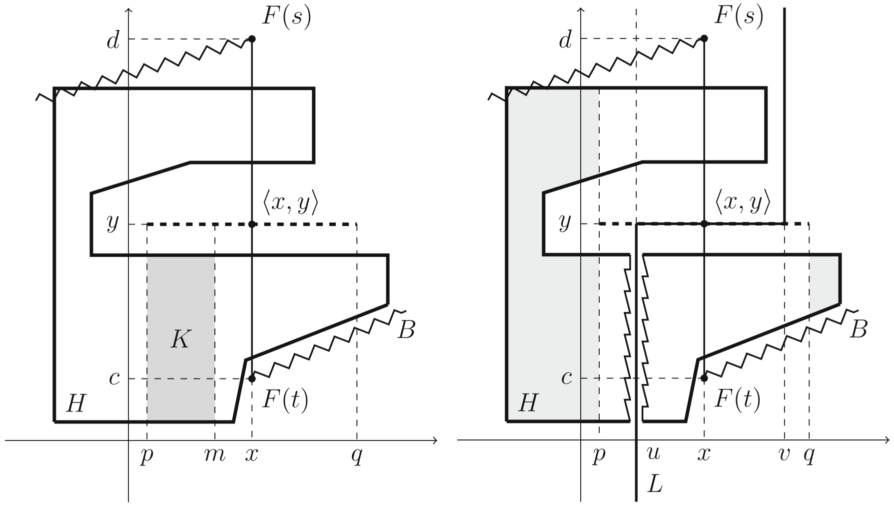

Fig. 3 Illustration for the proof of Lemma 5.3. Left figure corresponds to the case when $[p, x) \subset \pi[H \cap ([p, x) \times (-\infty, y])]$. The right figure addresses the case leading to the contradiction

has more than one element and is connected, there is an $x \in X$ that belongs to the interior of $\pi[H]$. Let $c < d$ be such that $B \cap \pi^{-1}(\{x\}) = \{x\} \times [c, d]$. See Fig. 3. Since H contains no vertical section of the set B, there is a $y \in (c, d)$ with $\langle x, y\rangle \notin H$. Let $s, t \in \mathbb{R}$ be such that $F(s) = \langle x, d\rangle$ and $F(t) = \langle x, c\rangle$. We assume that $s < t$, the other case being similar.[2] Then, $F_1(s) = d > y > F_1(t)$ and there exist $p \in (-1, x) \cap \pi[H]$ and $q \in (x, 1) \cap \pi[H]$ such that $F_1(u) > y > F_1(v)$ for every $u \in [p, x]$ and $v \in [x, q]$. See Fig. 3. Also, we can assume that $[p, q] \times \{y\}$ is disjoint with H (e.g. by imposing that the length of $[p, q]$ is less than the distance from $\langle x, y\rangle$ to H).

Now, if $[p, x) \subset \pi[H \cap ([p, x) \times (-\infty, y])]$, then the set $K := H \cap ([p, m] \times [y, \infty))$, for any $m \in (p, x)$, is as needed. See the left part of Fig. 3. Similarly, if $(x, q] \subset \pi[H \cap ((x, q] \times [y, \infty))]$, then the set $K := H \cap ([m', q] \times [y, \infty))$, for any $m' \in (x, q)$, satisfies the lemma. Therefore, by way of contradiction, assume that neither of this happens. Then, there are $u \in [p, x)$ and $v \in (x, q]$ such that the set

$$L := (\{u\} \times (-\infty, y]) \cup ([u, v] \times \{y\}) \cup (\{v\} \times [y, \infty))$$

is disjoint with H, see the right part of Fig. 3. But this is impossible, since such L separates $H \cap (\{p\} \times \mathbb{R}) \neq \emptyset$ from $H \cap (\{q\} \times \mathbb{R}) \neq \emptyset$, contradicting the connectedness of H. □

Theorem 5.4 *If* $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{Conn} \cap \mathrm{CIVP} \setminus \mathrm{AC} \neq \emptyset$.

