



Doubly paradoxical functions of one variable

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ABSTRACT

This paper concerns three kinds of seemingly paradoxical real valued functions of one variable. The first two, defined on \mathbb{R} , are the celebrated continuous nowhere differentiable functions, known as Weierstrass's monsters, and everywhere differentiable nowhere monotone functions—simultaneously smooth and very rugged—to which we will refer as differentiable monsters. The third kind was discovered only recently and consists of differentiable functions f defined on a compact perfect subset X of \mathbb{R} which has derivative equal zero on its entire domain, making it everywhere pointwise contractive, while, counterintuitively, f maps X onto itself. The goal of this note is to show that this pointwise shrinking globally stable map f can be extended to functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$ which are differentiable and Weierstrass's monsters, respectively. Thus, we pack three paradoxical examples into two functions. The construction of f is based on the following variant of Jarnik's Extension Theorem: *For every differentiable function f from a closed $P \subseteq \mathbb{R}$ into \mathbb{R} there exists its differentiable extension $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ such that \hat{f} is nowhere monotone on $\mathbb{R} \setminus P$.*

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1. Background

The number of counterintuitive examples that are known in mathematical analysis is very large, see e.g. book [9]. However, few have as much interesting history as *Weierstrass's monsters*—everywhere continuous nowhere differentiable functions from $[a, b]$ to \mathbb{R} —and *differentiable monsters*—the maps from $[a, b]$ to \mathbb{R} that are everywhere differentiable but monotone on no interval. Shortly, the first published example of Weierstrass's monster was given by K. Weierstrass and appeared in the 1872 paper, see [7] or [22]. At that time, mathematicians commonly believed that a continuous function must have a derivative at a “significant” set of points. Thus, the example was received with disbelief and such functions eventually became known as *Weierstrass's monsters*. One of the most elegant examples of such maps comes from the 1930 paper [21] of van der Waerden. It can be defined, on \mathbb{R} , as

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$$f(x) := \sum_{n=0}^{\infty} 4^n f_n(x), \tag{1}$$

where $f_n(x) := \min_{k \in \mathbb{Z}} |x - \frac{k}{8^n}|$ is the distance from $x \in \mathbb{R}$ to the set $\frac{1}{8^n}\mathbb{Z} = \{\frac{k}{8^n} : k \in \mathbb{Z}\}$. (See [4] or [19, thm. 7.18].) A large number of simple constructions of Weierstrass’s monsters can be also found in [20] or a recent book [10].

The history of differentiable monsters is described in detail in the 1983 paper of A. M. Bruckner [2]. The first construction of such a function was given in 1887 by A. Köpcke [13]. (A gap in [13] was corrected in [14,15].) The most influential study of this subject is the 1915 paper [6] of A. Denjoy. Two relatively simple constructions of differentiable monsters come from the 1970s papers [12,23]. A considerably simpler construction was recently found by the first author [4]. Specifically, a differentiable monster in [4] is defined on \mathbb{R} as

$$f(x) := h(x - t) - h(x), \tag{2}$$

where h a strictly increasing differentiable function from \mathbb{R} onto \mathbb{R} for which $G := \{x \in \mathbb{R} : f'(x) = 0\}$ contains a countable dense set¹ D and t is chosen from a dense G_δ -set $\bigcap_{d \in D} ((-d + G) \cap (d - G))$.

The third paradoxical example we consider was first constructed in the 2016 paper [5] of the first author and J. Jasinski. Since then, the construction was further generalized, in [1], and simplified, see [4]. The example is a differentiable self-homeomorphism f of a compact perfect subset \mathfrak{X} of \mathbb{R} with $f'(x) = 0$ for all $x \in \mathfrak{X}$. Thus, f is shrinking at every $x \in \mathfrak{X}$ and so, one would expect that the diameter of $f[\mathfrak{X}]$ should be smaller than that of \mathfrak{X} , which evidently is not the case. Of course, \mathfrak{X} must have Lebesgue measure zero, since $f' \equiv 0$ implies that $f[\mathfrak{X}]$ must have measure zero, see for example [8, p. 355]. The construction of f from [4] is also simple enough to be described in few lines. Specifically, it can be defined as

$$f := h \circ \sigma \circ h^{-1} \tag{3}$$

from $\mathfrak{X} := h[2^\omega]$ onto itself, where $\sigma : 2^\omega \rightarrow 2^\omega$ is the add-one-and-carry adding machine,² while $h(s) := \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s \upharpoonright n)}$, while $N(s \upharpoonright n)$ is defined as $N(s \upharpoonright n) := \sum_{i < n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n$. Notice, that $\mathfrak{X} = h[2^\omega]$ is a subset of the Cantor ternary set, denoted in what follows as \mathfrak{C} .

