

Krzysztof Chris Ciesielski, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310 and Department of Radiology, MIPG, University of Pennsylvania, Philadelphia, PA 19104-6021.
email: KCies@math.wvu.edu

LIPSCHITZ RESTRICTIONS OF CONTINUOUS FUNCTIONS AND A SIMPLE CONSTRUCTION OF ULAM-ZAHORSKI C^1 INTERPOLATION

Abstract

We present a simple argument that for every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ its restriction to some perfect set is Lipschitz. We will use this result to provide an elementary proof of the C^1 free interpolation theorem, that for every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ there exists a continuously differentiable function $g: \mathbb{R} \rightarrow \mathbb{R}$ which agrees with f on an uncountable set. The key novelty of our presentation is that no part of it, including the cited results, requires from the reader any prior familiarity with the Lebesgue measure theory.

1 Introduction and background

The main result we like to discuss here is the following 1985 theorem of Agronsky, Bruckner, Laczkovich, and Preiss [1]. It implies that every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ must have some traces of differentiability, even though there exist continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ that are nowhere differentiable (see e.g. [10, 22, 23]) or, even stronger, nowhere approximately and \mathcal{I} -approximately differentiable. In fact, the first coordinate of the classical Peano curve (i.e., $f_1: [0, 1] \rightarrow [0, 1]$, where $f = (f_1, f_2): [0, 1] \rightarrow [0, 1]^2$ is a

Mathematical Reviews subject classification: Primary: 26A24; Secondary: 26B05

Key words: differentiation of partial functions, extension theorems, Whitney extension theorem

Received by the editors June 28, 2017

Communicated by: Paul Humke

continuous surjection constructed by Peano) has these properties, see [6] or [7, Example 4.3.8]. Such a function cannot agree with a C^1 function on a set which is either of second category or of positive Lebesgue measure.

Theorem 1. *For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ there is a continuously differentiable function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that the set $[f = g] = \{x \in \mathbb{R}: f(x) = g(x)\}$ is uncountable. In particular, $[f = g]$ contains a perfect set P and the restriction $f \upharpoonright P$ is continuously differentiable.*

In the statement of Theorem 1 the differentiability of $h = f \upharpoonright P$ is understood as the existence of its derivative, that is, of the function $h': P \rightarrow \mathbb{R}$ defined, for every $p \in P$, as $h'(p) = \lim_{x \rightarrow p, x \in P} \frac{h(x) - h(p)}{x - p}$.

The story behind Theorem 1 spreads over a big part of the 20th century and is described in detail in [2] and [16]. Briefly, around 1940 S. Ulam asked, in Scottish Book, Problem 17.1, see [21], whether every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ agrees with some real analytic function on an uncountable set. Z. Zahorski showed, in his 1948 paper [25], that the answer is no: there exists a C^∞ (i.e., infinitely many times differentiable) function which can agree with every real analytic function on at most finite set of points. At the same paper Zahorski stated a problem, refereed to as Ulam-Zahorski problem: does every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ agrees with some C^∞ (or possibly C^n or D^n) function on some uncountable set? Clearly, Theorem 1 shows that Ulam-Zahorski problem has an affirmative answer for the C^1 class of functions. This is the best possible result in this direction, since A. Olevskii constructed, in his 1994 paper [16], a continuous function which can agree with every C^2 function on at most countable set of points.

The format of our proof of Theorem 1 is relatively straightforward. First we provide a simple argument that for every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ its restriction to some perfect set $P \subset \mathbb{R}$ is Lipschitz.¹ Here the key case, presented in Sec. 2, is when f is monotone. Then we will follow an argument of Morayne [15] to show that there is a perfect $Q \subset P$ for which $f \upharpoonright Q$ satisfies the assumptions of Whitney's C^1 extension theorem [24]. At this point, to make the argument more accessible, we point the reader to a version of Whitney's C^1 extension theorem from [4], whose proof is elementary and simple.

¹Of course this result follows immediately from Theorem 1, as g from Theorem 1 is Lipschitz on any bounded interval. However, we are after a simpler proof of Theorem 1, so using it to argue for our step to prove it is pointless.

2 Lipschitz restrictions of monotone continuous maps

In what follows f will always be a continuous function from \mathbb{R} into \mathbb{R} , Δ will stand for the set $\{\langle x, x \rangle : x \in \mathbb{R}\}$, and $q: \mathbb{R}^2 \setminus \Delta \rightarrow \mathbb{R}$ be the quotient function for f , that is, defined as $q(x, y) = \frac{f(x) - f(y)}{x - y}$. For $Q \subset \mathbb{R}$ we will use the symbol $q \upharpoonright Q^2$ to denote the restriction of q to the set $Q^2 \setminus \Delta$.

