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Differentiable extension theorem: A lost proof of V. Jarník

Monika Ciesielska [a], Krzysztof Chris Ciesielski [b,c]

[a] *Global Epidemiology & Evidence-based Statistics Global Branded R&D Teva Pharmaceuticals, Malvern, PA 19355, United States*
[b] *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, United States*
[c] *Department of Radiology, MIPG, University of Pennsylvania, Philadelphia, PA 19104-6021, United States*

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ABSTRACT

We present a beautiful but relatively unknown theorem that every differentiable function $f: P \to \mathbb{R}$, with $P \subset \mathbb{R}$ being closed, admits differentiable extension $F: \mathbb{R} \to \mathbb{R}$. We present an elementary proof of this result based on a construction sketched in a hard-to-access 1923 paper [4] of V. Jarník. Using this construction, we also obtain an elegant version of Whitney extension theorem characterizing when such an f admits continuously differentiable extension.

1. Introduction: Continuous and differentiable extension theorems

For a function $f: P \to \mathbb{R}$ with $P \subset \mathbb{R}$ let $\tilde{f}$ be its extension to the set $\tilde{P} = (-\infty, \inf(P) - 1] \cup P \cup [\sup(P) + 1, \infty)$ such that $\tilde{f} = 0$ on the (possibly empty) set $\tilde{P} \setminus P$. Also, if P is closed in $\mathbb{R}$, let $\bar{f}: \mathbb{R} \to \mathbb{R}$ be the *linear interpolation* of $\tilde{f}$, that is, the extension of $\tilde{f}$ such that $\bar{f}$ is linear on $[a, b]$ for every connected component (a, b) of $\mathbb{R} \setminus \tilde{P}$. (See Fig. 1.) If f is continuous, then so is its linear interpolation $\bar{f}$: its continuity

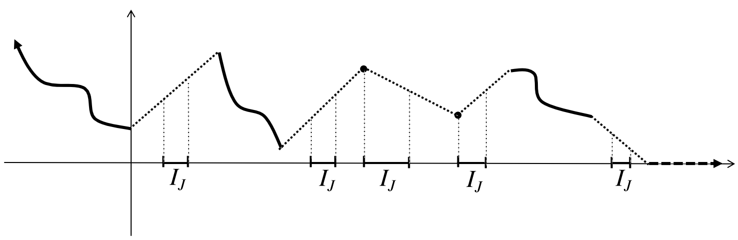

Fig. 1. A map f, represented by thick curves, is extended to $\tilde{f}$ via thick dashed ray and to the linear interpolation $\bar{f}$ by dotted segments. The intervals I_J (see Definition 1) marked on the x-axis are added to the domain $\tilde{P}$ of $\tilde{f}$ to obtain $\hat{P}$.

E-mail address: KCies@math.wvu.edu (K.C. Ciesielski).

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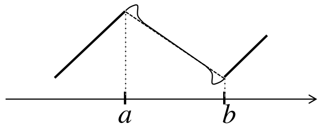

Fig. 2. A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus P$. The thick segments represent the parts of the graph of f.

at every $x \in \mathbb{R} \setminus P$ is obvious, while its continuity at points $x \in P$ follows from the fact that for every component (a, b) of $\mathbb{R} \setminus \tilde{P}$

$$\bar{f}(y) \text{ is between } f(a) \text{ and } f(b) \text{ for every } y \in (a, b), \tag{1}$$

so that $|\bar{f}(x) - \bar{f}(y)| \le \max\{|f(x) - f(a)|, |f(x) - f(b)|\}$.

We just proved the following instance of Tietze extension theorem:

Continuous Extension Theorem. *Every continuous map $f : P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is closed, admits continuous extension $F : \mathbb{R} \to \mathbb{R}$.*

Did you know, that this result has the following differentiable analog?

Differentiable Extension Theorem. *Every differentiable map $f : P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is closed, admits differentiable extension $F : \mathbb{R} \to \mathbb{R}$.*

Here differentiability of $f : P \to \mathbb{R}$ is understood as existence of its derivative: the function $f' : P' \to \mathbb{R}$ where $P' \subset P$ is the set of all accumulation points of P and $f'(p) = \lim_{x \to p,\ x \in P} \frac{f(x)-f(p)}{x-p}$ for every $p \in P'$.