To extend f to a differentiable monster, we will use the following result, that was first proved in the 1923 paper [11] of V. Jarník and independently rediscovered in the 1974 paper [17] of G. Petruska and M. Laczko. Its simplified proof, as well as a history of this result, can be found in a recent paper [3] of M. Ciesielska and the first author.

Proposition 1. (Jarník’s Extension Theorem) *Every differentiable function $f : P \rightarrow \mathbb{R}$, where $P \subseteq \mathbb{R}$ is closed, admits a differentiable extension $f : \mathbb{R} \rightarrow \mathbb{R}$.*

The differentiability of $f : P \rightarrow \mathbb{R}$ is understood as the existence of its derivative, that is, a function $f' : P' \rightarrow \mathbb{R}$ where $P' \subseteq P$ is the set of all accumulation points of P and $f'(p) := \lim_{x \rightarrow p, x \in P} \frac{f(x) - f(p)}{x - p}$ for every $p \in P'$.

In our proof we will also use the following well known result.

Proposition 2. (Folklore) *For every closed $K \subseteq \mathbb{R}$, there exists a C^∞ function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(x) = g'(x) = 0$ for all $x \in K$ and $g(x) > 0$ on $K^c = \mathbb{R} \setminus K$.*

¹ Such a map h was first constructed in the 1907 paper [18] of D. Pompeiu. It can be defined as the inverse of a function $g(x) := \sum_{i=1}^{\infty} 2^{-i}(x - q_i)^{1/3}$, where $\{q_i : i \in \mathbb{N}\}$ is an enumeration of rational numbers such that $|q_i| \leq i$ for all $i \in \mathbb{N}$.

² For $s = \langle s_0, s_1, s_2, \dots \rangle \in 2^\omega$ it is defined: $\sigma(s) := \langle 0, 0, 0, \dots \rangle$ when $s_i = 1$ for all $i < \omega$ and, otherwise, $\sigma(s) := \langle 0, 0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots \rangle$, where $s_k = 0$ and $s_i = 1$ for all $i < k$.

Proof. By [16, prop. 2.25], for every $n \in \mathbb{N}$ there is a C^∞ map $g_n: \mathbb{R} \rightarrow [0, 1]$ such that $g_n = 1$ on $\{x \in \mathbb{R}: \text{dist}(x, K) \geq \frac{1}{n}\}$ and $g_n = 0$ on $\{x \in \mathbb{R}: \text{dist}(x, K) \leq \frac{1}{n+2}\}$. Then $g = \sum_{n \in \mathbb{N}} c_n g_n$ is as needed, provided $c_n g_n^{(i)}[\mathbb{R}] \subset [0, 2^{-n}]$ for all $i \leq n$. \square

2. Differentiable monster extending f

The existence of such a function will be deduced from the following version of Jarník's Extension Theorem.

Theorem 3. *For every closed set $P \subseteq \mathbb{R}$ and differentiable $f: P \rightarrow \mathbb{R}$, there exists a differentiable extension $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ of f such that \hat{f} is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in \mathbb{R} , then \hat{f} is monotone on no interval.*

Of course, applying Theorem 3 to our pointwise shrinking globally stable map $f: \mathfrak{X} \rightarrow \mathfrak{X} \subseteq \mathbb{R}$, the resulting extension $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable monster.

Corollary 1. *There exists a differentiable monster $f: \mathbb{R} \rightarrow \mathbb{R}$ extending the mapping $f: \mathfrak{X} \rightarrow \mathfrak{X}$ from (3). \square*

Our proof of Theorem 3 will be based on the following two lemmas, the first of which is a variant of the squeeze theorem.