Theorem 2. *Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is monotone and continuous on a non-trivial interval $[a, b]$. For every $L > |q(a, b)|$ there exists a closed uncountable set $P \subset [a, b]$ such that $f \upharpoonright P$ is Lipschitz with constant L .*

The difficulty in proving Theorem 2 without measure theoretical tools comes from the fact that there exist strictly increasing continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ which possess finite or infinite derivative at every point, but that the derivative of f is infinite on a dense G_δ -set. The first example of such function was given by Pompeiu in [18]. More recent description of such functions can be found in [20, sec. 9.7] and [5]. These examples show that a perfect set in Theorem 2 should be nowhere dense. Thus we will use a measure theoretical approach, in which the measure theoretical tools will be present only implicitly or, as in case of Fact 5, given together with a simple proof.

We extract the proof of next theorem from the proof, presented in [8], of a Lebesgue theorem that every monotone function $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable almost everywhere.

Our proof of Theorem 2 is based on the following 1932 result of Riesz [19], known as the rising sun lemma. For reader's convenience we include its short proof.

Lemma 3. *If g is a continuous function from a non-trivial interval $[a, b]$ into \mathbb{R} , then the set $U = \{x \in [a, b] : g(x) < g(y) \text{ for some } y \in (x, b]\}$ is open in $[a, b]$ and $g(c) \leq g(d)$ for every open connected component (c, d) of U .*

PROOF. It is clear that U is open in $[a, b]$. To see the other part, let (c, d) be a component of U . By continuity of g , it is enough to prove that $g(p) \leq g(d)$ for every $p \in (c, d)$. Assume by way of contradiction that $g(d) < g(p)$ for some $p \in (c, d)$ and let $x \in [p, b]$ be a point at which $g \upharpoonright [p, b]$ achieves the maximum. Then $g(d) < g(p) \leq g(x)$ and so we must have $x \in [p, d) \subset U$, as otherwise d would belong to U . But $x \in U$ contradicts the fact that $g(x) \geq g(y)$ for every $y \in (x, b]$. \square

Remark 4. In Lemma 3 we also have $g(c) \geq g(d)$, since $c \in [a, b] \setminus U$. But we do not actually need this fact.

For an interval I let $\ell(I)$ be its length. We need the following simple well-known observations.

Fact 5. *Let $a < b$ and \mathcal{J} be a family of open intervals with $\bigcup \mathcal{J} \subset (a, b)$.*

(i) *If $[\alpha, \beta] \subset \bigcup \mathcal{J}$, then $\sum_{I \in \mathcal{J}} \ell(I) > \beta - \alpha$.*

(ii) *If the intervals in \mathcal{J} are pairwise disjoint, then $\sum_{I \in \mathcal{J}} \ell(I) \leq b - a$.*

PROOF. (i) By compactness of $[\alpha, \beta]$ we can assume that \mathcal{J} is finite, say of size n . Then (i) follows by an easy induction on n : if $(c, d) = J \in \mathcal{J}$ contains β , then either $c \leq \alpha$, in which case (i) is obvious, or $\alpha < c$ and, by induction, $\sum_{I \in \mathcal{J}} \ell(I) = \ell(J) + \sum_{I \in \mathcal{J} \setminus \{J\}} \ell(I) > \ell([c, \beta]) + \ell([\alpha, c]) = \beta - \alpha$.

(ii) Once again, it is enough to show (ii) for finite \mathcal{J} , say of size n , by induction. Then, there is $(c, d) = J \in \mathcal{J}$ to the right of any $I \in \mathcal{J} \setminus \{J\}$. Hence, by induction, $\sum_{I \in \mathcal{J}} \ell(I) = \ell(J) + \sum_{I \in \mathcal{J} \setminus \{J\}} \ell(I) \leq (b - c) + (c - a) = b - a$. \square

PROOF OF THEOREM 2. If there exists a nontrivial interval $[c, d] \subset [a, b]$ on which f is constant, then clearly $P = [c, d]$ is as needed. So, we can assume that f is strictly monotone on $[a, b]$. Also, replacing f with $-f$, if necessary, we can also assume that f is strictly increasing.