Differentiable Extension Theorem, to which we will refer as DE Theorem, was first proved in a 1923 paper [4] by a prominent Czech mathematician Vojtěch Jarník (1897–1970), in case when $P \subset \mathbb{R}$ is a compact *perfect set*, that is, when $P' = P$. It also appeared in print in several more recent papers with quite involved proofs, as we detail in Section 4.1. Nevertheless, the theorem and the proof we present below remain almost completely unknown, even among experts.

As we can see in Fig. 1, the linear interpolation $\bar{f}$ of a differentiable f need not to be differentiable. Nevertheless, $\bar{f}$ has actually very good differentiability properties, best expressed in terms of the unilateral derivatives $D^- \bar{f}(x) = \lim_{y \to x^-} \frac{\bar{f}(x) - \bar{}(y)}{x-y}$ and $D^+ \bar{f}(x) = \lim_{y \to x^+} \frac{\bar{f}(x) - \bar{}(y)}{x-y}$:

Proposition 1. *If $f : P \to \mathbb{R}$ is differentiable map, where $P \subset \mathbb{R}$ is closed, then the unilateral derivatives of its linear interpolation $\bar{f}$ exist and are finite everywhere. In particular, $\bar{f}$ is differentiable at all points $x \in \mathbb{R}$ that do not belonging to the set E_P of all end-points of connected components of $\mathbb{R} \setminus \tilde{P}$.*

Proof. Clearly $D^+ \bar{f}(x)$ and $D^- \bar{f}(x)$ exist for every $x \in \mathbb{R} \setminus P'$. They also exist for every $x \in P'$, since for every component $I = (a, b)$ of $\mathbb{R} \setminus \tilde{P}$ with $x \notin [a, b]$,

$$\frac{|\bar{f}(y) - \bar{f}(x)|}{y - x} \text{ is between } \frac{|f(a) - f(x)|}{a - x} \text{ and } \frac{|f(b) - f(x)|}{b - x} \text{ for every } y \in (a, b), \tag{2}$$

so that $\left| f'(x) - \frac{|\bar{f}(y) - \bar{f}(x)|}{y-x} \right| \le \max\left\{ \left| f'(x) - \frac{|f(a)-f(x)|}{a-x} \right|, \left| f'(x) - \frac{|f(b)-f(x)|}{b-x} \right| \right\}$.

The additional remark holds, since $D^- \bar{f}(x) = D^- \tilde{f}(x) = D^+ \tilde{f}(x) = D^+ \bar{f}(x)$ for every $x \in \tilde{P} \setminus E_P$. $\square$

Proposition 1 suggests that F from DE Theorem can be constructed by making small adjustment of $\bar{f}$, that is, defining F as $\bar{f} + g$ for some small adjustor map $g : \mathbb{R} \to \mathbb{R}$ such that $g = 0$ on $\tilde{P}$, see Fig. 2. Indeed, this is precisely the way we will proceed. For function g to have a chance to work properly, it must correct differentiability problems of $\bar{f}$ at the endpoints of the components of $\mathbb{R} \setminus \tilde{P}$. For this, it should have the following property for every component (a, b) of $\mathbb{R} \setminus \tilde{P}$:

($\tilde{*}$) $g \restriction [a, b]$ is C^1, $D^+g(a) = \tilde{f}'(a) - \frac{\tilde{f}(b)-\tilde{f}(a)}{b-a}$, and $D^-g(b) = \tilde{f}'(b) - \frac{\tilde{f}(b)-\tilde{f}(a)}{b-a}$.

This ensures that $D^+F(a) = \tilde{f}'(a)$ and $D^-F(b) = \tilde{f}'(b)$. The main difficulty in constructing an adjustor g that works properly is in ensuring that, after the adjustment, the derivatives D^+F and D^-F still exist on $\tilde{P}$. Also, a problem with ($\tilde{*}$) is that it is not well defined when either a or b is an isolated point in $\tilde{P}$.