Lemma 4. *Let $g: \mathbb{R} \rightarrow [0, \infty)$ and $\bar{f}: \mathbb{R} \rightarrow \mathbb{R}$ be differentiable such that $g'(x) = 0$ on the set $[g = 0] := \{x \in \mathbb{R}: g(x) = 0\}$. If $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ is such that $|\hat{f}(x) - \bar{f}(x)| \leq g(x)$ for every $x \in \mathbb{R}$, then \hat{f} is differentiable on $[g = 0]$.*

Proof. It is enough to show that $\lim_{h \rightarrow 0} \frac{\hat{f}(x+h) - \hat{f}(x)}{h} = \bar{f}'(x)$ for every $x \in [g = 0]$. Indeed, if $Q(x, h) = \frac{\hat{f}(x+h) - \hat{f}(x)}{h} - \frac{\bar{f}(x+h) - \bar{f}(x)}{h}$, then $\lim_{h \rightarrow 0} Q(x, h) = 0$ since $0 \leq |Q(x, h)| = \left| \frac{\hat{f}(x+h) - \bar{f}(x+h)}{h} \right| \leq \left| \frac{g(x+h) - g(x)}{h} \right| \xrightarrow{h \rightarrow 0} g'(x) = 0$. Therefore, $\lim_{h \rightarrow 0} \frac{\hat{f}(x+h) - \hat{f}(x)}{h} = \lim_{h \rightarrow 0} Q(x, h) + \lim_{h \rightarrow 0} \frac{\bar{f}(x+h) - \bar{f}(x)}{h} = \bar{f}'(x)$, as needed. \square

In the next lemma, we consider $C([a, b])$ with the sup norm $\|\cdot\|$.

Lemma 5. *For every $\varepsilon > 0$ and continuous function $\bar{f}: [a, b] \rightarrow \mathbb{R}$, there exists a differentiable nowhere monotone $f: [a, b] \rightarrow \mathbb{R}$ such that $\|f - \bar{f}\| < \varepsilon$, $f(a) = \bar{f}(a)$, $f(b) = \bar{f}(b)$, and $f'(a) = f'(b) = 0$.*

Proof. First notice that

(*) There exists a differentiable nowhere monotone $\varphi: [0, 1] \rightarrow [0, 1]$ such that $\varphi(0) = \varphi'(0) = \varphi'(1) = 0$ and $\varphi(1) = 1$.

To see this, take an arbitrary differentiable nowhere monotone function $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ (e.g. the map f from (2)) and notice that the set $[g' = 0] := \{x \in \mathbb{R}: \Phi'(x) = 0\}$ is dense. Since Φ is not constant, there exist $p, q \in [g' = 0]$ such that $\Phi(p) \neq \Phi(q)$. Let $L_1, L_2: \mathbb{R} \rightarrow \mathbb{R}$ be linear functions such that $L_1(0) = p$, $L_1(1) = q$, $L_2(\Phi(p)) = 0$ and $L_2(\Phi(q)) = 1$. Then $\varphi = L_2 \circ \Phi \circ L_1$ satisfies (*).

Let $M := \|\varphi\|$ and notice that $M \geq 1$. By uniform continuity of \bar{f} , there exists a $\delta > 0$ such that for every $x, y \in [a, b]$,

$$|x - y| < \delta \text{ implies } |\bar{f}(x) - \bar{f}(y)| < \frac{\varepsilon}{5M}. \quad (4)$$

Choose $a = x_0 < x_1 < \dots < x_n = b$ with $n \geq 2$ such that $x_{i+1} - x_i < \delta$ for every $i < n$. Then

$$|\bar{f}(x) - \bar{f}(x_i)| < \frac{\varepsilon}{5M} \leq \frac{\varepsilon}{5} \text{ for every } i < n \text{ and } x \in [x_i, x_{i+1}]. \tag{5}$$

Define $y_0 = \bar{f}(a)$, $y_n = \bar{f}(b)$, and choose, by induction, numbers y_1, \dots, y_{n-1} with the property that, for every $i < n$, we have $y_{i+1} \neq y_i$ and $|y_i - \bar{f}(x_i)| < \frac{\varepsilon}{5M} \leq \frac{\varepsilon}{5}$.³ In particular,

$$|y_{i+1} - y_i| \leq |y_{i+1} - \bar{f}(x_{i+1})| + |\bar{f}(x_{i+1}) - \bar{f}(x_i)| + |\bar{f}(x_i) - y_i| < \frac{3\varepsilon}{5M}. \tag{6}$$

For every $i < n$, let $L_i: \mathbb{R} \rightarrow \mathbb{R}$ be a linear function such that $L_i(x_i) = 0$ and $L_i(x_{i+1}) = 1$. Define $f_i: [x_i, x_{i+1}] \rightarrow \mathbb{R}$ via formula

$$f_i(x) := (y_{i+1} - y_i)\varphi(L_i(x)) + y_i \text{ for every } x \in [x_i, x_{i+1}]. \tag{7}$$

Then $f = \bigcup_{i < n} f_i$ is as needed.