Fix $L > |q(a, b)| = \frac{f(b) - f(a)}{b - a}$ and define $g: \mathbb{R} \rightarrow \mathbb{R}$ as $g(t) = f(t) - Lt$. Then $g(a) = f(a) - La > f(b) - Lb = g(b)$. Let $m = \sup\{g(x) : x \in [a, b]\}$ and $\bar{a} = \sup\{x \in [a, b] : g(x) = m\}$. Then $f(\bar{a}) - L\bar{a} = g(\bar{a}) \geq g(a) > g(b) = f(b) - Lb$, so $a \leq \bar{a} < b$ and we still have $L > |q(\bar{a}, b)| = \frac{f(b) - f(\bar{a})}{b - \bar{a}}$. Moreover, \bar{a} does not belong to the set

$$U = \{x \in [\bar{a}, b] : g(y) > g(x) \text{ for some } y \in (x, b]\}$$

from Lemma 3 applied to g on $[\bar{a}, b]$. In particular, U is open in \mathbb{R} and the family \mathcal{J} of all connected components of U contains only open intervals (c, d) for which, by Lemma 3, $g(c) \leq g(d)$.

The set $P = [\bar{a}, b] \setminus U \subset [a, b]$ is closed and for any $x < y$ in P we have $f(y) - Ly = g(y) \leq g(x) = f(x) - Lx$, that is, $|f(y) - f(x)| = f(y) - f(x) \leq Ly - Lx = L|y - x|$. In particular, f is Lipschitz on P with constant L . It is enough to show that P is uncountable.

To see this notice that for every $J = (c, d) \in \mathcal{J}$ we have $f(d) - Ld = g(d) \geq g(c) = f(c) - Lc$, that is, $\ell(f[J]) = f(d) - f(c) \geq L(d - c) = L\ell(J)$. Since the intervals in the family $\mathcal{J}^* = \{f[J] : J \in \mathcal{J}\}$ are pairwise disjoint and contained in the interval $(f(\bar{a}), f(b))$, by Fact 5(ii) we have $\sum_{J^* \in \mathcal{J}^*} \ell(J^*) \leq f(b) - f(\bar{a})$. So, $\sum_{J \in \mathcal{J}} \ell(J) \leq \frac{1}{L} \sum_{J \in \mathcal{J}} \ell(f[J]) = \frac{1}{L} \sum_{J^* \in \mathcal{J}^*} \ell(J^*) \leq \frac{f(b) - f(\bar{a})}{L} < b - \bar{a}$. Thus, by Fact 5(i), $P = [\bar{a}, b] \setminus U = [\bar{a}, b] \setminus \bigcup \mathcal{J} \neq \emptyset$. However, we need more,

that P cannot be contained in any countable set, say $\{x_n : n \in \mathbb{N}\}$. To see this, fix $\delta > 0$ such that $\frac{f(b)-f(\bar{a})}{L} + \delta < b - \bar{a}$, for every $n \in \mathbb{N}$ choose an interval $(c_n, d_n) \ni x_n$ of length $2^{-n}\delta$, and put $\hat{\mathcal{J}} = \mathcal{J} \cup \{(c_n, d_n) : n < \omega\}$. Then

$$\sum_{J \in \hat{\mathcal{J}}} \ell(J) = \sum_{J \in \mathcal{J}} \ell(J) + \sum_{n \in \mathbb{N}} \ell((c_n, d_n)) \leq \frac{f(b) - f(\bar{a})}{L} + \delta < \beta - \alpha$$

so, by Fact 5(i), $U \cup \bigcup_{n \in \mathbb{N}} (c_n, d_n) \supset U \cup \{x_n : n \in \mathbb{N}\}$ does not contain $[\bar{a}, b]$. In other words, $P = [\bar{a}, b] \setminus U$ is uncountable, as needed. \square

Remark 6. A presented proof of Theorem 2 actually gives a stronger result, that the set $[a, b] \setminus P$ can have arbitrary small Lebesgue measure.

3 Perfect set on which the difference quotient map is uniformly continuous

The next proposition is a version of a theorem of Morayne [15], which implies that the conclusion of Proposition 7 holds when f , defined on a perfect subset of \mathbb{R} , is Lipschitz (i.e., the quotient map for such f has bounded range). The key innovation in Proposition 7 is that we prove this result without assuming that f , or some restriction of it, is Lipschitz.

