On a Genocchi–Peano Example

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Early in a multivariable calculus class, students are asked to determine if

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{when } (x, y) \neq (0, 0), \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

is continuous. Although f is discontinuous (along the parabola $x = y^2$), some students are likely to think that this function is continuous since $f(0, 0)$ is equal to the limit along the x- and y-axes. Did you know that an 1821 calculus textbook of Augustin-Louis Cauchy [**1**] contains a theorem that seems to contradict the existence of an example with such properties and to agree with the naïve hypothesis that *a two variable function is continuous if it is continuous in each variable separately*?

The apparent contradiction comes from the fact that Cauchy's text is written for the set $\mathcal{R}$ of real numbers containing infinitesimals (i.e., numbers d with $0 < d < 1/n$ for every $n = 1, 2, 3, \ldots$), while the standard set $\mathbb{R}$ of real numbers does not contain such objects. The fact that Cauchy's result is false when $\mathcal{R}$ is replaced with the standard set $\mathbb{R}$ of real numbers was first observed by E. Heine and appeared in the 1870 calculus text of J. Thomae [**6**]; see [**5**]. The prominent example (1), which appears in many calculus books, comes from the 1884 treatise on calculus by A. Genocchi

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and G. Peano [**4**]. (See [**3**] on more history related to the above-mentioned Cauchy's result and Genocchi–Peano's example.) Not only is (1) continuous separately, it is also continuous when restricted to any straight line, including those passing through the origin.

This article will focus on the following questions.

Q1: What are other examples of two variable functions that are discontinuous but continuous along any straight line?

Q2: Can we generalize (1) to higher dimensions and, if so, in what sense?

Q3: What are the simplest examples of this sort?

In answering the question Q3, we will restrict our attention to the class of rational functions, one of the simplest classes containing removable discontinuities.

Another clarification: Before discussing question Q2, we need to decide whether to treat lines in $\mathbb{R}^2$ as the objects of dimension 1, or rather as hyperplanes, that is, objects of codimension 1 (e.g., hyperplanes in $\mathbb{R}^3$ are standard two-dimensional planes). In other words, do we want the functions of three or more variables to be continuous on all hyperplanes? Or just on all lines? The lines option does not lead to anything truly new, as a natural lift of the original Genocchi–Peano function $f(x, y)$ to the higher dimensions, defined by $G(x_1, x_2, \ldots, x_n) = f(x_1, x_2)$, is clearly discontinuous, while continuous on any straight line. Therefore, in what follows, we will require our examples to be continuous on all hyperplanes. Notice, that, for $n > 2$, this function G is not among such examples since it is discontinuous when restricted to the hyperplane $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_3 = x_2\}$.

Generalized Genocchi–Peano examples

The simplest rational functions $g\colon \mathbb{R}^n \to \mathbb{R}$ that may have a chance to lead to the examples we seek are in the form

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}} & \text{when } (x_1, \ldots, x_n) \neq (0, \ldots, 0), \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\alpha_i, \beta_i \in \mathbb{N} = \{1, 2, 3, \ldots\}$ for all $i \in \{1, \ldots, n\}$. For the rest of the article, we will assume that every function is 0 at $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$, the origin.

We say that $g\colon \mathbb{R}^n \to \mathbb{R}$ in the form of (2) and with $n > 1$ is a *Genocchi–Peano example* (abbreviated GPE) if g is discontinuous but has a continuous restriction $g \restriction H$ to any hyperplane H in $\mathbb{R}^n$. Of course, if all the β_i are even, then the maps (2) are continuous when restricted to the hyperplanes that do not contain the origin. Thus, in such case, we will restrict our attention to the hyperplanes that contain the origin, that is, expressible via equations $\sum_{i=1}^n b_k x_k = 0$. One of the main goals of this article is to investigate the following general question.

For any $n > 1$, for what values of $\alpha_i, \beta_i \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, is the function $g(x_1, x_2, \ldots, x_n)$ a Genocchi–Peano example?

