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General Theory of Fuzzy Connectedness Segmentations

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Abstract For any positive integer M, M-object *fuzzy connectedness* (*FC*) *segmentation* is a methodology for finding M objects in a digital image based on user-specified *seed points* and user-specified functions, called (*fuzzy*) *affinities*, which map each pair of image points to a value in the real interval [0, 1]. The theory of FC segmentation has proceeded along two tracks. One track, developed by researchers including the first author, has used two kinds of FC segmentations: RFC segmentation and IRFC segmentation. The other track, developed by researchers including the second and third authors, has used another kind of FC segmentation called MOFS segmentation. In RFC and IRFC segmentation the M delineated objects are pairwise disjoint. In contrast, the M objects delineated by MOFS segmentation may overlap, though in many practical applications the *tie-zone* (i.e., the set of points that do not lie in just one object) is extremely small. Another difference between (I)RFC and MOFS segmentation is that the former types of segmentation are defined in terms of just one affinity (regardless of the value of M), whereas MOFS segmentation is defined in terms of M different affinities with each of the M objects having its own affinity. Moreover, the affinity used in (I)RFC segmentation has almost always been assumed in the (I)RFC-track literature to be a symmetric function, but the affinities used in MOFS segmentation need not be symmetric. This paper presents the first unified mathematical study of FC segmentation that encompasses both (I)RFC and MOFS segmentation. We generalize the concepts of RFC and IRFC segmentation to the case where the affinity is not necessarily symmetric, explain just how the three different segmentation methods relate to each other, and give very concise mathematical (i.e., nonalgorithmic) path-based characterizations of the objects delineated by (I)RFC and MOFS segmentation. Our primary path-based characterization of MOFS objects depends on the concept of a *recursively optimal* path, which we introduce in this paper. Using another new concept—the *core* of an MOFS object—we prove results which show that MOFS segmentation is robust with respect to seed choice even when different affinities are used for different objects and the affinities are not necessarily symmetric. Two of these results substantially generalize known (I)RFC-track robustness results that previously had no MOFS-track counterpart. The fast MOFS algorithm in this paper (our Algorithm 5), which is reminiscent of Dijkstra's shortest path algorithm for weighted digraphs, is one of the most computationally efficient segmentation algorithms. It can be used to efficiently compute IRFC segmentations as well as MOFS segmentations: This is because it emerges quickly from our work that if a single affinity is used then IRFC objects are just MOFS objects from which all tie-zone points have been removed. When $M > 2$, this fast MOFS algorithm is likely to compute an M-object IRFC segmentation more quickly than commonly used IRFC segmentation algorithms that compute IRFC objects one at a time (except possibly when the tie-zone of the segmentation is very large, in

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which case we show that the IRFC segmentation must be unstable).

Keywords Image Processing · Segmentation · Fuzzy connectedness · Robustness

1 Introduction

Image segmentation is an important and challenging task for which a multitude of different techniques have been developed; see, e.g., Sect. 1.6 of [23] and the survey articles in Part IV of that book. This paper deals with the segmentation methodology known as *fuzzy connectedness* (or *FC*) *segmentation*, which has been used with considerable success—see, e.g., Fig. 1—on biomedical and other images [5–21,25–27,29,30,32–39]. The earliest uses of FC segmentation that the authors are aware of were in geophysical data processing [12–15]. Udupa and Samarasekera [38] were the first to apply FC segmentation to medical imaging.

Much of the theory of FC segmentation has developed along two different tracks. In one of the tracks [5,6,16–21,25,32,33,37,39] two kinds of segmentation are used: RFC segmentation and IRFC segmentation. The other track [7–11,26,29] uses a third kind of segmentation that is called MOFS segmentation. Accordingly, we will refer to the former track as the (I)RFC-track, and refer to the latter track as the MOFS-track.[1,2] In this paper, we present a general theory of FC segmentation that encompasses both tracks and unifies them.

Let V be the set of all points of a digital image (so that V is finite and nonempty), let M be a positive integer, and let $S_1, \ldots, S_M$ be pairwise disjoint nonempty subsets of V. Then FC segmentation can be understood as one method of identifying M subsets $O_1, \ldots, O_M$ of V such that $S_i \subseteq O_i \subseteq S_i \cup (V \setminus \bigcup_j S_j)$ for $1 \leq i \leq M$. Each of the sets $O_1, \ldots, O_M$ that is identified is called an *object*, and (for $1 \leq i \leq M$) each point in the originally specified set S_i is called a *seed point* or simply a *seed* for the ith object O_i. In practical applications each of the seed sets $S_1, \ldots, S_M$ is usually small and might well consist of just a single point. In many applications one of the M objects is called the *background*.

In addition to using the term *FC segmentation* to refer to the process by which the objects $O_1, \ldots, O_M$ are found, we will also call the sequence of objects $O_1, \ldots, O_M$ an *FC segmentation* or an *M-object FC segmentation* of the set V of image points.

This terminology implies that an FC segmentation is not necessarily a segmentation in the most typical sense because it is not necessarily a partition of the set V of image points: It is not required that the O_i be pairwise disjoint nor that their union be the whole of V. However, FC segmentation is the only kind of segmentation we discuss in this paper, and we will often refer to FC segmentations as "segmentations."

We also note here that our concept of *object* is simpler than that used in much of the literature on FC segmentation: In the FC segmentation literature objects are often *fuzzy* sets defined by a membership function valued in the real unit interval [0, 1]—see, for example, the definition of an M-semisegmentation in [10]—whereas our objects O_i are sets in the ordinary sense (i.e., they are "crisp" or "hard" sets in the language of fuzzy set theory). While it would be reasonably straightforward to reformulate our work in terms of fuzzy sets[3] this would complicate our notation and terminology unnecessarily: Our goal is to give a unified theory of FC segmentation in which our mathematical results are stated as simply and concisely as possible.

The objects O_i that are found by FC segmentation depend on user-specified mappings called *fuzzy affinities* or just *affinities*. We define an *affinity* (on V) to be a mapping[4] $\psi : V \times V \to [0, 1]$ such that $\psi(v, v) = 1$ for all $v \in V$. For all $u, v \in V$ we call the value $\psi(u, v) \in [0, 1]$ the *ψ-affinity value* of (u, v).

An affinity on V may be regarded as an edge-weight function of the complete digraph (with loops) on V. Affinity values are described in [38] and elsewhere as (user-specified) measures of the "hanging togetherness" of pairs of image

[1] A simpler FC segmentation method, called *absolute* fuzzy connectedness (AFC) segmentation, predates (I)RFC and MOFS segmentation. In AFC segmentations each object consists of those points that are connected to the object's seed set by a path whose strength is no less than a user-specified threshold. An important motivation for the development of (I)RFC and MOFS segmentation was to eliminate the need for users to specify such thresholds.