Proposition 7. *For every continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ there exists a perfect set $Q \subset \mathbb{R}$ such that the quotient map $q \upharpoonright Q^2$ is bounded and uniformly continuous.*

PROOF. If f is monotone on some non-trivial interval $[a, b]$, then, by Theorem 2, there exists a perfect set $P \subset \mathbb{R}$ such that $f \upharpoonright P$ is Lipschitz. Thus, by Morayne's theorem applied to $f \upharpoonright P$, there exists a perfect $Q \subset P$ for which the quotient map q is as needed. On the other hand, if f is monotone on no non-trivial interval, then, by a 1953 theorem of Padmavally [17] (compare also [14, 13, 9]) there exists a perfect set $Q \subset \mathbb{R}$ on which f is constant. Of course, the quotient map on such Q is as desired. \square

4 The main result

The following theorem is a restatement of Theorem 1 in a slightly different language.

Theorem 8. *For every continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ there exists a perfect set $Q \subset \mathbb{R}$ such that $f \upharpoonright Q$ can be extended to C^1 function $F : \mathbb{R} \rightarrow \mathbb{R}$.*

Let $Q \subset \mathbb{R}$ be as Proposition 7. It is well known, see e.g. [12], that uniform continuity of $q \upharpoonright Q^2$ implies that the assumptions of the Whitney's C^1 extension theorem (see [24]) are satisfied, that is, $f \upharpoonright Q$ has a desired C^1 extension $F: \mathbb{R} \rightarrow \mathbb{R}$. The problem with the citation [12], and many other papers containing needed extension result, is that the proofs presented there can hardly be considered simple. Thus, we like conclude the extendability of $f \upharpoonright Q$, having uniformly continuous $q \upharpoonright Q^2$, to C^1 extension $F: \mathbb{R} \rightarrow \mathbb{R}$ from the following recent result of Ciesielska and Ciesielski [4] which has simple elementary proof.

For a bounded open interval J let I_J be the closed middle third of J and for a perfect set $Q \subset \mathbb{R}$ let

$$\hat{Q} = Q \cup \bigcup \{I_J : J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}.$$

Proposition 9. [4] *Let $f: Q \rightarrow \mathbb{R}$, where Q is a perfect subset of \mathbb{R} , and put $\hat{f} = \bar{f} \upharpoonright \hat{Q}$, where $\bar{f}: \mathbb{R} \rightarrow \mathbb{R}$ is a linear interpolation of $f \upharpoonright Q$. If $f \upharpoonright Q$ is differentiable, then there exists a differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$ of \hat{f} . Moreover, F is C^1 if, and only if, \hat{f} is continuously differentiable.*

PROOF OF THEOREM 8. If $Q \subset \mathbb{R}$ is from Proposition 7, then $q \upharpoonright Q^2$, defined on $Q^2 \setminus \Delta$, can be extended to uniformly continuous \bar{q} on Q^2 and $f: Q \rightarrow \mathbb{R}$ is continuously differentiable with $(f \upharpoonright Q)'(x) = \bar{q}(x, x)$ for every $x \in Q$. By Proposition 9, \hat{f} is differentiable (as a restriction of differentiable F). In particular, $\hat{f}'(x) = F'(x) = (f \upharpoonright Q)'(x)$ for every $x \in Q$ and $\hat{f}'(x) = \bar{q}(c, d)$ whenever $x \in I_J$, where $J = (c, d)$ is a bounded connected component of $\mathbb{R} \setminus Q$.

By Proposition 9, we need to show that \hat{f}' is continuous. Clearly \hat{f}' is continuous on $\hat{Q} \setminus Q$, as it is locally constant on this set. So, let $x \in Q$ and let $\varepsilon > 0$. We need to find an open U containing x such that $|\hat{f}'(x) - \hat{f}'(y)| < \varepsilon$ whenever $y \in \hat{Q} \cap U$. Since \bar{q} is continuous, there exists an open $V \in \mathbb{R}^2$ containing $\langle x, x \rangle$ such that $|\hat{f}'(x) - \bar{q}(y, z)| = |\bar{q}(x, x) - \bar{q}(y, z)| < \varepsilon$ whenever $\langle y, z \rangle \in Q^2 \cap V$. Let U_0 be open interval containing x such that $U_0^2 \subset V$ and let $U \subset U_0$ be an open set containing x such that: if $U \cap I_J \neq \emptyset$ for some bounded connected component $J = (c, d)$ of $\mathbb{R} \setminus Q$, then $c, d \in U_0$. We claim that U is as needed. Indeed, let $y \in \hat{Q} \cap U$. If $y \in Q$, then $\langle y, y \rangle \in U^2 \subset V$ and $|\hat{f}'(x) - \hat{f}'(y)| = |\bar{q}(x, x) - \bar{q}(y, y)| < \varepsilon$. Also, if $y \in I_J$ for some bounded connected component $J = (c, d)$ of $\mathbb{R} \setminus Q$, then $\langle c, d \rangle \in U_0^2 \subset V$ and, once again, $|\hat{f}'(x) - \hat{f}'(y)| = |\bar{q}(x, x) - \bar{q}(c, d)| < \varepsilon$. \square