We first address this second problem, since it has an easy solution: we extend $\tilde{P}$ to a perfect set $\hat{P} \subset \mathbb{R}$ for which $\bar{f} \restriction \hat{P}$ is differentiable and notice that it is enough to find a differentiable extension F of $\bar{f} \restriction \hat{P}$. This way we will avoid the problem of isolated points a or b in ($\tilde{*}$). We will also obtain, as an additional benefit, a new way to characterize functions f that admit *continuously* differentiable extensions $F : \mathbb{R} \to \mathbb{R}$, an alternative formulation of the Whitney's extension theorem [10], see Section 3.

Definition 1. Let $\mathcal{J}$ be the family of all components of $\mathbb{R} \setminus \tilde{P}$. For each $J = (a, a + h) \in \mathcal{J}$ put $I_J = [a + \frac{1}{3}h, a + \frac{2}{3}h]$ when $a \in P'$ and let $I_J = [a, a + \frac{h}{2}]$, the left half of J, otherwise.[1] Then we define $\hat{P} = \tilde{P} \cup \bigcup_{J\in\mathcal{J}} I_J$ and $\hat{f} = \bar{f} \restriction \hat{P}$. We will refer to $\hat{f}$ as *the canonical extension of* f. See Fig. 1.

Notice that $\hat{P}$ is perfect and $E_P \subset E_{\hat{P}}$. Thus, the unilateral differentiability of $\bar{f}$, Proposition 1, implies immediately the following result.

Proposition 2. *If P is a closed subset of $\mathbb{R}$ and $f : P \to \mathbb{R}$ is differentiable, then $\hat{P}$ is a perfect subset of $\mathbb{R}$ unbounded from both sides and $\hat{f} : \hat{P} \to \mathbb{R}$ is differentiable. In particular, in order to prove DE Theorem it is enough to show that it holds for perfect sets $P \subset \mathbb{R}$ unbounded from both sides.*

2. Proof of DE Theorem

Let f be a differentiable function from a closed set $P \subset \mathbb{R}$ into $\mathbb{R}$. We need to show that it admits differentiable extension $F : \mathbb{R} \to \mathbb{R}$. By Proposition 2 we can assume, without loss of generality, that P is perfect and unbounded from both sides.

Let $\bar{f} : \mathbb{R} \to \mathbb{R}$ be the linear interpolation of f. Let $\kappa \le \omega$ be the cardinality of the family $\mathcal{J}$ of all connected components of $\mathbb{R} \setminus P$ and let $\{(a_i, b_i) : 1 \le i \le \kappa\}$ be an enumeration of $\mathcal{J}$. For every $1 \le i \le \kappa$ define $\ell_i = \min\{1, b_i - a_i\}$ and let $\varepsilon_i \in (0, 3^{-i}\ell_i)$ be such that

(a) $\left| f'(a_i) - \frac{f(x)-f(a_i)}{x-a_i} \right| < 3^{-i}$ for every $x \in P \cap [a_i - \varepsilon_i, a_i)$;
(b) $\left| f'(b_i) - \frac{f(x)-f(b_i)}{x-b_i} \right| < 3^{-i}$ for every $x \in P \cap (b_i, b_i + \varepsilon_i]$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a map such that $g = 0$ on P and, for every $1 \le i \le \kappa$,

($*$) $g \restriction [a_i, b_i]$ is C^1, $D^+g(a_i) = f'(a_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i}$, $D^-g(b_i) = f'(b_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i}$;
(c) $g = 0$ on $[a_i + \varepsilon_i^2, b_i - \varepsilon_i^2]$ and $|g(x)| \le \varepsilon_i^2$ for $x \in [a_i, b_i]$;
(d) $|g(x)| \le |g'(a_i)(x - a_i)|$ for $x \in [a_i, a_i + \varepsilon_i^2]$; and
(e) $|g(x)| \le |g'(b_i)(x - b_i)|$ for $x \in [b_i - \varepsilon_i^2, b_i]$.

[1] This definition of set I_J is chosen for our version of Whitney extension theorem from Section 3. The proof of DE Theorem will also work if we define $I_J = \emptyset$ for $J = (a, a + h) \in \mathcal{J}$ with $a \in P'$.

