Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: https://www.tandfonline.com/loi/umma20

Sierpiński's Topological Characterization of Q

Krzysztof Chris Ciesielski

To cite this article: Krzysztof Chris Ciesielski (2020) Sierpiński's Topological Characterization of Q , Mathematics Magazine, 93:2, 136-138, DOI: 10.1080/0025570X.2020.1708187

To link to this article: https://doi.org/10.1080/0025570X.2020.1708187

Sierpiński's Topological Characterization of $\mathbb{Q}$

KRZYSZTOF CHRIS CIESIELSKI
West Virginia University
Morgantown, WV 26506-6310
KCies@math.wvu.edu

In a 1920 paper [**5**], Wacław Sierpiński proved the following result characterizing the space $\mathbb{Q}$ of rational numbers considered with the standard topology:

Theorem 1. *Any countable metric space $\langle X, d\rangle$ without isolated points is homeomorphic to $\mathbb{Q}$.*

Its simple and natural form seems to indicate that its proof could be included in an undergraduate topology course as soon as the notion of "homeomorphism" is introduced. This result can help to illuminate the difference between the standard topologies on $\mathbb{R}$ and on $\mathbb{Q}$. According to Theorem 1, $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}^2$ and to $\mathbb{Q}_\ell$ (that is, $\mathbb{Q}$ with the "Sorgenfrey topology," generated by all left closed intervals $[p, q)$). In contrast, their real counterparts $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}_\ell$—obtained by replacing $\mathbb{Q}$ with $\mathbb{R}$ in their respective definitions—are pairwise topologically different. Also, Theorem 1 implies that the Furstenberg topology on the integers $\mathbb{Z}$, which has been used to prove the infinitude of primes (see Aigner [**1**] or Furstenberg [**4**]), is actually homeomorphic to the standard topology on $\mathbb{Q}$.

Nevertheless, so far Theorem 1 could not have been included early in topological education for a simple reason—all of the published proofs are too complicated for such a purpose. The proofs presented by Błaszczyk [**2**] and Eberhart [**3**] are relatively simple, but they are both considerably longer than our proof, and they are not self contained, since they both rely on Cantor's characterization of the linear structure of $\mathbb{Q}$: *Any linearly ordered dense set with neither smallest nor greatest element is order-isomorphic to* $(\mathbb{Q}, \leq)$.

We now prove Theorem 1.

Proof. Let S be the set of all infinite sequences $s = \langle s_1, s_2, \ldots\rangle$ of natural numbers that are eventually zero, that is, such that $0 = s_n = s_{n+1} = \cdots$ for some $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Notice that S is countable, and so is the set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers. Equip S with a topology τ generated by a basis formed by all sets $[t]$, with $t \in \mathbb{N}^{<\omega}$, defined as

$$[t] := \{s \in S : t \subset s\},$$

where "$t \subset s$" means "the sequence s extends t."* To finish the proof it is enough to show that there is a homeomorphism $h : X \to S$, since then there also exists a homeomorphism $H : \mathbb{Q} \to S$, implying that $H^{-1} \circ h : X \to \mathbb{Q}$ is a homeomorphism and proving Theorem 1.

Let $\{x_n : n < \omega\}$ be an enumeration, with no repetitions, of the set X and let

$$D := \{d(x, y) : x, y \in X\}$$

be the set of all distances between the elements in X. Notice that D is countable. Also, for any $r > 0$ with $r \notin D$, the open ball

$$B_d(c, r) := \{x \in X : d(c, x) < r\}$$

Math. Mag. **93** (2020) 136–138. doi:10.1080/0025570X.2020.1708187
MSC: Primary 54B99

*Of course, $\langle S, \tau\rangle$ is actually a subspace of $\mathbb{N}^{\mathbb{N}}$ considered with the product topology.

centered in $c \in X$ and with radius r is also a closed set in X, since it is equal to

$$B_d[c, r] := \{x \in X : d(c, x) \leq r\}.$$

All open balls in X considered below will be with radii *not* in D. So, they will be clopen sets.

In the construction of h we will repeatedly use the following simple fact:

(*) For every $k \in \mathbb{N}$ and nonempty open subset U of X, there exists a sequence

$$S_k(U) := \langle B_i : i \in \mathbb{N} \rangle$$

of pairwise disjoint clopen balls contained in U, each of radius $\leq 2^{-k}$, such that $U = \bigcup_{i \in \mathbb{N}} B_i$. Moreover, we assume that B_0 contains $x_{n(U)}$, where

$$n(U) := \min\{i < \omega : x_i \in U\}.$$

The balls are chosen by induction on $i \in \mathbb{N}$: each B_i is centered at the point $x_{n(U_i)}$, where

$$U_i := U \setminus \bigcup_{j<i} B_j,$$

and has radius $r_i \in (0, 2^{-k}) \setminus D$ small enough so that

$$B_i \subsetneq U \setminus \bigcup_{j<i} B_j.$$

More specifically, each U_i is open, as a difference of open U and a finite union of balls, each of which is a closed set according to our choice of balls of radii not in D. Each nonempty U_{i-1}, including $U_0 = U$, has more than one point, since X has no isolated points. This allows us to choose each r_i small enough so that $U_i = U_{i-1} \setminus B_i$ is nonempty, as long as $U_{i-1} \neq \emptyset$. Finally, each $x_k \in U$ belongs to $\bigcup_{j \leq i} B_j$, according to our rule of choosing the center of each B_i with the smallest possible available index.