[2] IRFC segmentation is closely related to one version of the watershed transform. Specifically, in the case where each seed set S_i consists of just one seed point, Audigier and Lotufo observed in [2] that the objects of an IRFC segmentation are the catchment basins of the tie-zone IFT watershed transform generated by the same seeds and a path cost function that is a strictly decreasing function of the path's strength (with respect to the affinity used to create the segmentation). The set of points that do not lie in any of the IRFC objects is the *tie-zone* of the same watershed transform. Tie-zone IFT watershed transforms are discussed in [1–4], though the path cost function used in [1] and [4] is not directly relevant to IRFC segmentation.

[3] This can be done by replacing each of our crisp objects O_i with the fuzzy set whose membership value at each point $v \in O_i$ is the strength of the strongest O_i-path from O_i's seed set S_i to v, and whose membership value at each point $v \in V \setminus O_i$ is 0. This definition of the membership value at each point v is quite simple, but other definitions (e.g., definitions which depend directly on the image intensity value at v) may give membership values that are more useful in some applications. One reason to define objects as crisp sets rather than fuzzy sets is that there is no standard way to define the membership value at a point.

[4] Affinities whose values need not be numbers (e.g., affinities whose values are n-tuples of real numbers) are considered in [17,18,31]. In particular, in [31] affinity values may be elements of any partially ordered set and the strength of connectedness of one point to another is an element of a free distributive lattice over the partially ordered set.

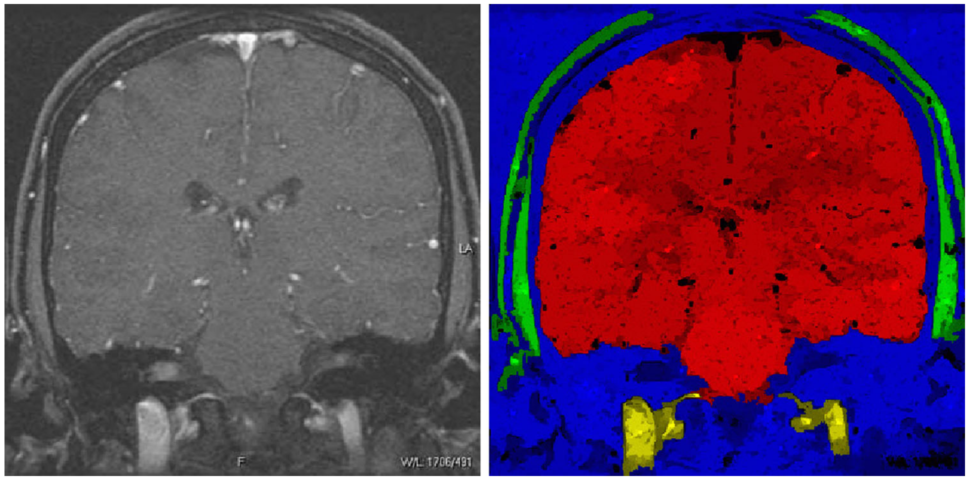

Fig. 1 A slice of an MRI image of a patient's brain and a 4-object MOFS segmentation of the slice. The four objects are shown in *red, blue, green*, and *yellow* in the electronic version of this paper. (Reproduced from [10]) (Color figure online)

points. Most affinities ψ that are used for FC segmentation in imaging have the property that $\psi(u, v)$ can be nonzero only when the points u and v are near each other. For example, when V is the set of elements of a 3D image array $I[\,][\,][\,]$, it is quite common to use affinities ψ such that $\psi(I[i][j][k], I[i'][j'][k']) = 0$ whenever $\max(|i - i'|, |j - j'|, |k - k'|) > 1$.

FC segmentation is unlikely to identify useful objects unless the affinity or affinities we use are appropriate for our application. The important problem of how to define appropriate affinities is discussed, e.g., in [7,11,17,18,28,29,34] but will not be considered here. In this paper, we make no assumptions regarding the affinities that are used; our mathematical results are valid for all affinities.

RFC segmentation and IRFC segmentation differ from MOFS segmentation in two ways. One difference is that the objects in any RFC or IRFC segmentation are pairwise disjoint, whereas objects in an MOFS segmentation may overlap (though in many practical applications, including all applications in which segmentations are stable in the sense of Sect. 2.5, the overlap areas are extremely small). The other difference is that, for any seed sets $S_1, \ldots, S_M$, the RFC and IRFC segmentations of V are determined by a single affinity[5] $\psi : V \times V \to [0, 1]$, whereas the MOFS segmentation of V depends on M affinities $\psi_1, \ldots, \psi_M$—one affinity for each of the M objects. Moreover, in the MOFS-track literature affinities have *not* been assumed to be symmetric functions (i.e., affinities ψ have *not* been assumed to satisfy $\psi(u, v) = \psi(v, u)$ for all image points u and v) and nonsymmetric affinities have sometimes been used in MOFS segmentation [7], but in the (I)RFC-track literature affinities have almost always been assumed to be symmetric (though nonsymmetric affinities are considered in [6]).

[5] The single affinity used for (I)RFC segmentation is often created from M distinct components, each specific to one object.

In spite of these differences, RFC, IRFC, and MOFS segmentations are very closely related. In the sequel we will explain just how they relate to each other, and give concise path-based characterizations of these segmentations that are purely mathematical in the sense that they make no reference to any algorithm. (Our first path-based characterization of MOFS segmentations is stated in terms of *recursively optimal* paths, a new concept that will be introduced in Sect. 3.2.) We will then establish new results which imply that MOFS segmentation is robust with respect to seed choice even when different affinities are used for different objects and the affinities are not symmetric. Two of these results can be viewed as substantial generalizations of (I)RFC-track robustness results that previously had no counterpart in the MOFS-track literature.

Some Key Results of This Paper

Given an affinity $\psi : V \times V \to [0, 1]$ on V and $A, B, W \subseteq V$, a *W-path* from A to B of length l is any sequence $p = \langle w_0, \ldots, w_l \rangle$ of points in W such that $w_0 \in A$ and $w_l \in B$; the *ψ-strength* of $p = \langle v_0, \ldots, v_l \rangle$, denoted by $\psi(p)$, is defined by $\psi(p) = \min_{1 \le j \le l} \psi(v_{j-1}, v_j)$ if $l > 0$ and $\psi(p) = 1$ if $l = 0$; the *ψ-strength of connectedness of $A \neq \emptyset$ to $B \neq \emptyset$ via W* is defined as $\psi^W(A, B) = \max\{\psi(p) \mid p \text{ is a } (W \cup A \cup B)\text{-path from } A \text{ to } B\}$. We say the seed sets $S_1, \ldots, S_M$ are *consistent* with the affinity ψ if $\psi^V(S_i, S_j) < 1$ for all distinct i and j in $\{1, \ldots, M\}$; similarly, we say $S_1, \ldots, S_M$ are *consistent* with the affinities $\psi_1, \ldots, \psi_M$ if $\psi_i^V(S_i, S_j) < 1$ for all distinct i and j in $\{1, \ldots, M\}$.