It is worth noting that any GPE is, in particular, continuous on any straight line.

Clearly, the function f given by (1) is a GPE. It is also easy to see that

$$h(x, y) = \frac{xy^2}{x^2 + y^6} \tag{3}$$

constitutes another such example since it is discontinuous on the curve $x = y^3$. These two examples are essentially different: $f[\mathbb{R}^2]$ is bounded, as

$$|f(x, y)| = \sqrt{\frac{x^2}{x^2 + y^4}}\sqrt{\frac{y^4}{x^2 + y^4}} \leq 1,$$

while $h[\mathbb{R}^2]$ is not (since $\lim_{y\to 0^+} h(y^3, y) = \infty$).

A simple GPE for $n = 3$ is given by

$$j(x_1, x_2, x_3) = \frac{x_1 x_2 x_3^2}{x_1^2 + x_2^4 + x_3^8}. \tag{4}$$

Indeed, j is discontinuous on $\{(t^4, t^2, t) \mid t \in \mathbb{R}\}$. To see that it is continuous on any hyperplane (containing the origin), notice that

$$|j(x_1, x_2, x_3)| = \frac{|x_1|}{d^{1/2}}\frac{|x_2|}{d^{1/4}}\left(\frac{|x_3|}{d^{1/8}}\right)^2$$

where $d = x_1^2 + x_2^4 + x_3^8$. Now, each of the three quotients is bounded above by 1. Moreover, for a hyperplane $x_3 = ax_1 + bx_2$, we have

$$\frac{|x_3|}{d^{1/8}} \leq |a|\frac{|x_1|}{d^{1/8}} + |b|\frac{|x_2|}{d^{1/8}} \leq |a|\frac{d^{1/2}}{d^{1/8}} + |b|\frac{d^{1/4}}{d^{1/8}} \to 0$$

as $d \to 0$. So, j is continuous at $(0, 0, 0)$ on this hyperplane. Similarly, j is continuous on a hyperplane $x_2 = ax_1$ since

$$\frac{|x_2|}{d^{1/4}} \leq |a|\frac{d^{1/2}}{d^{1/4}} \to 0$$

as $d \to \infty$. Hence, j is indeed a GPE.

This justification for $j(x_1, x_2, x_3)$ exemplifies well the general argument for our main theorem characterizing GPEs, stated next. First, note that none of the β_i can be odd if g of the form (2) is to be a GPE. For if β_i were odd, then g would be discontinuous on any hyperplane containing a point $y = (y_1, \ldots, y_n) \in (\mathbb{R} \setminus \{0\})^n$ satisfying $\sum_{i=1}^n y_i^{\beta_n} = 0$. (To see this more clearly, set $y_j = 1$ for $j \neq i$ and $y_i = \sqrt[\beta_i]{1 - n}$.) Therefore, in the rest of the article, we will assume that all the β_i are even, and because of symmetry in the definition of g, we will also assume $\beta_1 \leq \cdots \leq \beta_n$.

Theorem (Characterization of GPEs). *Let g be given by* (2) *with $\beta_1 \leq \cdots \leq \beta_n$ positive even numbers.*

(i) *g is discontinuous if and only if $\sum_{i=1}^n \alpha_i/\beta_i \leq 1$.*

(ii) *g has a continuous restriction to every hyperplane if and only if*

$$\left(\sum_{i=1}^n \frac{\alpha_i}{\beta_i}\right) - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > 1 \text{ for every } k \in \{2, \ldots, n\}.$$

In particular, g is a Genocchi–Peano example if and only if the conditions from (i) and (ii) hold.

Notice, that the values required for the condition in (ii) can be calculated by replacing β_k with β_{k-1} in the expression $\sum_{i=1}^{n} \alpha_i/\beta_i$. The proof of the theorem is at the end of this article.

It is worth pointing out that, for a fixed sequence $(\beta_1, \ldots, \beta_n)$, the theorem implies that there can only be finitely many GPEs, namely those that satisfy $\sum_{i=1}^{n} \alpha_i/\beta_i \leq 1$.