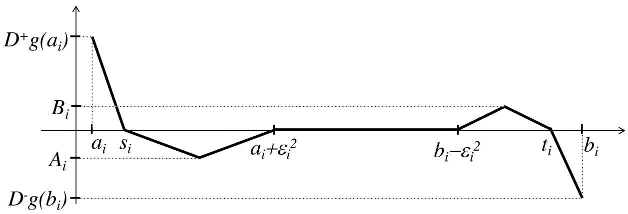

Fig. 3. A format of a map h_i.

Such g can be defined on each $[a_i, b_i]$ as $g(x) = \int_{a_i}^{x} h_i(r)\, dr$, where $h_i\colon [a_i, b_i] \to \mathbb{R}$, depicted in Fig. 3, is such that $h_i = 0$ on $[a_i + \varepsilon_i^2, b_i - \varepsilon_i^2]$,

- there is an $s_i \in (a_i, a_i+\varepsilon_i^2)$ such that h_i is linear on $[a_i, s_i]$ with $h_i(s_i) = 0$ and $h_i(a_i) = f'(a_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i}$ while $\int_{a_i}^{s_i} |h_i(r)|\, dr = \frac{1}{2}|h_i(a_i)|(s_i - a_i) < \varepsilon_i^2$; on $[s_i, a_i+\varepsilon_i^2]$ it is defined as $h_i(x) = A_i \mathrm{dist}(x, \{s_i, a_i+\varepsilon_i^2\})$, where constant A_i is chosen so that $\int_{a_i}^{a_i+\varepsilon_i^2} h_i(r)\, dr = 0$;
- there is a $t_i \in (b_i - \varepsilon_i^2, b_i)$ such that h_i is linear on $[t_i, b_i]$ with $h_i(t_i) = 0$ and $h_i(b_i) = f'(b_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i}$ while $\int_{t_i}^{b_i} |h_i(r)|\, dr = \frac{1}{2}|h_i(b_i)|(b_i - t_i) < \varepsilon_i^2$; on $[b_i - \varepsilon_i^2, t_i]$ it is defined as $h_i(x) = B_i \mathrm{dist}(x, \{b_i - \varepsilon_i^2, t_i\})$, where constant B_i is chosen so that $\int_{b_i-\varepsilon_i^2}^{b_i} h_i(r)\, dr = 0$.

We claim, that if g satisfies all these requirements, then $F = \bar{f} + g$ is a differentiable extension of f. To see this, fix an $x \in \mathbb{R}$. We need to show that $F'(x)$ exists.

For $x \in \mathbb{R} \setminus P$ this follows from the first part of $(*)$ and differentiability of $\bar{f}$ on $\mathbb{R} \setminus P$. So, in what follows we assume that $x \in P$. Next notice that it is enough to show that the unilateral derivatives $D^+F(x)$ and $D^-F(x)$ exist. Indeed, if they exist, than they are equal: for $x \notin \bigcup_{1 \le i \le \kappa}\{a_i, b_i\}$ this is ensured by the fact that $D^+F(x) = D^+f(x) = D^-f(x) = D^-F(x)$, while for $x \in \bigcup_{1 \le i \le \kappa}\{a_i, b_i\}$ by the condition $(*)$ imposed on g.

Since the cases are symmetric, we will show only the existence of $D^+F(x)$. Clearly, it exists when x is in the set $E = \{a_i\colon 1 \le i \le \kappa\}$. So, assume that $x \notin E$ and fix an $\varepsilon > 0$. Then $F(x) = f(x)$ and it is enough to find a $\delta > 0$ such that

$$\left| f'(x) - \frac{F(y) - f(x)}{y - x} \right| < 5\varepsilon \text{ whenever } y \in (x, x + \delta). \tag{3}$$