The following theorem gives some concise mathematical characterizations of RFC, IRFC, and MOFS objects; O_i^{RFC}, O_i^{IRFC}, and O_i^{MOFS}, respectively, denote the RFC, IRFC, and MOFS objects associated with the ith seed set.

Theorem 1.1 *Suppose that $S_1, \ldots, S_M \subset V$ are pairwise disjoint nonempty seed sets. Then:*

1. *Assuming $S_1, \ldots, S_M$ are consistent with the affinity ψ, the RFC object O_i^{RFC} given by $S_1, \ldots, S_M$ and ψ satisfies*

$$O_i^{\mathrm{RFC}} = \{v \in V \mid \max_{j \neq i} \psi^V(S_j, v) < \psi^V(S_i, v)\} \tag{1.1}$$

and also satisfies

$$O_i^{\mathrm{RFC}} = \{v \in V \mid \max_{j \neq i} \psi^V(S_j, v) < \psi^{O_i^{\mathrm{RFC}}}(S_i, v)\}. \tag{1.2}$$

2. *Assuming $S_1, \ldots, S_M$ are consistent with the affinity ψ, the IRFC object O_i^{IRFC} given by $S_1, \ldots, S_M$ and ψ is the unique set O that satisfies*

$$O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^V(S_i, v)\}. \tag{1.3}$$

Moreover, O_i^{IRFC} is also the unique set O that satisfies

$$O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^O(S_i, v)\}. \tag{1.4}$$

3. *Assuming $S_1, \ldots, S_M$ are consistent with the affinities $\psi_1, \ldots, \psi_M$, the sequence of MOFS objects $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}} \rangle$ given by $S_1, \ldots, S_M$ and $\psi_1, \ldots, \psi_M$ is the unique sequence of sets $\langle O_1, \ldots, O_M \rangle$ such that*

$$O_i = \{v \in V \mid \max_{j \neq i} \psi_j^{O_j}(S_j, v) \leq \psi_i^{O_i}(S_i, v) \neq 0\} \quad \text{for } 1 \leq i \leq M. \tag{1.5}$$

Moreover, if $\psi_1 = \cdots = \psi_M = \psi$, then the IRFC object O_i^{IRFC} given by $S_1, \ldots, S_M$ and ψ is

$$O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}. \tag{1.6}$$

Even though some readers who are familiar with the theory of FC segmentation might feel they recognize (1.1) and the first assertion of statement 3 as facts they were already aware of, no part of the theorem is an immediate consequence of results that have been explicitly stated in the literature, for the following reason: The unified theory of FC segmentation we present in this paper defines RFC, IRFC, and MOFS segmentations in a new way (as the results of Algorithms 1–3 below) that is intended to immediately reveal (1.6), to convey a quick understanding of the nature of the objects found by each type of segmentation, and to facilitate parallel development of the mathematics of (I)RFC and MOFS segmentations. So all parts of the above theorem must be understood as mathematical results *about the segmentations defined by our new definitions*. However, as a result of proving (1.1) and the first assertions of statements 2 and 3 of the above theorem, we are able to conclude that our new definitions of RFC, IRFC, and MOFS segmentations are in fact equivalent to definitions of these segmentations that have been used in previous work.

The fact (1.6), which is restated as Corollary 2.7 below, implies that efficient algorithms for computing MOFS objects, such as Algorithm 5 in Sect. 4, can be used to efficiently compute IRFC objects as well. In the past, IRFC segmentations have often been computed using algorithms that compute IRFC objects one at a time. For segmentation into more than two objects, we believe the use of Algorithm 5 (which computes all the objects simultaneously) will typically be a faster way to compute IRFC segmentations. When the affinity is symmetric, the *TZWS by union-find* method of Audigier and Lotufo [2] is another way to compute all the objects of an IRFC segmentation simultaneously. That method will be briefly described in Sect. 4.2.

Interestingly, there seems to be no really easy way to deduce (1.6) from results in the literature about MOFS and IRFC segmentations, even though this fact is almost immediately evident from our new definitions of these segmentations.

Statement 1 follows from the second statements of Theorem 3.6 and its corollary. Statement 2 and the first assertion of statement 3 are parts of Theorems 3.8 and 3.10. These three theorems are key results of this paper that give concise path-based characterizations of RFC, IRFC, and MOFS segmentations.

The first statements of Theorems 3.6, 3.8, and 3.10 give alternative characterizations, which we consider to be at least as useful as the characterizations stated in Theorem 1.1 above, but which we did not include in the above theorem to avoid having to define certain concepts (specifically, the concepts of hereditarily optimal and recursively optimal paths) in this subsection.

Since $\psi^V(S_j, v) \geq \psi^{V \setminus O}(S_j, v)$ for any set O and point $v \in V$, it is easy to see from (1.1), (1.3), and (1.6) that, for any given affinity ψ and seed sets consistent with the affinity, $O_i^{\mathrm{RFC}} \subseteq O_i^{\mathrm{IRFC}} \subseteq O_i^{\mathrm{MOFS}}$ for all $i \in \{1, \ldots, M\}$ (assuming the same affinity ψ is used for all objects in the MOFS segmentation).

From (1.2), (1.4), and (1.5) we see at once that that any element v belonging to one of these objects, say O_i, is connected to the object's seed set via an internal path (i.e., an O_i-path) of strength $\psi^{O_i}(S_i, v) > 0$. This is an important property. Indeed, one reason (I)RFC segmentation has not been extended to use different affinities for different objects is that the resulting (I)RFC objects would not always have this property. The fact that MOFS segmentation allows different affinities for different objects but (I)RFC segmentation does not has been a major obstacle to the reconciliation of the two tracks of FC segmentation theory.

Another fundamental result of this paper is the following theorem (which follows from Proposition 5.3 and Theorem 5.4). As will be seen in Sect. 5, this theorem implies Corollaries 5.5 and 5.6, and Remark 5.7, which tell us that MOFS segmentations are totally robust when we make "small" changes in the seed sets; here "totally robust" means that the segmentations do not change at all.

In the statement of the theorem we use the notation $O_i^{\text{MOFS}}(\Psi, \mathcal{S})$ for the ith MOFS object given by the sequences $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ of affinities and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ of seed sets. Also, $\text{Core}_i^{\Psi,\mathcal{S}} \subseteq O_i^{\text{MOFS}}(\Psi, \mathcal{S})$ is a set of considerable size that we will describe in more details after stating the theorem.