An exhaustive example. We determine all GPEs of the form

$$f(x_1, x_2) = \frac{x_1^{\alpha_1} x_2^{\alpha_2}}{x_1^6 + x_2^{10}} \tag{5}$$

for $\alpha_1, \alpha_2 \in \mathbb{N}$. By the theorem, the exponents must satisfy the inequalities from (i) and (ii), namely $\alpha_1/6 + \alpha_2/10 \leq 1$ and $\alpha_1/6 + \alpha_2/6 > 1$. The reader can verify that there are 23 pairs satisfying $5\alpha_1 + 3\alpha_2 \leq 30$, equivalent to the first inequality, of which eight satisfy $\alpha_1 + \alpha_2 > 6$, equivalent to the second. These eight solutions correspond to the GPEs

$$\frac{x_1 x_2^6}{x_1^6 + x_2^{10}}, \quad \frac{x_1 x_2^7}{x_1^6 + x_2^{10}}, \quad \frac{x_1 x_2^8}{x_1^6 + x_2^{10}}, \quad \frac{x_1^2 x_2^5}{x_1^6 + x_2^{10}},$$
$$\frac{x_1^2 x_2^6}{x_1^6 + x_2^{10}}, \quad \frac{x_1^3 x_2^4}{x_1^6 + x_2^{10}}, \quad \frac{x_1^3 x_2^5}{x_1^6 + x_2^{10}}, \quad \frac{x_1^4 x_2^3}{x_1^6 + x_2^{10}}.$$

A difficulty behind using the characterization is that, for each GPE candidate function with a fixed denominator, we need to check n inequalities: $\sum_{i=1}^{n} \alpha_i/\beta_i \leq 1$ and $n - 1$ from the theorem condition (ii). While it is fairly easy to write a program that, for fixed $\beta_1, \ldots, \beta_n$, finds all α_i satisfying these inequalities in one of the common symbolic algebra systems (e.g., Mathematica, Maple, Matlab), this could be a challenging task without a computer. The following corollary, though more restrictive than our characterization of GPEs theorem, reduces the task of checking whether a candidate is a GPE to a verification of a single equation.

Corollary (Sufficient condition for GPEs). *Let g be as in* (2) *with $\beta_1 \leq \cdots \leq \beta_n$.*

(a) *If g is a Genocchi–Peano example, then the β_i must be distinct.*

(b) *If all β_i are even and $\sum_{i=1}^{n} \alpha_i/\beta_i = 1$, then g is a Genocchi–Peano example if and only if all β_i are distinct.*

Moreover, the functions as in (b) are the only GPEs with $g[\mathbb{R}^n]$ bounded.

This follows quite easily from our characterization of GPEs theorem. Indeed, if g is a GPE, then the β_i must be distinct since otherwise there would exist $k \in \{2, \ldots, n\}$ with $\alpha_k/\beta_k = \alpha_k/\beta_{k-1}$, and so the inequalities from parts (i) and (ii) of the characterization cannot simultaneously hold. On the other hand, if all the β_i are distinct and $\sum_{i=1}^{n} \alpha_i/\beta_i = 1$, then $\alpha_k/\beta_k < \alpha_k/\beta_{k-1}$ for every $k \in \{2, \ldots, n\}$ and

$$\left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i}\right) - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = 1.$$

Thus, g is a GPE.

We leave the last part of the sufficient condition result as an exercise.

Exercise 1. Prove, using the characterization result, that the only bounded GPEs are those for which $\sum_{i=1}^{n} \alpha_i/\beta_i = 1$. *Hint:* Follow the arguments using (3) and (1). For general n, (11) below might be useful.