Indeed, differentiability of f follows from the differentiability of each map f_i and the fact that they, as well as their derivatives, agree on the end points: for every $i < n$ we have $f_i(x_i) = y_i$, $f_i(x_{i+1}) = y_{i+1}$, and $f'_i(x_i) = f'_i(x_{i+1}) = 0$. It satisfies $\|f - \bar{f}\| < \varepsilon$ since, we see that for every $x \in [x_i, x_{i+1}]$,

$$\begin{aligned} |f(x) - \bar{f}(x)| &\leq |f(x) - y_i| + |y_i - \bar{f}(x_i)| + |\bar{f}(x_i) - \bar{f}(x)| \\ &= |(y_{i+1} - y_i)\varphi(L_i(x))| + |y_i - \bar{f}(x_i)| + |\bar{f}(x_i) - \bar{f}(x)| \\ &< |y_{i+1} - y_i| \|\varphi\| + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \leq \frac{3\varepsilon}{5M}M + \frac{2\varepsilon}{5} = \varepsilon. \end{aligned}$$

This also ensures $f(a) = \bar{f}(a)$, $f(b) = \bar{f}(b)$, and $f'(a) = f'(b) = 0$. \square

Proof of Theorem 3. Let $\bar{f}: \mathbb{R} \rightarrow \mathbb{R}$ be the differentiable extension of f , which exists by Jarník’s theorem, see Proposition 1. Let $K := P \cup \mathbb{Z}$, where \mathbb{Z} is the set of all integers, and let $g: \mathbb{R} \rightarrow [0, \infty)$ be as in Proposition 2. We will define $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ so that

$$|\hat{f}(x) - \bar{f}(x)| \leq g(x) \text{ for every } x \in \mathbb{R}. \tag{8}$$

This, by Lemma 4, will ensure that \hat{f} is differentiable on the set $[g = 0] = K$.

Since (8) demands $\hat{f} = \bar{f}$ on K , we need to define \hat{f} only on $K^c = \mathbb{R} \setminus K$. Let \mathcal{J} be the collection of all connected components of K^c . Notice that each interval in \mathcal{J} is bounded, since $\mathbb{Z} \subseteq K$. So, fix a $J = (a, b) \in \mathcal{J}$. We will define \hat{f} on J as the following function f_J .

Choose an increasing sequence $\langle c_k : k \in \mathbb{Z} \rangle$ such that $\lim_{k \rightarrow -\infty} c_k = a$ and $\lim_{k \rightarrow \infty} c_k = b$. Then $(a, b) = \bigcup_{k \in \mathbb{Z}} [c_k, c_{k+1}]$. For each $k \in \mathbb{Z}$, let $\varepsilon_k := \inf g[c_k, c_{k+1}]$ and notice that $\varepsilon_k > 0$ since g is positive on (a, b) . Using Lemma 5 to $\bar{f} \upharpoonright [c_k, c_{k+1}]$ and ε_k choose differentiable nowhere monotone functions $f_k: [c_k, c_{k+1}] \rightarrow \mathbb{R}$ such that $f_k(c_k) = \bar{f}(c_k)$, $f_k(c_{k+1}) = \bar{f}(c_{k+1})$, $f'_k(c_k) = f'_k(c_{k+1}) = 0$, and

$$|f_k(x) - \bar{f}(x)| < \varepsilon_k \text{ for all } x \in [c_k, c_{k+1}].$$

This ensures that functions f_k and their derivatives agree at the endpoints, so that $f_J = \bigcup_{k \in \mathbb{Z}} f_k$ is a differentiable function. It is clearly nowhere monotone, since so is each f_k . Finally, notice that

³ We can choose $y_i = \bar{f}(x_i)$, unless some consecutive numbers $\bar{f}(x_i)$ are equal.

$$|f_J(x) - \bar{f}(x)| \leq g(x) \text{ for every } x \in J \tag{9}$$

since for every $k \in Z$ and $x \in [c_k, c_{k+1}]$, we have $|f_J(x) - \bar{f}(x)| = |f_k(x) - \bar{f}(x)| < \varepsilon_k \leq g(x)$.

To finish the proof, define \hat{f} on K as $\bar{f} \upharpoonright K$ and on each $J \in \mathcal{J}$ as f_J . We claim, that such \hat{f} is as needed.