Theorem 1.2 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be a sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ a sequence of pairwise disjoint nonempty seed sets consistent with the affinities. Let $\mathcal{R} = \langle R_1, \ldots, R_M \rangle$ be such that $S_i \subseteq R_i \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$. Then the sequence $\mathcal{R}$ is consistent with the affinities and $O_i^{\text{MOFS}}(\Psi, \mathcal{R}) = O_i^{\text{MOFS}}(\Psi, \mathcal{S})$ for $1 \le i \le M$.*

We prove (as Proposition 5.9) that the set $\text{Core}_i^{\Psi,\mathcal{S}}$ contains the set

$$Q_i^{\Psi,\mathcal{S}} = \Big\{ v \in O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \ne i} O_j^{\text{MOFS}}(\Psi, \mathcal{S}) \mid \psi_i^{O_i^{\text{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \ge \psi_i^V\big(v, \bigcup_{j \ne i} O_j^{\text{MOFS}}(\Psi, \mathcal{S})\big) \Big\},$$

and that, for symmetric ψ_i, $\text{Core}_i^{\Psi,\mathcal{S}}$ is exactly $Q_i^{\Psi,\mathcal{S}}$. Moreover, if all the affinities ψ_i are equal to the same symmetric affinity ψ, then (as stated in Proposition 5.11) $\text{Core}_i^{\Psi,\mathcal{S}}$ is exactly the IRFC object $O_i^{\text{IRFC}}(\psi, \mathcal{S})$. This fact, together with the above Theorem 1.2, implies immediately robustness results for IRFC segmentation that constitute one of the pillars of (I)RFC segmentation theory.

Notice that the results discussed in this subsection do not require affinities to be symmetric (not even in the case of (I)RFC segmentations), except where symmetry of affinities is an explicitly stated hypothesis. This is significant because almost all of the theory of (I)RFC-track FC segmentation has been developed solely for symmetric affinities.

2 RFC, IRFC, and MOFS Segmentations

2.1 Preliminaries

In the rest of this paper V will denote an arbitrary nonempty finite set. As in Sect. 1, we think of V as the set of all points of a digital image.

We have already defined the concepts of a W-path from a nonempty set A to a nonempty set B and also defined (for any affinity ψ on V) the ψ-strength of connectedness of A to B via a (possibly empty) set X (which is denoted by $\psi^X(A, B)$). For $a, b \in V$, a W-path from $\{a\}$ to $\{b\}$ will also be called a W-path from a to b. Similarly, we write $\psi^X(a, B)$, $\psi^X(A, b)$, and $\psi^X(a, b)$ for $\psi^X(\{a\}, B)$, $\psi^X(A, \{b\})$, and $\psi^X(\{a\}, \{b\})$, respectively. Note that $\psi(a, b) = \psi^{\emptyset}(a, b) \le \psi^X(a, b) \le \psi^V(a, b)$, that $\psi^X(A, B) = 1$ if $A \cap B \ne \emptyset$, and that $\psi^{\emptyset}(A, B) = \max_{a \in A, b \in B} \psi(a, b)$.

If p is a V-path of length $l \ge 0$ from u to v, and q is a V-path of length $l' \ge 0$ from v to w, then $p \cdot q$ will denote the V-path of length $l + l'$ from u to w that is obtained by concatenating p and q. Thus $p \cdot q$ is obtained by removing the initial point v from q and then appending the rest of q to p. (For example, $\langle a, b, c \rangle \cdot \langle c, d, e, f \rangle = \langle a, b, c, d, e, f \rangle$.) Note that $p \cdot q$ is undefined if the final point of p is different from the initial point of q. The $\cdot$ operation is associative on V-paths in the sense that if $p \cdot q$ and $q \cdot r$ are defined then $(p \cdot q) \cdot r = p \cdot (q \cdot r)$. A property of $\psi(p)$ which will be used many times in the sequel is that if the final point of each of the V-paths p' and p'' is the initial point of the V-path q, and $\psi(p') \ge \psi(p'')$, then $\psi(p' \cdot q) \ge \psi(p'' \cdot q)$. This follows from the fact that $\psi(p \cdot q) = \min(\psi(p), \psi(q))$ whenever $p \cdot q$ is defined.

The next two propositions state important properties of $\psi^X(A, B)$ that will be used later.

Proposition 2.1 *Let ψ be any affinity on V and let A, A', B, B', X, and X' be subsets of V such that $A \supseteq A'$, $B \supseteq B'$, and $X \cup A \cup B \supseteq X' \cup A' \cup B'$. Then $\psi^X(A, B) \ge \psi^{X'}(A', B')$.*

Proof The conclusion follows from the definition of $\psi^X(A, B)$, since the set of $(X \cup A \cup B)$-paths from A to B contains the set of $(X' \cup A' \cup B')$-paths from A' to B'. □

Proposition 2.2 *Let ψ be any affinity on V, let A, B, and X be subsets of V, and let $u \in X \cup A \cup B$. Then $\psi^X(A, B) \ge \min(\psi^{X \cup B}(A, u), \psi^{X \cup A}(u, B)) \ge \min(\psi^X(A, u), \psi^X(u, B))$. In particular, if $\psi^X(A, u) \ge \alpha$ and $\psi^X(u, B) \ge \alpha$, then $\psi^X(A, B) \ge \alpha$.*

Proof As the third sentence is an immediate consequence of the second we need only verify the second sentence. Let p_1 be an $(X \cup A \cup B)$-path from A to u such that $\psi(p_1) = \psi^{X \cup B}(A, u)$, let p_2 be an $(X \cup A \cup B)$-path from u to B such that $\psi(p_2) = \psi^{X \cup A}(u, B)$, and let p be the $(X \cup A \cup B)$-path $p_1 \cdot p_2$ from A to B. Then $\psi(p) = \min(\psi^{X \cup B}(A, u), \psi^{X \cup A}(u, B))$ and so the first inequality holds (by the definition of $\psi^X(A, B)$). The second inequality holds because $\psi^{X \cup B}(A, u) \ge \psi^X(A, u)$ and $\psi^{X \cup A}(u, B)) \ge \psi^X(u, B)$ (by Proposition 2.1). □

2.2 Simple Algorithms That Compute RFC, IRFC, and MOFS Segmentations

Algorithms 1–3 below compute the M-object RFC, IRFC, and MOFS segmentations of the set V for pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$ and an affinity ψ on V (in the RFC and IRFC cases) or M affinities $\psi_1, \ldots, \psi_M$ on V (in the MOFS case): In this paper we *define* RFC, IRFC, and MOFS segmentations as the results of these three algorithms.