Notice, that among the eight GPEs of the form (5) only one, $x_1^3 x_2^5/(x_1^6 + x_2^{10})$, satisfies the sufficient condition. The corollary also implies that each map

$$g_n(x_1, \ldots, x_n) = \frac{x_1 x_2 \cdots x_{n-1} x_n^2}{x_1^2 + x_2^4 + \cdots + x_{n-1}^{2^{n-1}} + x_n^{2^n}} \tag{6}$$

is a GPE, as

$$\frac{1}{2} + \cdots + \frac{1}{2^{n-1}} + \frac{2}{2^n} = 1.$$

(The fact that the g_n are GPEs was first noticed, without a proof of correctness, in [**2**].) Note that the original GPE given in (1) is g_2, while j from (4) is g_3. Another general class of GPEs, each for $n > 1$ variables, is given by

$$h_n(x_1, \ldots, x_n) = \frac{x_1^2 \cdots x_i^{2i} \cdots x_n^{2n}}{x_1^{2n} + \cdots + x_i^{2in} + \cdots + x_n^{2n^2}} \tag{7}$$

where the assumptions of the corollary hold since $\sum_{i=1}^{n} 2i/(2in) = n(1/n) = 1$. In particular, these give the following GPEs of 2, 3, and 4 variables, respectively.

$$g_2(x, y) = \frac{xy^2}{x^2 + y^4}, \quad h_2(x, y) = \frac{x^2 y^4}{x^4 + y^8},$$

$$g_3(x, y, z) = \frac{xyz^2}{x^2 + y^4 + z^8}, \quad h_3(x, y, z) = \frac{x^2 y^4 z^6}{x^6 + y^{12} + z^{18}},$$

$$g_4(x, y, z, w) = \frac{xyzw^2}{x^2 + y^4 + z^8 + w^{16}}, \quad h_4(x, y, z, w) = \frac{x^2 y^4 z^6 w^{12}}{x^8 + y^{16} + z^{24} + w^{32}}.$$

In these examples, the degrees of the denominators of GPEs given by (6) are smaller than those given by (7). However, 2^n is bigger than $2n^2$ for large values of n, that is, this trend reverses for as $n \to \infty$. (In fact, already for any $n \geq 7$.) These observations open up a discussion of the simplest GPEs.

The simplest Genocchi–Peano examples

So far, we answered the questions Q1 and Q2, in the class of functions of the form (2). Now, we tackle the question Q3 about the simplest Genocchi–Peano examples. But how do you define simplest, even just in the class of the GPEs? We decided to express simplicity in terms of the degree of the denominator of (2) (i.e., β_n) so that the smaller β_n corresponds to the simpler GPE. In general, little is known about the minimal degrees β_n for GPEs of n variables. Of course, we must have $\beta_n \geq 2n$ since we know from the corollary that all the β_i are even and distinct. Also, we have GPEs with $\beta_n \leq \min\{2^n, 2n^2\}$ considering the maps g_n and h_n from (6) and (7). Thus, for any GPE of n variables, the minimal β_n satisfies

$$2n \leq \beta_n \leq \min\{2^n, 2n^2\}. \tag{8}$$

However, in general, the upper bound $\min\{2^n, 2n^2\}$ is far from optimal, as we will see in some of the following investigations of GPEs for small values of n.

• For $n = 2$, the inequalities (8) immediately imply that $\beta_n = 4$. In particular, the original GPE (1), which we know now as g_2, has the denominator of minimal degree. Moreover, it is easy to see that g_2 is the only GPE of two variables with $\beta_n = 4$.

• For $n = 3$, we know by (8) that $6 \le \beta_n \le 8$. Any GPE map (2) with $\beta_n < 8$ would need to be of the form

$$\frac{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}{x_1^2 + x_2^4 + x_3^6}.$$