For this, pick an $m \in \mathbb{N}$ such that $3^{-m} < \varepsilon$ and choose $\delta > 0$ such that $(x, x+\delta)$ is disjoint with $\bigcup_{i<m}[a_i, b_i]$ and $\left|f'(x) - \frac{\bar{f}(y)-\bar{f}(x)}{y-x}\right| < \varepsilon$ whenever $0 < |y-x| < \delta$. This choice is possible by Proposition 1. We claim that (3) holds for such δ. Since $\left|f'(x) - \frac{F(y)-f(x)}{y-x}\right| = \left|f'(x) - \frac{\bar{f}(y)+g(y)-\bar{f}(x)}{y-x}\right| \le \left|f'(x) - \frac{\bar{f}(y)-\bar{f}(x)}{y-x}\right| + \left|\frac{g(y)}{y-x}\right| < \varepsilon + \left|\frac{g(y)}{y-x}\right|$, it is enough to show that $\left|\frac{g(y)}{y-x}\right| < 4\varepsilon$.

If $y \notin \bigcup_{1 \le i \le \kappa}[a_i, a_i+\varepsilon_i^2] \cup [b_i - \varepsilon_i^2, b_i]$, then $\left|\frac{g(y)}{y-x}\right| < 4\varepsilon$ holds, since then $g(y) = 0$. So, we can assume that there is an $1 \le i \le \kappa$ such that $y \in [a_i, a_i + \varepsilon_i^2] \cup [b_i - \varepsilon_i^2, b_i]$. This implies that $i \ge m$ and $3^{-i} \le 3^{-m} < \varepsilon$.

If $y \in [b_i - \varepsilon_i^2, b_i]$, then $y - x \ge b_i - \varepsilon_i^2 - a_i \ge \ell_i - \varepsilon_i^2 \ge \varepsilon_i$ and $\left|\frac{g(y)}{y-x}\right| < \frac{\varepsilon_i^2}{\varepsilon_i} < 3^{-i} \le 3^{-m} < \varepsilon$. So, assume that $y \in [a_i, a_i + \varepsilon_n^2]$. If $a_i - x \ge \varepsilon_i$, then again $\left|\frac{g(y)}{y-x}\right| < \frac{\varepsilon_i^2}{\varepsilon_i} < 3^{-i} \le 3^{-m} < \varepsilon$. Thus, assume that $a_i - x < \varepsilon_n$. Then, by (a), $\left|f'(a_i) - \frac{f(x)-f(a_i)}{x-a_i}\right| < 3^{-i} \le 3^{-m} < \varepsilon$ and $|f'(a_i) - f'(x)| \le \left|f'(a_i) - \frac{f(x)-f(a_i)}{x-a_i}\right| + \left|\frac{\bar{f}(x)-\bar{f}(a_i)}{x-a_i} - f'(x)\right| < 2\varepsilon$. So, by (d),

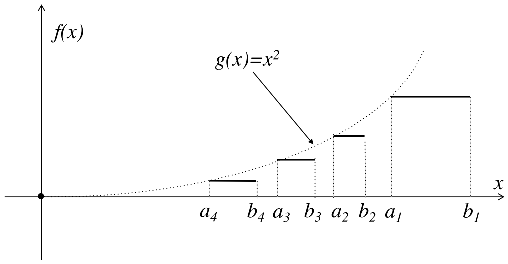

Fig. 4. Graph of a map $f\colon P \to \mathbb{R}$, horizontal thick segments, with $f' = 0$ on P. No differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f has continuous derivatives, unless $\frac{f(a_n)-f(b_{n+1})}{a_n-b_{n+1}} \to_{n\to\infty} 0$.

$$\begin{aligned}
\left|\frac{g(y)}{y-x}\right| &\le \left|\frac{g'(a_i)(y-a_i)}{y-x}\right| = \left|\frac{\left(f'(a_i)-\frac{f(b_i)-f(a_i)}{b_i-a_i}\right)(y-a_i)}{y-x}\right| \\
&= \left|\frac{f'(a_i)(y-a_i)-\frac{\bar f(y)-f(a_i)}{y-a_i}(y-a_i)}{y-x}\right| = \left|\frac{f'(a_i)(y-a_i)-(\bar f(y)-f(a_i))}{y-x}\right| \\
&= \left|\frac{[f'(a_i)(x-a_i)+f(a_i)-f(x)]+[f'(a_i)-f'(x)](y-x)+[f'(x)(y-x)+f(x)-\bar f(y)]}{y-x}\right| \\
&\le \left|f'(a_i)-\frac{f(x)-f(a_i)}{x-a_i}\right|\left|\frac{x-a_i}{y-x}\right| + |f'(a_i)-f'(x)| + \left|f'(x)-\frac{\bar f(y)-\bar f(x)}{y-x}\right| \\
&< \varepsilon+2\varepsilon+\varepsilon = 4\varepsilon.
\end{aligned}$$

This completes the proof of DE Theorem.