Proof Similarly as in the proof of Theorem 4.2, take the SZ-functions: $h\colon \hat{M} \to \mathbb{R}$ from Lemma 4.1, f from Lemma 2.5, this time with $M := \hat{M} \cup X$ where X is from Proposition 5.2, and an arbitrary $\sigma \in \mathrm{SZ}$. Let

$$\bar{f} = f \cup h \cup \sigma \restriction \big(\mathbb{R} \setminus \mathrm{dom}(f \cup h)\big).$$

Then, by Proposition 1.4, we have $\bar{f} \in \mathrm{SZ}$. So, the set $D := X \cup \mathrm{dom}(\gamma \cap \bar{f})$ has cardinality $< \mathfrak{c}$, where $\gamma \in \mathcal{G}$ is as in part (c) of Proposition 5.2. Moreover, $\mathrm{dom}(\bar{Z} \cap \bar{f}) \subset D$. Let $\hat{\chi}: D \to \mathbb{R}$ be such that $\hat{\chi} \cap \bar{Z} = \emptyset$ and $\hat{\chi} \restriction X \subseteq B \setminus \bar{Z}$, where $\bar{Z}$ and B are from Proposition 5.2. Define

$$\hat{f} := \hat{\chi} \cup \bar{f} \restriction \mathbb{R} \setminus D.$$

We claim that $\hat{f}$ is as needed, that is, that $\hat{f} \in \mathrm{SZ} \cap \mathrm{Conn} \cap \mathrm{CIVP} \setminus \mathrm{AC}$.

Indeed, $\hat{f} \in \mathrm{SZ}$ by Proposition 1.4, as a union of two SZ-functions: $\hat{f} \restriction D$ and $\hat{f} \restriction \mathbb{R} \setminus D = \bar{f} \restriction \mathbb{R} \setminus D$. Clearly, $\hat{f} \notin \mathrm{AC}$, since $\hat{f} \cap \bar{Z} = \emptyset$, while $\bar{Z}$ is a blocking set.

An argument that $\hat{f} \in \mathrm{CIVP}$ is identical to one presented in Theorem 4.2.

Finally, to see that $\hat{f} \in \mathrm{Conn}$, fix a compact connected subset H of $\mathbb{R}^2$ with $|\pi[H]| > 1$. By Proposition 5.1 it is enough to show that $\hat{f} \cap H \neq \emptyset$. This is clear when H contains a vertical section of the set $\bar{Z} \cap \pi^{-1}(X)$, since we ensured that $\hat{\chi} \subset \hat{f}$ intersects every such H. But otherwise, by Lemma 5.3, there is a $K \in \mathbb{K}$ contained in $H \setminus \bar{Z}$. Also, by Lemma 2.5, there an $x \in \mathrm{dom}(f)$ such that $\langle x, f(x)\rangle \in K \subset H$. So,

[2] Actually, the other case cannot happen in the actual construction of the curve from Proposition 5.2.

to finish the proof, it is enough to notice that $\hat{f}(x) = f(x)$. Indeed, this is the case, since $x \notin D$: $x \notin X$, as $\mathrm{dom}(f)$ is disjoint from $M = \hat{M} \cup X$; and $x \notin \mathrm{dom}(\gamma \cap \bar{f})$ since K is disjoint from $\bar{Z} \supset \gamma$. □

The proof of the next theorem is a simple mix of the elements of the proofs of Theorems 5.4 and 4.4.

Theorem 5.5 *If* $\mathrm{cov_M} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{AC} \cup \mathrm{CIVP}) \neq \emptyset$.

Proof Similarly as in the proof of Theorem 4.4, take the SZ-functions: $h: \hat{M} \to \mathbb{R}$ from Lemma 4.3, f from Lemma 2.5 with $M := \hat{M} \cup X$ where set X is from Proposition 5.2, and $\sigma : \mathbb{R} \to \mathbb{R}$. Let

$$\bar{f} = f \cup h \cup \sigma \restriction \big(\mathbb{R} \setminus \mathrm{dom}(f \cup h)\big).$$