Next, we construct the family

$$\{B_s : s \in \mathbb{N}^{<\omega}\}$$

of nonempty clopen sets in X. The construction is by induction on the length of sequences s. Thus, for the sequence $\emptyset$ of length 0 we put $B_\emptyset := X$. Also, if $\mathbb{N}^k$ is the set of all sequences in $\mathbb{N}^{<\omega}$ of length k (possibly 0) and for every $s \in \mathbb{N}^k$ and $i \in \mathbb{N}$ the symbol $s\hat{\ }i$ denotes the sequence s extended by one more term with value i, then we define

$$\langle B_{s\hat{\ }i} : i \in \mathbb{N} \rangle := S_k(B_s).$$

Notice that

$$\text{for every } x \in X \text{ and } k \in \mathbb{N} \text{ there exists a unique } s \in \mathbb{N}^k \text{ so that } x \in B_s. \tag{1}$$

This is justified by an easy inductive argument. For $k = 1$ this holds, since the sets $\{B_{\emptyset\hat{\ }i} : i \in \mathbb{N}\}$ form a partition of $B_\emptyset = X$. Also, if $x \in B_s$ for some $s \in \mathbb{N}^k$, then there is a unique $t \in \mathbb{N}^{k+1}$, which must be of the form $t = s\hat{\ }i$, for which $x \in B_{s\hat{\ }i}$, as the sets $\{B_{s\hat{\ }i} : i \in \mathbb{N}\}$ form a partition of B_s.

Notice that, by (1), for every $x \in X$ there is a unique sequence $s = s_x \in \mathbb{N}^{\mathbb{N}}$ such that

$$x \in \bigcap_{k \in \mathbb{N}} B_{s \restriction k},$$

where $s \restriction k$ is the restriction of s to its first k elements. Define $h\colon X \to \mathbb{N}^{\mathbb{N}}$ by letting $h(x) := s_x$ for every $x \in X$. We claim that this is the desired homeomorphism.

Clearly h is one-to-one. To see that h is onto S, first notice that $h[X] \subset S$. Indeed, by $(*)$, for every $x_j \in X$ we have $h(x_j) \restriction k = 0$ for every $k > n$, where n is such that

$$d(x_i, x_j) > 2^{-n},$$

for every $i < j$. Thus, $h(x_j) \in S$, since $h(x_j)$ is eventually 0. Also, h is onto S, since for every $s \in S$ there exists a $k \in \mathbb{N}$ such that $s_m = 0$ for all $m \geq k$. Let $j = n(B_{s \restriction k})$. Then, by $(*)$, $x_j \in B_{s \restriction m}$ for all $m \geq k$ and so $h(x_j) = s$.

Finally, we need to show that both h and h^{-1} are continuous. Clearly, h is continuous, since for every basic open set $[s]$ in S, $s \in \mathbb{N}^{<\omega}$, we have that

$$h^{-1}([s]) = B_s$$

is open in X. Also, h^{-1} is continuous, since

$$\{B_s : s \in \mathbb{N}^{<\omega}\}$$

is a basis for X and, for every $s \in \mathbb{N}^{<\omega}$,

$$(h^{-1})^{-1}(B_s) = h(B_s) = [s]$$

is open in S.

■

REFERENCES

[1] Aigner, M., Ziegler, G. (2001). *Proofs from THE BOOK*. Heidelberg: Springer-Verlag.

[2] Błaszczyk, A. (forthcoming). A simple proof of Sierpiński's theorem. *Amer. Math. Monthly*. 126(5): 464–466. doi.org/10.1080/00029890.2019.1577103

[3] Eberhart, C. (1977). Some remarks on the irrational and rational numbers. *Amer. Math. Monthly*. 84(1): 32–35. doi.org/10.2307/2318303

[4] Furstenberg, H. (1955). On the infinitude of primes. *Amer. Math. Monthly*. 62(5): 353. doi.org/10.2307/2307043

[5] Sierpiński, W. (1920). Sur une propriété topologique des ensembles dénombrables denses en soi. *Fundam. Math.* 1: 11–16.

Summary. In a 1920 paper, Sierpiński proved the following theorem characterizing the space $\mathbb{Q}$ of rational numbers considered with the standard topology: *Any countable metric space* $\langle X, d\rangle$ *without isolated points is homeomorphic to* $\mathbb{Q}$. In this note, we provide a simple proof of this result, that requires only basic topological background. As such, it can be incorporated into an undergraduate topology curriculum.

KRZYSZTOF CHRIS CIESIELSKI (MR Author ID: 49415) received his Masters and Ph.D. degrees in Pure Mathematics from Warsaw University, Poland, in 1981 and 1985, respectively. He has worked at West Virginia University since 1989. Since 2006, he has been an adjunct professor in the Department of Radiology at the University of Pennsylvania. He is the author of three books and over 140 journal research articles. He is an editor of *Real Analysis Exchange*, *Journal of Applied Analysis*, and *Journal of Mathematical Imaging and Vision*.