3. Alternative formulation of Whitney extension theorem

It is well known and easy to see that function f from DE Theorem need not to admit C^1 extension, even when f' is constant. See, for example, Fig. 4. The theorem characterizing functions $f\colon P \to \mathbb{R}$ that admit C^1 extension $F\colon \mathbb{R} \to \mathbb{R}$ was proved in 1938 by Whitney [10]. The usual characterization of such functions f either involves complicated conditions on difference quotients of f, or requires a use of a non-standard notion of the derivative of f, see [6, thm **W**]. Below we present an alternative variant of Whitney extension theorem.

Theorem 3. *A differentiable map $f\colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is closed, admits a continuously differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ if, and only if, the canonical extension $\hat f\colon \hat P \to \mathbb{R}$ of f (see Definition 1) is continuously differentiable.*

Proof. Let $\mathcal{J}$ be the family of all connected components of $\mathbb{R} \setminus \tilde P$.

To show necessity of continuous differentiability of $\hat f$ assume that there exists a continuously differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f. Clearly, by Proposition 2, $\hat f$ is differentiable. Also, $\hat f'$ is continuous on $\hat P \setminus P'$, since $\hat f$ is locally linear on $\hat P \setminus P' = \bigcup_{J\in\mathcal{J}} I_J$. Thus, we need to show that $\hat f'$ is continuous on P'. Notice that $F = f$ and $F' = f'$ on P'.

Fix an $x \in P'$ and $\varepsilon > 0$. It is enough to find a $\delta > 0$ such that

$$|\hat f'(x) - \hat f'(y)| < \varepsilon \text{ whenever } y \in \hat P \cap (x-\delta, x+\delta). \tag{4}$$

Let $\delta_0 \in (0,1)$ be such that $|F'(x) - F'(y)| < \varepsilon$ whenever $|x-y| < \delta_0$. Choose $\delta \in (0, \delta_0)$ such that for every $J = (a,b) \in \mathcal{J}$: if $x \in [a,b]$, then $\delta < \frac{b-a}{3}$; if $x \notin [a,b]$ and $[a,b] \not\subset (x-\delta_0, x+\delta_0)$, then $(x-\delta, x+\delta)$

is disjoint with $[a,b]$. To see that such δ satisfies (4) pick $y \in \hat{P} \cap (x-\delta, x+\delta)$. If $y \in P$, then (4) holds, since then $|\hat{f}'(x) - \hat{f}'(y)| = |F'(x) - F'(y)| < \varepsilon$. So, assume that $y \notin P$. Then, $y \notin \tilde{P}$, as $\delta < 1$. Thus, there exists a $J = (a,b) \in \mathcal{J}$ such that $y \in I_J$. Note that $x \notin [a,b]$, since in such case $\delta < \frac{b-a}{3}$, preventing $y \in I_J$. Therefore, $[a,b] \subset (x-\delta_0, x+\delta_0)$, as $(x-\delta, x+\delta)$ is not disjoint with $[a,b]$, both containing y. By the mean value theorem, there exists a $\xi \in (a,b) \subset (x-\delta_0, x+\delta_0)$ such that $F'(\xi) = \frac{F(b)-F(a)}{b-a}$. So $|F'(x) - F'(\xi)| < \varepsilon$. Also, $\hat{f}'(y) = \bar{f}'(y) = \frac{f(b)-f(a)}{b-a} = F'(\xi)$. Therefore, $|\hat{f}'(x) - \hat{f}'(y)| = |F'(x) - F'(\xi)| < \varepsilon$, proving (4).