Then $\bar{f} \in \mathrm{ES}$, since by (c) of Lemma 2.5, f intersects any constant map g defined on any non-trivial interval. Also, by Proposition 1.4, we have $\bar{f} \in \mathrm{SZ}$. So, the set $D := X \cup \mathrm{dom}(\gamma \cap \bar{f})$ has cardinality $< \mathfrak{c}$, where $\gamma \in \mathcal{G}$ is as in part (c) of Proposition 5.2. Moreover, $\mathrm{dom}(\bar{Z} \cap \bar{f}) \subset D$. Let $\hat{\chi} : D \to \mathbb{R}$ such that $\hat{\chi} \cap \bar{Z} = \emptyset$ and $\hat{\chi} \restriction X \subseteq B \setminus \bar{Z}$, where $\bar{Z}$ and B are from Proposition 5.2. Define

$$\hat{f} := \hat{\chi} \cup \bar{f} \restriction (\mathbb{R} \setminus D).$$

We claim that $\hat{f}$ is as needed, that is, that $\hat{f} \in \mathrm{SZ} \cap \mathrm{Conn} \cap \mathrm{PR} \setminus (\mathrm{AC} \cup \mathrm{CIVP})$.

Indeed, $\hat{f} \in \mathrm{SZ}$ by Proposition 1.4, as being a union of two SZ-functions: $\hat{\chi}$ and $\bar{f} \restriction (\mathbb{R} \setminus D)$. Clearly, $\hat{f} \notin \mathrm{AC}$, since $\hat{f} \cap \bar{Z} = \emptyset$, while $\bar{Z}$ is a blocking set.

The arguments for $\hat{f} \notin \mathrm{CIVP}$ and $\hat{f} \in \mathrm{PR}$ are the same that those used in Theorem 4.4, while $\hat{f} \in \mathrm{Conn}$ can be argued as in Theorem 5.4. □

Similarly, the proof of this last theorem in this section is a mix of the elements of proofs of Theorems 5.4 and 3.1.

Theorem 5.6 *If* $\mathrm{cov_M} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{Conn} \setminus (\mathrm{AC} \cup \mathrm{PR}) \neq \emptyset$.

Proof Take the SZ-functions: f from Lemma 2.5 with $M := X$ from Proposition 5.2, $\sigma \in \mathrm{SZ}$, and $\hat{X}: X \to \mathbb{R}$ such that $\hat{X} \subset B \setminus \bar{Z}$. Let

$$\bar{f} = f \cup \hat{X} \cup \sigma \restriction \big(\mathbb{R} \setminus \mathrm{dom}(f \cup \hat{X})\big).$$

Then, by Proposition 1.4, we have $\bar{f} \in \mathrm{SZ}$. So, the set $D := \mathrm{dom}(\gamma \cap \bar{f})$ has cardinality $< \mathfrak{c}$, where $\gamma \in \mathcal{G}$ is as in part (c) of Proposition 5.2. Moreover, we have $\mathrm{dom}(\bar{Z} \cap \bar{f}) \subset D$. Define

$$\hat{f} := \bar{f} + \chi_D,$$

where χ_D is the characteristic function of D. We claim that $\hat{f}$ is as needed, that is, that $\hat{f} \in \mathrm{SZ} \cap \mathrm{Conn} \cap \setminus (\mathrm{AC} \cup \mathrm{PR})$.

Indeed, $\hat{f} \in \mathrm{SZ}$ by Proposition 1.4 as both $\hat{f} \restriction (\mathbb{R}\setminus D) = \bar{f} \restriction (\mathbb{R}\setminus D)$ and $\hat{f} \restriction D$ are SZ. Clearly, $\hat{f} \notin \mathrm{AC}$, since $\hat{f} \cap \bar{Z} = \emptyset$, while $\bar{Z}$ is a blocking set.

To show $\hat{f} \notin \mathrm{PR}$, fix any perfect $P \subset \mathbb{R}$. By Proposition 2.2, it is enough to show that $\hat{f}$ is unbounded on P. To see this, choose a perfect $C \subset P\setminus X$. Then, by (b) of Lemma 2.5, $\bar{f} \supset f$ is unbounded on $C \subset P$. Therefore $\hat{f}$ is unbounded on P as being the sum of an unbounded function and the other bounded.