Among such functions, only $\alpha_1 = \alpha_2 = \alpha_3 = 1$ gives the necessary inequality $\alpha_1/2 + \alpha_2/4 + \alpha_3/6 \le 1$. However, for this choice, condition (ii) of the characterization fails for $k = 3$. Hence, there is no GPE of this form, and $\beta_n = 8$. This means that, again, the g_n family entry, here

$$g_3(x_1, x_2, x_3) = \frac{x_1 x_2 x_3^2}{x_1^2 + x_2^4 + x_3^8},$$

is a GPE with the denominator of minimal degree. In fact, g_3 has also the smallest degree numerator among all three-variable GPEs with minimal degree denominator. This is the case since each of the functions

$$\frac{x_1 x_2 x_3}{x_1^2 + x_2^4 + x_3^8}, \quad \frac{x_1 x_2 x_3}{x_1^2 + x_2^6 + x_3^8}, \quad \frac{x_1 x_2 x_3}{x_1^4 + x_2^6 + x_3^8}$$

fails the theorem condition (ii) for $k = 3$, as

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} > \frac{1}{2} + \frac{1}{6} + \frac{1}{6} > \frac{1}{4} + \frac{1}{6} + \frac{1}{6}.$$

Moreover, g_3 is the only GPE with $\beta_n = 8$ and the numerator of degree 4 since none of the following potential candidates is a GPE:

$$\frac{x_1^2 x_2 x_3}{x_1^2 + x_2^4 + x_3^8}, \quad \frac{x_1 x_2^2 x_3}{x_1^2 + x_2^4 + x_3^8}, \quad \frac{x_1^2 x_2 x_3}{x_1^2 + x_2^6 + x_3^8} \tag{9}$$

fail (i) of the characterization while one can show that

$$\frac{x_1 x_2^2 x_3}{x_1^2 + x_2^6 + x_3^8}, \frac{x_1 x_2 x_3^2}{x_1^2 + x_2^6 + x_3^8}, \frac{x_1^2 x_2 x_3}{x_1^4 + x_2^6 + x_3^8}, \frac{x_1 x_2^2 x_3}{x_1^4 + x_2^6 + x_3^8}, \frac{x_1 x_2 x_3^2}{x_1^4 + x_2^6 + x_3^8} \tag{10}$$

fail (ii) for $k = 3$.

Exercise 2. Prove that the maps listed in (9) fail to be GPEs by showing explicitly (without using the characterization) that they are continuous at the origin.

Exercise 3. Verify that the functions listed in (10) indeed fail the theorem condition (ii) with $k = 3$.

• For $n = 4$, the inequalities (8) give bounds $8 \le \beta_n \le 16$. We will show that, in this case, the smallest possible β_n of a GPE is 10 if we allow unbounded maps and 12 otherwise.

First notice that $\beta_n > 8$ since otherwise

$$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \geq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} > 1,$$

and condition (i) of the characterization fails. Thus, $\beta_n \geq 10$. The possibility of equality is justified by the GPE

$$\frac{x_1 x_2 x_3^2 x_4^3}{x_1^4 + x_2^6 + x_3^8 + x_4^{10}},$$

which satisfies (i) from the characterization since

$$\frac{1}{4} + \frac{1}{6} + \frac{2}{8} + \frac{3}{10} = \frac{29}{30} < 1$$

and the inequalities from (ii) for every $k \in \{2, 3, 4\}$, respectively

$$\frac{1}{4} + \frac{1}{4} + \frac{2}{8} + \frac{3}{10} = \frac{21}{20}, \quad \frac{1}{4} + \frac{1}{6} + \frac{2}{6} + \frac{3}{10} = \frac{63}{60}, \quad \frac{1}{4} + \frac{1}{6} + \frac{2}{8} + \frac{3}{8} = \frac{25}{24},$$

each greater than 1.

To show that there is no bounded GPE with $\beta_n = 10$, by the corollary, it is enough to that show $a/2 + b/4 + c/6 + d/8 + e/10 = 1$ for no $a, b, c, d, e \in \{0, 1, 2, \ldots\}$ with precisely one of a, b, c, d being zero. Writing

$$\frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{e}{10} = \frac{120a + 60b + 40c + 30d + 24e}{240},$$

we see that this number can only be an integer when e is divisible by 5. But if e is divisible by 5 and precisely one of $a, b, c, d \in \{0, 1, 2, \ldots\}$ is zero, then

$$\frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{e}{10} \geq \frac{a}{2} + \frac{b}{4} + \frac{c}{6} + \frac{d}{8} + \frac{5}{10} \geq \frac{0}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{5}{10} > 1.$$