The sufficiency of continuous differentiability of $\hat{f}$ is proved by finding continuously differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of $\hat{f}$. This F is constructed by a small refinement of the construction of F in Section 2, in which we extend function f equal to $\hat{f}$ from Theorem 3. More specifically, let $\{(a_i, b_i)\colon 1 \le i \le \kappa\}$ be an enumeration of the family of all connected components of $\mathbb{R} \setminus \hat{P}$. For every $1 \le i \le \kappa$, let α_i and β_i be the endpoints of $[a_i, b_i]$ such that $\hat{f}'(\alpha_i) \le \hat{f}'(\beta_i)$ and, when choosing maps h_i, ensure that their range is contained in $\left[\hat{f}'(\alpha_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i} - 3^{-i}, \hat{f}'(\beta_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i} + 3^{-i}\right]$. This can be achieved by choosing s_i and t_i so close to, respectively, a_i and b_i that the resulted constants A_i and B_i have magnitude $\le 3^{-i}$. We claim, that such constructed F has continuous derivative. To see this, choose an $x \in \mathbb{R}$. We will show only that F' is right-continuous at x, the argument for left-continuity being similar.

Clearly, the definition of F ensures that F' is right-continuous at x if there exists a $y > x$ such that $(x,y) \cap \hat{P} = \emptyset$. So, assume that there is no such y. Choose an $\varepsilon > 0$. It is enough to find a $\delta > 0$ such that

$$|F'(x) - F'(y)| < 2\varepsilon \text{ whenever } y \in (x, x+\delta). \tag{5}$$

Let $\delta_0 > 0$ be such that $|\hat{f}'(x) - \hat{f}'(y)| < \varepsilon$ whenever $y \in (x, x+\delta_0) \cap \hat{P}$. Choose $n \in \mathbb{N}$ such that $3^{-n} < \varepsilon$ and let $\delta \in (0, \delta_0)$ such that: $(0,\delta)$ is disjoint with every (a_i, b_i) for which $i < n$; if (a_i, b_i) intersects $(0,\delta)$, then $[a_i, b_i] \subset (0, \delta_0)$. To see that such δ satisfies (5) pick $y \in (x, x+\delta)$. If $y \in \hat{P}$, then (5) holds, since then $|F'(x) - F'(y)| = |\hat{f}'(x) - \hat{f}'(y)| < \varepsilon$. So, assume that $y \notin \hat{P}$. Then, $y \in (a_i, b_i)$ for some $i \ge n$. Since $\beta_i \in [a_i, b_i] \subset (0, \delta_0)$, we have $|\hat{f}'(x) - \hat{f}'(\beta_i)| < \varepsilon$ and $\hat{f}'(\beta_i) < \hat{f}'(x) + \varepsilon$. So, $F'(y) = \bar{f}'(y) + g'(y) = \frac{f(b_i)-f(a_i)}{b_i-a_i} + h_i(y) \le \hat{f}'(\beta_i) + 3^{-i} < \hat{f}'(x) + \varepsilon + 3^{-i} \le F'(x) + 2\varepsilon$. Similarly, $F'(y) \ge \hat{f}'(\alpha_i) - 3^{-i} > \hat{f}'(x) - \varepsilon - 3^{-i} \ge F'(x) - 2\varepsilon$. So, (5) holds. $\square$

4. History and related results

4.1. Differential extension theorem in literature

DE Theorem first appeared in print in 1923 paper [4] of Vojtěch Jarník, for the case when $P \subset \mathbb{R}$ is compact perfect. Unfortunately, [4] appeared in little known journal, *Bull. Internat. de l'Académie des Sciences de Bohême* is written in French, and only sketches the construction of the extension. A more complete version of the proof, that appeared in [5], is written in Czech and is even less accessible. Therefore, this result of Jarník was unnoticed by the mathematical community until mid 1980s. DE Theorem was rediscovered by György Petruska and Miklós Laczkovich and published in 1974 paper [9]. Its proof in [9] is quite involved and embedded in a deeper, more general research. A simpler proof of the theorem appeared in 1984 paper [7] of Jan Mařík; however, it is considerably more complicated than the one we presented above and it uses Lebesgue integration tools. Apparently, the authors of neither [9] nor [7] have been aware of Jarník's paper [4] at the time of publication of their articles. However [4] is cited in 1985 paper [1] that discusses multivariable version of DE Theorem. Also, two recent papers [8,6] that address generalizations of DE Theorem cite [4].