The argument for $\hat{f} \in \mathrm{Conn}$ is identical to one used for it in Theorem 5.4. □

6 Easy examples in SZ ∩ AC ∩ CIVP and SZ ∩ AC ∩ PR \ CIVP

The examples of maps in these classes, being additionally *additive*, have been constructed under the same set theoretical assumption earlier: for the class SZ ∩ AC ∩ CIVP in a 1999 paper [3] of K. Banaszewski and T. Natkaniec while for the class SZ ∩ AC ∩ PR \ CIVP in a 2004 paper [25] of T. Natkaniec and H. Rosen. Nevertheless, we have crafted a powerful Lemma 2.5 as the core of all theorems above. In particular, the lemma produces a partial function f such that any extension of f is almost continuous, and $\mathrm{dom}(f)$ avoids an arbitrary F_σ meager subset of $\mathbb{R}$. Along with Lemmas 4.1 and 4.3, the same machinery allows us to easily build the maps in SZ ∩ AC ∩ CIVP and SZ ∩ AC ∩ PR \ CIVP.

Theorem 6.1 *If* $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{CIVP} \neq \emptyset$.

Proof The function $\bar{f}$ used in the proof of Theorem 4.2 is as needed. That is, if

$$\bar{f} := f \cup h \cup \sigma \restriction \big(\mathbb{R}\setminus \mathrm{dom}(f \cup h)\big),$$

where $h \colon \hat{M} \to \mathbb{R}$ is from Lemma 4.1, f is from Lemma 2.5 with $M := \hat{M}$ and $\sigma \in \mathrm{SZ}$, then $\bar{f} \in \mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{CIVP}$.

Indeed, $\bar{f} \in \mathrm{AC}$ by part (c) of Lemma 2.5. Also, by Proposition 1.4, we have $\bar{f} \in \mathrm{SZ}$.

To see that $\bar{f} \in \mathrm{CIVP}$, fix $a < b$ with $\bar{f}(a) \neq \bar{f}(b)$ and a perfect set K between $\bar{f}(a)$ and $\bar{f}(b)$. We need to find a perfect $P \subset (a, b)$ with $\bar{f}[P] \subset K$. But, by part (b) of Lemma 4.1, there exists a perfect $P \subset \hat{M} \cap (a, b)$ such that $h[P] \subset K$. Since $\bar{f} = h$ on P, $\bar{f}[P] = h[P] \subset K$, as needed. □

Theorem 6.2 *If* $\mathrm{cov}_{\mathrm{M}} = \mathfrak{c}$ *holds, then* $\mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP} \neq \emptyset$.

Proof The function $\bar{f}$ used in the proof of Theorem 4.4 is as needed. That is, if

$$\bar{f} := f \cup h \cup \sigma \restriction \big(\mathbb{R}\setminus \mathrm{dom}(f \cup h)\big),$$

where $h \colon \hat{M} \to \mathbb{R}$ is from Lemma 4.3, f is from Lemma 2.5 with $M := \hat{M}$ and $\sigma \in \mathrm{SZ}$, then $\bar{f} \in \mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}$.

Indeed, $\bar{f} \in \mathrm{AC} \cap \mathrm{ES}$ by part (c) of Lemma 2.5. Also, by Proposition 1.4, we have $\bar{f} \in \mathrm{SZ}$.

For $\bar{f} \notin$ CIVP, notice that $\bar{f} \in$ ES implies the existence of $a < b$ for which $\mathfrak{C}$ is between $\bar{f}(a)$ and $\bar{f}(b)$. Thus, it is enough to show, that $\bar{f}[P] \not\subset \mathfrak{C}$ for every perfect $P \subset \mathbb{R}$. So, by way of contradiction, assume that there is a perfect $P \subset \mathbb{R}$ with $\bar{f}[P] \subset \mathfrak{C}$. Then, $P \cap \hat{M} = \emptyset$, since, by part (b) of Lemma 4.3, for every $x \in \hat{M}$ we have $\bar{f}(x) = h(x) \notin \mathfrak{C}$. So, $\bar{f} = f$ on P and, by part (b) of Lemma 2.5, $\bar{f}[P] = f[P]$ is unbounded, contradicting $\hat{f}[P] \subset \mathfrak{C}$.

To see $\bar{f} \in$ PR, fix an $s \in \mathbb{R}$ and let $t = \bar{f}(s)$. By part (c) of Lemma 4.3, there is a perfect set $P \subset \hat{M} \cup \{s\}$ having s as a bilateral limit point and such that $\lim_{x\to s,\, x\in P} h(x) = t$. Since $\lim_{x\to s,\, x\in P} \bar{f}(x) = \lim_{x\to s,\, x\in P} h(x) = t = \bar{f}(s)$, we conclude that $\bar{f} \restriction P$ is continuous at s. □

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