The maps

$$\frac{x_1 x_2 x_3 x_4}{x_1^2 + x_2^4 + x_3^6 + x_4^{12}}, \quad \frac{x_1 x_2 x_3^2 x_4^4}{x_1^4 + x_2^6 + x_3^8 + x_4^{12}}$$

are both bounded GPEs with $\beta_n = 12$ since, using the sufficiency corollary,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = 1, \quad \frac{1}{4} + \frac{1}{6} + \frac{2}{8} + \frac{4}{12} = 1.$$

Exercise 4. Use the characterization theorem to show that

$$\frac{x_1 x_2 x_3 x_4^2}{x_1^2 + x_2^6 + x_3^8 + x_4^{10}}$$

is another example of GPE with $\beta_n = 10$. Notice that this numerator has smaller degree than the example above.

Exercise 5. Use the characterization theorem to show that there is no GPE of four variables with denominator of degree 10 and numerator of degree less than 5.

Proof of the GPE characterization theorem

Let $\gamma = \sum_{i=1}^{n} \alpha_i/\beta_i$ and $d = x_1^{\beta_1} + \cdots + x_n^{\beta_n}$. Then g of the form $(x_1^{\alpha_1} \cdots x_n^{\alpha_n})/d$,

$$g(x_1, \ldots, x_n) = \frac{1}{d^{1-\gamma}} \frac{x_1^{\alpha_1}}{d^{\alpha_1/\beta_1}} \cdots \frac{x_n^{\alpha_n}}{d^{\alpha_n/\beta_n}} = d^{\gamma-1} \prod_{i=1}^{n} \frac{x_i^{\alpha_i}}{d^{\alpha_i/\beta_i}}. \tag{11}$$

To see (i), first assume that $\gamma \leq 1$. Then, for $t > 0$,

$$h(t) = g\left(t^{1/\beta_1}, \ldots, t^{1/\beta_n}\right) = \frac{t^{\alpha_1/\beta_1} \cdots t^{\alpha_n/\beta_n}}{nt} = \frac{t^{\gamma-1}}{n}$$

so that h does not converge to $g(0, \ldots, 0) = 0$ as $t \to 0$. Thus, g is discontinuous.

Conversely, assume that $\gamma > 1$. Since

$$\left| \frac{x_i^{\alpha_i}}{d^{\alpha_i/\beta_i}} \right| \leq \frac{|x_i|^{\alpha_i}}{\left(x_i^{\beta_i}\right)^{\alpha_i/\beta_i}} = 1$$

for every $i \in \{1, \ldots, n\}$, the expression for g in (11) implies $|g(x_1, \ldots, x_n)| \leq d^{\gamma-1}$. But $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} d^{\gamma-1} = 0$ since $\gamma - 1 > 0$. So by the squeeze theorem, $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} g(x_1, \ldots, x_n) = 0$, i.e., g is continuous at the origin. So indeed, g is continuous, as desired.

To see that (ii) holds, let

$$\delta_k = \left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \right) - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}$$

for $k \in \{2, \ldots, n\}$.

First assume that $\delta_k \leq 1$ for some $k \in \{2, \ldots, n\}$ and consider the hyperplane $H = \{x \in \mathbb{R}^n \mid x_k = x_{k-1}\}$. We will show that $g \restriction H$, the restriction g to H, is discontinuous. Indeed, for every $t > 0$ and $i \in \{1, \ldots, n\}$ let

$$f_i(t) = \begin{cases} t^{1/\beta_i} & \text{if } i \neq k, \\ t^{1/\beta_{k-1}} & \text{if } i = k. \end{cases}$$