4.2. DE Theorem for functions on $\mathbb{R}^n$

There is no straightforward generalization of DE Theorem to differentiable functions f defined on closed subsets P of $\mathbb{R}^n$. This is the case, since the derivative of any extension $F\colon \mathbb{R}^n \to \mathbb{R}$ is Baire class one, as it is a pointwise limit of continuous functions $F_n(x) = n\left(F\left(x+\frac{1}{n}\right) - F(x)\right)$. Therefore the derivative f' of any differentiable extendable function $f'\colon P \to \mathbb{R}$ must be also Baire class one. However, there exists a differentiable function $f\colon P \to \mathbb{R}$, with $P \subset \mathbb{R}^2$ being closed, for which f' is not Baire class one, see [1, thm 5]. Clearly this f admits no differentiable extension to $\mathbb{R}^2$. However, in [1] the authors prove that this is the only obstacle to generalize DE Theorem to multivariable functions. More specifically, they prove that differentiable function $f\colon P \to \mathbb{R}$, with P being a closed subset of $\mathbb{R}^n$, admits differentiable extension $F\colon \mathbb{R}^n \to \mathbb{R}$ if, and only if, $f'\colon P \to \mathbb{R}$ is Baire class one.

4.3. What else is known about differentiable maps on closed $P \subset \mathbb{R}$?

If we exclude results assuming that P has positive Lebesgue measure, what we discussed above includes essentially all "positive" results. Everything else seems to be concentrated on different examples, some surprising, even weird. Perhaps, the most unexpected among them is one described in the following 2016 theorem.

Theorem 4. *(Ciesielski, Jasinski [3]) There exists a compact perfect set $\mathfrak{X} \subset \mathbb{R}$ and a differentiable bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}'(x) = 0$ for every $x \in \mathfrak{X}$. Moreover, the forward $\mathfrak{f}$-orbit of every $x \in \mathfrak{X}$ is dense in $\mathfrak{X}$.*

Of course, such $\mathfrak{X}$ cannot be an interval. In fact, $\mathfrak{X}$ must be of Lebesgue measure 0, as $\mathfrak{f}[\mathfrak{X}]$ is of measure 0 whenever $\mathfrak{f}' \equiv 0$. By DE Theorem, the map $\mathfrak{f}$ can be extended to a differentiable $F\colon \mathbb{R} \to \mathbb{R}$. However, such extension cannot be C^1, since for C^1 maps $F\colon \mathbb{R} \to \mathbb{R}$ we have (see [2, lemma 3.3]): *If P is a compact perfect subset of $\mathbb{R}$ such that $P \subset F[P]$, then there exists an $x \in P$ such that $|F'(x)| \geq 1$.*

Another strange example comes from 2014 paper [2] and concerns Peano-like maps, that is, continuous maps from a space P onto its square P^2.

Theorem 5. *(Ciesielski, Jasinski [2]) There exists a perfect subset P of $\mathbb{R}$ and a differentiable $f = \langle f_1, f_2\rangle$ from P onto P^2 with $f_1' = f_2' = 0$. Moreover, f can be extended to a C^∞ map $F\colon \mathbb{R} \to \mathbb{R}^2$.*

Once again, P must be of measure 0, since $P^2 = f[P]$ have plane measure 0. The set from Theorem 5 is unbounded. It is unknown, if such an example can exists when P is compact, even when we assume only that f is differentiable, see [2]:

Problem 1. Let $P \subset \mathbb{R}$ be compact perfect and let f be a function from P onto P^2. Can f be differentiable? Continuously differentiable?

By DE Theorem the differentiable version of the problem can be rephrased as: *Do there exist a differentiable function $F\colon \mathbb{R} \to \mathbb{R}^2$ and a compact perfect set $P \subset \mathbb{R}$ such that $f[P] = P^2$?* In this last version of the problem, the map $F\colon \mathbb{R} \to \mathbb{R}^2$ cannot be C^1, as proved in [2, theorem 3.1].

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