These algorithms are not intended to be efficient. Rather, they are intended to be simple and concise, so as to give readers who are new to the subject a quick (yet completely accurate) understanding of the nature of the objects that are found by each of these types of segmentation. More efficient algorithms that compute the same segmentations will be presented in Sect. 4.

Recall that the seed sets $S_1, \ldots, S_M$ are said to be *consistent* with ψ or $\psi_1, \ldots, \psi_M$ just if there do *not* exist distinct S_i and S_j such that there is a V-path of ψ-strength or ψ_i-strength 1 from S_i to S_j. The seed sets would normally satisfy this consistency condition in all practical applications of FC segmentation that the authors are aware of. Assuming the condition is satisfied, the segmentations found by Algorithms 1–3 in this subsection will be the same as the segmentations found by RFC, IRFC, and MOFS algorithms in the literature; this will follow from the results of Sect. 3.3. (However, whereas the literature on RFC and IRFC segmentations has almost always assumed that the affinity ψ is symmetric, our discussion of these segmentations will not assume this.) Regardless of whether or not the consistency condition is satisfied, the objects $O_1, \ldots, O_M$ found by Algorithms 1–3 will have the property that $S_i \subseteq O_i \subseteq S_i \cup (V \setminus \bigcup_j S_j)$, for $1 \leq i \leq M$. But the objects found by some FC segmentation algorithms in the literature will not have this basic property if the seed sets do not satisfy the consistency condition.

The three algorithms can be formally stated as follows, where $|A|$ stands for the number of elements in the set A:

Algorithm 1: RFC Segmentation of a Nonempty Finite Set V into M Objects

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Data: M pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$; an affinity ψ on V
Result: The RFC segmentation $\langle O_1^{\mathrm{RFC}}, \ldots, O_M^{\mathrm{RFC}} \rangle$ of V

1 **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow S_i$
2 **sort** $A = \psi[V \times V] \setminus \{0\}$ **into** $1 = \alpha_1 > \ldots > \alpha_{|A|}$
3 **for** $n \leftarrow 1$ **to** $|A|$ **do** /* the main loop */
4   **for** $i \leftarrow 1$ **to** M **do**
        $newT_i \leftarrow T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi^V(T_i, v) \geq \alpha_n\}$
5   **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow newT_i$
6 **for** $i \leftarrow 1$ **to** M **do** $O_i^{\mathrm{RFC}} \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$

Algorithm 2: IRFC Segmentation of a Nonempty Finite Set V into M Objects

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Data: M pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$; an affinity ψ on V
Result: The IRFC segmentation $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}} \rangle$ of V

1 **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow S_i$
2 **sort** $A = \psi[V \times V] \setminus \{0\}$ **into** $1 = \alpha_1 > \ldots > \alpha_{|A|}$
3 **for** $n \leftarrow 1$ **to** $|A|$ **do** /* the main loop */
4   **for** $i \leftarrow 1$ **to** M **do**
        $newT_i \leftarrow T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi^{V \setminus \bigcup_j T_j}(T_i, v) \geq \alpha_n\}$
5   **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow newT_i$
6 **for** $i \leftarrow 1$ **to** M **do** $O_i^{\mathrm{IRFC}} \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$

Algorithm 3: MOFS Segmentation of a Nonempty Finite Set V into M Objects

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Data: M pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$; M affinities $\psi_1, \ldots, \psi_M$ on V
Result: The MOFS segmentation $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}} \rangle$ of V

1 **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow S_i$
2 **sort** $A = \bigcup_j \psi_j[V \times V] \setminus \{0\}$ **into** $1 = \alpha_1 > \ldots > \alpha_{|A|}$
3 **for** $n \leftarrow 1$ **to** $|A|$ **do** /* the main loop */
4   **for** $i \leftarrow 1$ **to** M **do**
        $newT_i \leftarrow T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi_i^{V \setminus \bigcup_j T_j}(T_i, v) \geq \alpha_n\}$
5   **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow newT_i$
6 **for** $i \leftarrow 1$ **to** M **do** $O_i^{\mathrm{MOFS}} \leftarrow T_i$

It will be obvious from even a cursory inspection of the three algorithms that they are extremely similar. For example, the only difference between the RFC and IRFC algorithms is on the fourth line, where $\psi^V(T_i, v)$ in the RFC algorithm is replaced by $\psi^{V \setminus \bigcup_j T_j}(T_i, v)$ in the IRFC algorithm. The IRFC and MOFS algorithms are also very similar. In RFC and IRFC segmentation we do not use different affinities for different objects. But if we assume that each of the M affinities $\psi_1, \ldots, \psi_M$ in the MOFS algorithm (i.e., Algorithm 3) is equal to the single affinity ψ in the IRFC algorithm (Algorithm 2), then the only difference between the IRFC and MOFS algorithms is that $O_i^{\mathrm{IRFC}} \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$ on the last line of the IRFC algorithm is replaced by $O_i^{\mathrm{MOFS}} \leftarrow T_i$ in the MOFS algorithm: Each IRFC object consists of those points of the corresponding MOFS object which do not lie in any of the other $M - 1$ MOFS objects.

The underlying idea of Algorithms 1–3 is to have the M objects compete with each other to occupy image points over a series of iterations. At any given time, each image point will either be unoccupied or be occupied by one or more objects. The set of points that are occupied by the ith object at that time will be referred to as the ith object's *territory* and is represented by the variable T_i. Thus a point is unoccupied if, and only if, that point lies in $V \setminus \bigcup_j T_j$.

A fundamental property of these algorithms is that, once a point has been occupied by one or more objects at an iteration of the main loop, the point will be occupied by just those objects during all subsequent iterations of that loop; it can never be occupied by any other object. Equivalently, a point that lies in $T_i \setminus T_j$ (for some i and j) at the end of one iteration of the main loop will lie in $T_i \setminus T_j$ during all subsequent iterations.

In each algorithm, the territory of each object is initialized to consist just of that object's seed points. Every iteration of the main loop expands the ith object's territory (for $1 \le i \le M$) to also include all currently unoccupied points which can be reached from the ith object's current territory via a V-path that has the following properties:

1. The ψ-strength (in the RFC and IRFC cases) or ψ_i-strength (in the MOFS case) of the V-path is not less than the affinity threshold used at that iteration.
2. In the IRFC and MOFS algorithms, the V-path is a $T_i \cup (V \setminus \bigcup_j T_j)$-path—i.e., each of its points either was already occupied by the ith object or was unoccupied at the beginning of the current iteration.