References

- [1] S. Agronsky, A.M. Bruckner, M. Laczkovich, and D. Preiss, *Convexity conditions and intersections with smooth functions*, Trans. Amer. Math. Soc., **289** (1985), 659–677.
- [2] J.B. Brown, *Restriction theorems in real analysis*, Real Anal. Exchange, **20(1)** (1994/1995), 510–526.
- [3] J.B. Brown, *Differentiable restrictions of real functions*, Proc. Amer. Math. Soc., **108(2)** (1990), 391–398.
- [4] M. Ciesielska and K.C. Ciesielski, *Differentiable extension theorem; a lost proof of V. Jarník*, J. Math. Anal. Appl., **454(2)** (2017), 883–890. <http://dx.doi.org/10.1016/j.jmaa.2017.05.032>.
- [5] K.C. Ciesielski, *Monsters in calculus*, Amer. Math. Monthly, to appear; www.math.wvu.edu/~kcies/prepF/131.DifferentiableMonster.pdf.
- [6] K. Ciesielski and L. Larson, *The Peano curve and I-approximate differentiability*, Real Anal. Exchange, **17(2)** (1991/1992), 608–621.
- [7] K. Ciesielski, L. Larson, and K. Ostaszewski, *I-density continuous functions*, Mem. Amer. Math. Soc., **107 (515)** (1994).
- [8] C.-A. Faure, *The Lebesgue differentiation theorem via the rising sun lemma*, Real Anal. Exchange, **29(2)** (2003), 947–951.
- [9] K.M. Garg, *On level sets of a continuous nowhere monotone function*, Fund. Math., **52** (1963), 60–68.
- [10] M. Jarnicki and P. Pflug, *Continuous Nowhere Differentiable Functions*, Springer Monographs in Mathematics, New York, 2015.
- [11] Y. Katznelson and K. Stromberg, *Everywhere differentiable, nowhere monotone, functions*, Amer. Math. Monthly, **81(4)** (1974), 349–354.
- [12] M. Koc and L. Zajíček, *A joint generalization of Whitney's C^1 extension theorem and Aversa–Laczkovich–Preiss' extension theorem*, J. Math. Anal. Appl., **388** (2012), 1027–1039.
- [13] S. Marcus, *Sur les fonctions continues qui ne sont monotones en aucun intervalle*, Rev. Math. Pures Appl., **3** (1958), 101–105.
- [14] S. Minakshisundaram, *On the roots of a continuous non-differentiable function*, J. Indian M. Soc., **4** (1940), 31–33.

- [15] M. Morayne, *On continuity of symmetric restrictions of Borel functions*, Proc. Amer. Math. Soc., **93** (1985), 440–442.
- [16] A. Olevskiĭ, *Ulam-Zahorski problem on free interpolation by smooth functions*, Trans. Amer. Math. Soc., **342(2)** (1994), 713–727.
- [17] K. Padmavally, *On the roots of equation $f(x) = \xi$ where $f(x)$ is real and continuous in (a, b) , but monotonic in no subinterval of (a, b)* , Proc. Amer. Math. Soc., **4** (1953), 839–841.
- [18] D. Pompeiu, *Sur les fonctions dérivées*, Math. Ann., **63(3)** (1907), 326–332.
- [19] F. Riesz, *Sur un Théorème de Maximum de Mm. Hardy et Littlewood*, J. Lond. Math. Soc., **7(1)** (1932), 10–13.
- [20] B. S. Thomson, J. B. Bruckner, and A. M. Bruckner, *Elementary Real Analysis*, 2008.
- [21] S. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.
- [22] B. L. van der Waerden, *Ein einfaches Beispiel einer nicht-differenzierbare Stetige Funktion*, Math. Z., **32** (1930), 474–475.
- [23] K. Weierstrass, *Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen*, Gelesen Akad. Wiss. 18 Juli 1872; English translation: On continuous functions of a real argument that do not possess a well-defined derivative for any value of their argument, in: G.A. Edgar, *Classics on Fractals*, Addison-Wesley Publishing Company, 1993, 3–9.
- [24] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., **36** (1934), 63–89.
- [25] Z. Zahorski, *Sur l'ensemble des points singulière d'une fonction d'une variable réelle admettant des dérivées des tous orders*, Fund. Math., **34** (1947), 183–245.