Then $(f_1(t), \ldots, f_n(t)) \in H$. Moreover, since

$$(f_i(t))^{\alpha_i} = \begin{cases} t^{\alpha_i/\beta_i} & \text{if } i \neq k, \\ t^{\alpha_i/\beta_{k-1}} & \text{if } i = k \end{cases} \quad \text{and} \quad (f_i(t))^{\beta_i} = \begin{cases} t & \text{if } i \neq k, \\ t^{\beta_k/\beta_{k-1}} & \text{if } i = k \end{cases}$$

we have

$$g(f_1(t), \ldots, f_n(t)) = \frac{t^{\gamma - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}}}{(n-1)t + t^{\beta_k/\beta_{k-1}}} = \frac{1}{(n-1) + t^{(\beta_k/\beta_{k-1})-1}} t^{\delta_k - 1}.$$

Thus, $\lim_{t\to 0} g(f_1(t), \ldots, f_n(t)) \neq 0$ since $\lim_{t\to 0^+} t^{\delta_k-1} \geq 1$ (as $\delta_k - 1 \leq 0$) and

$$\lim_{t\to 0^+} \frac{1}{(n-1) + t^{(\beta_k/\beta_{k-1})-1}}$$

is either $1/(n-1)$ (when $\beta_{k-1} < \beta_k$) or $1/n$ (when $\beta_{k-1} = \beta_k$). Therefore, $g \upharpoonright H$ is discontinuous at the origin.

To complete the argument, assume that $\delta_k > 1$ for every $k \in \{2, \dots, n\}$ and let H be a hyperplane. We need to show that $g \upharpoonright H$ is continuous. This is obvious when H does not contain the origin. So, assume that it does and that $\sum_{i=1}^{n} b_i x_i = 0$ on H. Let $k \in \{1, \dots, n\}$ be the largest for which $b_k \neq 0$.

If $k = 1$, then $g \upharpoonright H$ is identically 0 and thus continuous. Assume $k > 1$. Then the equation $\sum_{i=1}^{n} b_k x_k = 0$ can be written as $x_k = \sum_{i=1}^{k-1} a_i x_i$. In particular, since $1/\beta_i \geq 1/\beta_{k-1}$ for every $i \in \{1, \dots, k-1\}$, for every $d \in (0, 1)$ we have

$$|x_k| = \left|\sum_{i=1}^{k-1} a_i x_i\right| \leq \sum_{i=1}^{k-1} |a_i||x_i| \leq \sum_{i=1}^{k-1} |a_i| d^{\frac{1}{\beta_i}} \leq \sum_{i=1}^{k-1} |a_i| d^{\frac{1}{\beta_{k-1}}} = A d^{\frac{1}{\beta_{k-1}}} \tag{12}$$

where $A = \sum_{i=1}^{k-1} |a_i|$. Since

$$\left|\frac{x_i^{\alpha_i}}{d^{\alpha_i/\beta_i}}\right| = \left|\frac{x_i^{\beta_i}}{d}\right|^{\alpha_i/\beta_i} \leq 1$$

for every $i \in \{1, \dots, n\}$, by (11) and (12), we have

$$\begin{aligned} |g(x_1, \dots, x_n)| &\leq \frac{1}{d^{1-\gamma}} \frac{|x_k|^{\alpha_k}}{d^{\alpha_k/\beta_k}} \\ &\leq \frac{\left(A d^{\frac{1}{\beta_{k-1}}}\right)^{\alpha_k}}{d^{1-\gamma+\frac{\alpha_k}{\beta_k}}} = A^{\alpha_k} d^{\gamma - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} - 1} = A^{\alpha_k} d^{\delta_k - 1}. \end{aligned}$$

But $A^{\alpha_k} d^{\delta_k - 1} \to 0 = g(0, \dots, 0)$ as $d \to 0^+$ since $\delta_k - 1 > 0$. Thus, $g \upharpoonright H$ is continuous at the origin and continuous overall.

Summary. We characterize the simple rational functions of arbitrarily many real variables that are discontinuous but continuous when restricted to any hyperplane. The characterization is expressed by simple inequalities with respect to the exponents of each variable. Examples include two infinite families of such Genocchi–Peano examples. We also investigate the smallest degree of the denominators of such examples.

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