The affinity threshold in property 1 decreases from one iteration to the next: The affinity threshold at the nth iteration is α_n, which in the RFC and IRFC cases is the nth-largest nonzero value in the range of our affinity ψ; in the MOFS case α_n is the nth-largest nonzero value in the union of the ranges of our M affinities $\psi_1, \ldots, \psi_M$.

We mention that "$\ge \alpha_n$" on line 4 of all three algorithms really means "$= \alpha_n$," because no point v in $V \setminus \bigcup_j T_j$ can satisfy the condition with "$> \alpha_n$" in place of "$\ge \alpha_n$." (This fact will be more formally stated in Sects. 6.1 as (6.2)–(6.3) and (6.6)–(6.7).)

We have already mentioned that, for any given affinity ψ and seed sets consistent with the affinity, $O_i^{\mathrm{RFC}} \subseteq O_i^{\mathrm{IRFC}} \subseteq O_i^{\mathrm{MOFS}}$ for all $i \in \{1, \ldots, M\}$ (assuming the same affinity ψ is used for all objects in the MOFS segmentation). In the following simple example (for which the affinity is as shown in Fig. 2) we have that $O_1^{\mathrm{RFC}} \subsetneq O_1^{\mathrm{IRFC}} \subsetneq O_1^{\mathrm{MOFS}}$:

Example 2.3 Let $M = 2$, $V = \{d, s_1, c, s_2\}$, $S_1 = \{s_1\}$, and $S_2 = \{s_2\}$. Let ψ be the symmetric affinity on V such that $\psi(d, s_1) = \psi(s_1, d) = \psi(c, s_1) = \psi(s_1, c) = \psi(c, s_2) = \psi(s_2, c) = 0.5$ and $\psi(x, y) = 0$ in all other cases where x and y are distinct points of V. (See Fig. 2.) For Algorithm 3, let $\psi_1 = \psi_2 = \psi$. Then, readily, $O_1^{\mathrm{RFC}} = \{s_1\} \subsetneq O_1^{\mathrm{IRFC}} = \{s_1, d\} \subsetneq O_1^{\mathrm{MOFS}} = \{s_1, c, d\}$ and $O_2^{\mathrm{RFC}} = O_2^{\mathrm{IRFC}} = \{s_2\} \subsetneq O_2^{\mathrm{MOFS}} = \{s_2, c\}$.

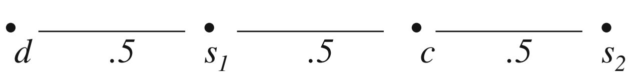

Fig. 2 Affinity values for Example 2.3

The *tie-zone* of an IRFC segmentation is the set of all points of V that do not lie in any of the objects of the segmentation. The *tie-zone* of an MOFS segmentation is the set of all points of V that do not lie in just one object of the segmentation. (We will not define the tie-zone of an RFC segmentation.) Looking again at Algorithms 2 and 3, we see that the tie-zone of the IRFC segmentation given by the affinity ψ and seed sets $S_1, \ldots, S_M$ is the same as the tie-zone of the corresponding MOFS segmentation (which is given by executing Algorithm 3 with the same seed sets and $\psi_1 = \cdots = \psi_M = \psi$) and consists of the points that lie in two or more of the MOFS objects as well as the points that lie in none of those MOFS objects. (It is fairly easy to see from Algorithm 3 that a point v lies in none of those MOFS objects if, and only if, there is no V-path from $\bigcup_j S_j$ to v of nonzero ψ-strength.[6])

2.3 Simple Nonalgorithmic Characterizations and Related Properties of RFC, IRFC, and MOFS Segmentations

While the RFC, IRFC, and MOFS segmentations of V have been defined as the segmentations created by Algorithms 1, 2, and 3 above, it is not difficult to characterize these segmentations nonalgorithmically. Statement 1 of each of Theorems 2.4–2.6 below shows how this can be done in terms of the values assumed by the variables $T_1, \ldots, T_M$ during execution of Algorithms 1, 2, and 3: Statement 1(a) of each theorem shows how the sequence $T_i^0 \subseteq T_i^1 \subseteq \cdots \subseteq T_i^{|A|}$ of values assumed by each variable T_i could be defined inductively, without reference to Algorithms 1–3, while statement 1(b) of each theorem shows how the objects of the corresponding segmentation are determined by the final values $T_1^{|A|}, \ldots, T_M^{|A|}$ of $T_1, \ldots, T_M$. These characterizations of the RFC, IRFC, and MOFS segmentations will play an important role in our derivations of the concise path-based characterizations of the same segmentations that we present in Sect. 3.

The rest of Theorems 2.4–2.6 state related properties of the segmentations. Statement 2 of each theorem implies that "$= \alpha_n$" in the second part of statement 1(a) could be replaced by "$\ge \alpha_n$." In Theorems 2.5 and 2.6, statement 3 gives variants of the second part of statement 1(a) that are more concise but cannot be used to give an inductive definition of T_i^n (as the right sides of these variants involve T_i^k for some $k \ge n$).

Theorems 2.4–2.6 will be proved in Sect. 6.1.

Theorem 2.4 *Let $\langle O_1^{\mathrm{RFC}}, \ldots, O_M^{\mathrm{RFC}} \rangle$ be the segmentation of V found by Algorithm 1 for an affinity ψ on V and pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$. Let*

[6] To see the "only if" part, consider the final iteration of Algorithm 3's main loop in the case $\psi_1 = \cdots = \psi_M = \psi$ and then observe that $\psi(u, v) = 0$ whenever u lies in an MOFS object but v lies in no MOFS object.

$A = \psi[V \times V]\backslash\{0\}$ and let $1 = \alpha_1 > \ldots > \alpha_{|A|}$ be the sequence obtained by sorting A into decreasing order. For $1 \leq i \leq M$ and $0 \leq n < |A|$, let T_i^n be the value of the variable T_i at the beginning of the $n+1$st iteration of the main loop when Algorithm 1 is executed, and let $T_i^{|A|}$ be the value of T_i at the end of the $|A|$th iteration of the main loop (which is the value of T_i when Algorithm 1 terminates). Then $S_i \subseteq O_i^{\mathrm{RFC}} \subseteq S_i \cup (V\backslash \bigcup_j S_j)$ for $1 \leq i \leq M$. Moreover:

1. For $1 \leq i \leq M$ we have that
   (a) $T_i^0 = S_i$, and, for every $1 \leq n \leq |A|$, $T_i^n = T_i^{n-1} \cup \{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) = \alpha_n\}$.
   (b) $O_i^{\mathrm{RFC}} = T_i^{|A|}\backslash \bigcup_{j\neq i} T_j^{|A|}$.
2. $\{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) > \alpha_n\} = \emptyset$ for every $1 \leq i \leq M$ and $1 \leq n \leq |A|$.