Indeed, this definition and (9) ensure that (8) holds so, by Lemma 4, \hat{f} is differentiable on K . It is differentiable on K^c , since it is differentiable on each of its components. Also, since \hat{f} is nowhere monotone on each $J \in \mathcal{J}$, it is also nowhere monotone on every interval I for which $I \cap K$ is nowhere dense in I (as then, the sets $\{x \in I: \hat{f}'(x) < 0\}$ and $\{x \in I: \hat{f}'(x) > 0\}$ are dense in I). In particular, \hat{f} is nowhere monotone on $\mathbb{R} \setminus P$ and, in the case when P is nowhere dense in \mathbb{R} , also on \mathbb{R} . \square

3. Weierstrass’s monster extending f

The main goal of this section is to prove the following theorem, where \mathfrak{C} is the Cantor ternary set.

Theorem 6. *There exist Weierstrass’s monsters $f, h: \mathbb{R} \rightarrow \mathbb{R}$ such that $h(x) = 0$ for all $x \in \mathfrak{C}$ and f extends the function $f: \mathfrak{X} \rightarrow \mathfrak{X}$ from (1).*

The construction of these functions will be based on the following lemma.

Lemma 7. *Let $f: [0, 1] \rightarrow \mathbb{R}$ be non-constant, continuous, with $f(0) = 0 = f(1)$. Let \mathcal{K} be the family of all connected components of $[0, 1] \setminus \mathfrak{C}$. If $h_0: [0, 1] \rightarrow \mathbb{R}$ is defined as*

$$h_0(x) := \begin{cases} (b - a)f\left(\frac{x-a}{b-a}\right) & \text{for } x \in (a, b) \in \mathcal{K}, \\ 0 & \text{for } x \in \mathfrak{C}, \end{cases}$$

then h_0 is continuous but not differentiable at any $x \in \mathfrak{C}$.

Proof. For every $n \in \mathbb{N}$ let $\mathcal{K}_n := \{(a, b) \in \mathcal{K}: b - a = \frac{1}{3^n}\}$ and $h_n: [0, 1] \rightarrow \mathbb{R}$ be defined as

$$h_n(x) := \begin{cases} (b - a)f\left(\frac{x-a}{b-a}\right) & \text{for } x \in (a, b) \in \mathcal{K}_n \\ 0 & \text{otherwise.} \end{cases}$$

Then h_n is continuous, $\|h_n\| = \frac{\|f\|}{3^n}$, and so $h_0 = \sum_{n=1}^\infty h_n$ is continuous by Weierstrass M-test.

Next choose an arbitrary $x \in \mathfrak{C}$. Since f is non-constant, $M = \|f\| > 0$ and $M = f(x_0)$ for some $x_0 \in (0, 1)$. For every $n \in \mathbb{N}$ choose an $x_n \in (a, b)$ with $|h_0(x_n)| = M$. Then, by the construction of \mathfrak{C} , $0 < |x_n - x| < 2(b - a) = \frac{2}{3^n}$ so that $x_n \rightarrow_n x$. Moreover,

$$\left| \frac{h_0(x_n) - h_0(x)}{x_n - x} \right| = \frac{|h_0(x_n)|}{|x_n - x|} = \frac{(b - a)M}{|x_n - x|} \geq \frac{(b - a)M}{2(b - a)} = \frac{M}{2},$$

so the finite derivative $h'_0(x)$ indeed does not exist. \square

Proof of Theorem 6. Let f be the restriction of the Weierstrass’s monster from (1) to $[0, 1]$ and notice that $f(0) = f(1) = 0$. Let h_0 be the function from Lemma 7. Then h_0 is a Weierstrass’s monster with $h_0 \upharpoonright \mathfrak{C} \equiv 0$. It is easy to extend it to a Weierstrass’s monster h on \mathbb{R} .

To construct f , let $\bar{f}: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary differentiable extension of the function $f: \mathfrak{X} \rightarrow \mathfrak{X}$ from (1). Such an extension exists by Jarník’s theorem, see Proposition 1. Define $f := \bar{f} + h$, where h is as above. Clearly f is continuous and, since $\mathfrak{X} \subseteq \mathfrak{C}$ and $h \upharpoonright \mathfrak{C} \equiv 0$, we also have $f \upharpoonright \mathfrak{X} = \bar{f} \upharpoonright \mathfrak{X} + h \upharpoonright \mathfrak{X} = f$. Finally, f cannot be differentiable at any $x \in \mathbb{R}$, since otherwise $h = f - \bar{f}$ would be. \square

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