Theorem 2.5 *Let $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}}\rangle$ be the segmentation of V found by Algorithm 2 for an affinity ψ on V and pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$. Let $A = \psi[V \times V]\backslash\{0\}$ and let $1 = \alpha_1 > \ldots > \alpha_{|A|}$ be the sequence obtained by sorting A into decreasing order. For $1 \leq i \leq M$ and $0 \leq n < |A|$, let T_i^n be the value of the variable T_i at the beginning of the $n+1$st iteration of the main loop when Algorithm 2 is executed, and let $T_i^{|A|}$ be the value of T_i at the end of the $|A|$th iteration of the main loop (which is the value of T_i when Algorithm 2 terminates). Then $S_i \subseteq O_i^{\mathrm{IRFC}} \subseteq S_i \cup (V\backslash \bigcup_j S_j)$ for $1 \leq i \leq M$. Moreover:*

1. *For $1 \leq i \leq M$ we have that*
   (a) $T_i^0 = S_i$, and $T_i^n = T_i^{n-1} \cup \{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi^{T_i^{n-1}\cup(V\backslash \bigcup_j T_j^{n-1})}(S_i, v) = \alpha_n\}$ for $1 \leq n \leq |A|$.
   (b) $O_i^{\mathrm{IRFC}} = T_i^{|A|}\backslash \bigcup_{j\neq i} T_j^{|A|}$.
2. *$\{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi^{T_i^{n-1}\cup(V\backslash \bigcup_j T_j^{n-1})}(S_i, v) > \alpha_n\} = \emptyset$ for $1 \leq i \leq M$ and $1 \leq n \leq |A|$.*
3. *$T_i^n = T_i^{n-1} \cup \{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi^{T_i^k}(S_i, v) = \alpha_n\}$ for $1 \leq i \leq M$ and $1 \leq n \leq k \leq |A|$.*

Theorem 2.6 *Let $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}}\rangle$ be the segmentation of V found by Algorithm 3 for affinities $\psi_1, \ldots, \psi_M$ on V and pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$. Let $A = \bigcup_j \psi_j[V \times V]\backslash\{0\}$ and let $1 = \alpha_1 > \ldots > \alpha_{|A|}$ be the sequence obtained by sorting A into decreasing order. For $1 \leq i \leq M$ and $0 \leq n < |A|$, let T_i^n be the value of the variable T_i at the beginning of the $n+1$st iteration of the main loop when Algorithm 3 is executed, and let $T_i^{|A|}$ be the value of T_i at the end of the $|A|$th iteration of the main loop (which is the value of T_i when Algorithm 3 terminates). Then $S_i \subseteq O_i^{\mathrm{MOFS}} \subseteq S_i \cup (V\backslash \bigcup_j S_j)$ for $1 \leq i \leq M$. Moreover:*

1. *For $1 \leq i \leq M$ we have that*
   (a) $T_i^0 = S_i$, and $T_i^n = T_i^{n-1} \cup \{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi_i^{T_i^{n-1}\cup(V\backslash \bigcup_j T_j^{n-1})}(S_i, v) = \alpha_n\}$ for $1 \leq n \leq |A|$.
   (b) $O_i^{\mathrm{MOFS}} = T_i^{|A|}$.
2. *$\{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi_i^{T_i^{n-1}\cup(V\backslash \bigcup_j T_j^{n-1})}(S_i, v) > \alpha_n\} = \emptyset$ for $1 \leq i \leq M$ and $1 \leq n \leq |A|$.*
3. *$T_i^n = T_i^{n-1} \cup \{v \in V\backslash \bigcup_j T_j^{n-1} \mid \psi_i^{T_i^k}(S_i, v) = \alpha_n\}$ for $1 \leq i \leq M$ and $1 \leq n \leq k \leq |A|$.*

2.4 An MOFS Algorithm Can be Used to Compute IRFC Segmentations

IRFC segmentations can be easily found using any algorithm that computes MOFS segmentations. This follows from the following corollary, which is an immediate consequence of statement 1 of Theorem 2.5 and statement 1 of Theorem 2.6 (and which is also evident from a quick inspection of Algorithms 2 and 3):

Corollary 2.7 *Let $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}}\rangle$ be the segmentation of V found by Algorithm 2 for an affinity ψ on V and pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$. Let $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}}\rangle$ be the segmentation of V found by Algorithm 3 for the same M seed sets $S_1, \ldots, S_M$ in the case where each of the M affinities $\psi_1, \ldots, \psi_M$ is the affinity ψ. Then $O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}}\backslash \bigcup_{j\neq i} O_j^{\mathrm{MOFS}}$ for $1 \leq i \leq M$.*

IRFC segmentations have commonly been computed one object at a time (e.g., by executing the $\mathrm{GC}_{\max}$ or IRFC-IFT algorithm of [20] a total of M times as explained in footnote 14 below), whereas MOFS segmentations have been computed using methods akin to Dijkstra's shortest path algorithm [24] to compute all of the M objects simultaneously. Algorithm 5 below is an MOFS algorithm of this kind, and when $M > 2$ it is likely to be more efficient to compute an M-object IRFC segmentation using this MOFS algorithm than using a one-object-at-a-time IRFC algorithm M times, except possibly when a large proportion (e.g., more than 20 %) of the points of V lie in the IRFC segmentation's tie-zone. Moreover, in cases where the tie-zone of an IRFC segmentation does constitute such a large fraction of V, the segmentation is unstable with respect to tiny changes in affinity values, as we will see in Sect. 3.4.

Using an MOFS algorithm to compute the IRFC segmentation provides additional information about the segmentation's tie-zone, because it identifies the IRFC objects (if any) that are "in contention for" each tie-zone point: Once the MOFS segmentation has been computed, we may regard an IRFC object as being in contention for a tie-zone point p just if p belongs to the corresponding MOFS object.

As mentioned earlier, in the case where the affinity is symmetric the TZWS by union-find method of [2], which we will discuss further in Sect. 4.2, is another method of computing all the objects of an IRFC segmentation simultaneously.

2.5 Perturbing MOFS Affinity Values to Eliminate Small Overlaps Between Objects, and Instability of MOFS Segmentations in Which Different Objects Overlap Substantially

An advantage of MOFS segmentation over (I)RFC segmentation is that the former allows different affinities to be used for different objects. But distinct objects of an MOFS segmentation may possibly intersect, whereas the objects of any RFC or IRFC segmentation are pairwise disjoint.

In many applications of MOFS segmentation, any intersections of distinct objects are so small as to be negligible. However, complete disjointness of objects may be desirable (e.g., because we wish to use software tools which assume different objects never intersect). For this reason one might consider "hybrid" segmentations that would be produced by a modified version of Algorithm 3 in which the assignment $O_i^{\text{MOFS}} \leftarrow T_i$ is replaced by $O_i \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$. While the objects O_i of the resulting segmentations would of course be pairwise disjoint, a drawback of this kind of segmentation would be that the objects might sometimes have parts which are *disconnected* from their seeds: The ith object O_i might contain points which *cannot* be reached from its seed set S_i via an O_i-path whose ψ_i-strength is nonzero. A simple example of this is shown in Fig. 3. As mentioned above, the same thing cannot happen in RFC, IRFC, and MOFS segmentations.

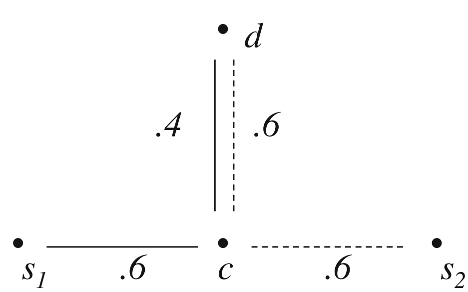

Fig. 3 Let $V = \{s_1, s_2, c, d\}$, $M = 2$, $S_1 = \{s_1\}$, and $S_2 = \{s_2\}$. Let ψ_1 and ψ_2 be the symmetric affinities on V such that $\psi_1(c, d) = \psi_1(d, c) = 0.4$, $\psi_1(s_1, c) = \psi_1(c, s_1) = \psi_2(s_2, c) = \psi_2(c, s_2) = \psi_2(c, d) = \psi_2(d, c) = 0.6$, and $\psi_i(x, y) = 0$ in all other cases where $i \in \{1, 2\}$ and x and y are distinct points in V. Then execution of Algorithm 3 terminates with $T_1 = \{s_1, c\}$ and $T_2 = \{s_2, c, d\}$. So if in Algorithm 3 we replace $O_i^{\text{MOFS}} \leftarrow T_i$ with $O_i \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$ then the resulting algorithm will produce an object $O_2 = T_2 \setminus T_1 = \{s_2, d\}$ that is "disconnected" (as $c \notin O_2$ and so there is no O_2-path of nonzero ψ_2-strength from S_2 to $d \in O_2$)

Another way to eliminate small intersections of distinct objects, which does not have the above-mentioned drawback, is to perturb very slightly the MOFS affinity values $\psi_i(u, v)$ in such a way that the resulting perturbed values $\psi'_i(u, v)$ satisfy the condition

$$\{\psi'_{i_1}(u, v) \mid u, v \in V \text{ and } u \neq v\} \cap \{\psi'_{i_2}(u, v) \mid u, v \in V \text{ and } u \neq v\} \setminus \{0\} = \emptyset \text{ if } i_1 \neq i_2. \tag{2.1}$$

For example, if affinity values are represented as floating point binary values and $M = 4$, and if $\psi_i(u, v) < 1$ $(1 \leq i \leq 4)$ whenever $u \neq v$,[7] then whenever $u \neq v$ and $\psi_i(u, v) \neq 0$ we can set the two least significant bits of $\psi_i(u, v)$ to 00, 01, 10, or 11 according to whether $i = 1$, 2, 3, or 4. The condition (2.1) ensures that the objects of the MOFS segmentation computed from the perturbed affinity values will be pairwise disjoint.[8]

This method is appropriate for any application in which MOFS segmentations are stable with respect to tiny changes in affinity values—i.e., any application in which tiny changes in affinity values are very unlikely to appreciably affect the segmentations that are produced.

We can also conclude from the same line of thought that different objects of an MOFS segmentation cannot overlap in any substantial way if the segmentation is stable in this sense (because overlapping of objects can always be entirely eliminated by making arbitrarily small changes in the affinity

[7] The condition that $\psi_i(u, v) < 1$ $(1 \leq i \leq M)$ whenever $u \neq v$ is needed only because we do not want any perturbed affinity value to exceed 1. From a mathematical perspective there is no real loss of generality when we assume this condition. Indeed, if the condition is not satisfied we can define affinities $\psi_i^*(u, v)$ such that $\psi_i^*(u, v) = \psi_i(u, v)/2$ whenever $u \neq v$, and then use the ψ^*s in place of the ψs: The ψ^*s evidently satisfy the condition, and it is not hard to see from Algorithm 3 that using the ψ^*s in place of the ψs will not change the segmentations that are produced, because for all V-paths p and q of nonzero length and all $i, j \in \{1, \ldots, M\}$ we have that $\psi_i^*(p) < \psi_j^*(q)$ if and only if $\psi_i(p) < \psi_j(q)$. (See [17, Prop. 1] for more examples of different affinities that are equivalent for FC segmentation purposes.) From a computational perspective, we mention that many MOFS segmentation algorithms, including our Algorithm 5 below, can be easily modified to produce correct segmentations even if some affinity values exceed 1. If we use a modified algorithm of this kind, then the perturbation described here can be applied even if the condition is not satisfied.

[8] Indeed, suppose not: Suppose $v \in O_i^{\text{MOFS}} \cap O_j^{\text{MOFS}}$ where $\langle O_1^{\text{MOFS}}, \ldots, O_M^{\text{MOFS}} \rangle$ is the MOFS segmentation derived from $\psi'_1, \ldots, \psi'_M$ and pairwise disjoint seed sets $S_1, \ldots, S_M$, and $i \neq j$. Then we see from the main loop of Algorithm 3 that v must be incorporated into T_i and T_j at the same iteration of that loop. Assuming v is incorporated into T_i and T_j at the nth iteration of the loop, and using the notation of Theorem 2.6, we have that $v \in T_i^n \setminus T_i^{n-1}$ and $v \in T_j^n \setminus T_j^{n-1}$, whence we see from Theorem 2.6 that there is a V-path from S_i to v of ψ'_i-strength α_n and there is also a V-path from S_j to v of ψ'_j-strength α_n. But this would imply $\alpha_n \in \{\psi'_i(u, v) \mid u, v \in V \text{ and } u \neq v\} \cap \{\psi'_j(u, v) \mid u, v \in V \text{ and } u \neq v\}$, which contradicts (2.1).

values). Equivalently, if different MOFS objects have substantial overlap, then the MOFS segmentation is unstable.

Affinity perturbation as described here is proposed as a method of eliminating overlaps between objects only when all such overlaps are small; it is *not* recommended when different objects have substantial overlap.

An unexpected substantial overlap of MOFS objects may indicate that an affinity or seed was poorly chosen. In any application where different MOFS objects would be expected to overlap substantially, the instability of such MOFS segmentations with respect to tiny changes in affinity values should be borne in mind when considering whether MOFS is a suitable method for that application.

In Sect. 3.4 we will see that these remarks about substantial overlaps of objects in MOFS segmentations apply as well to large tie-zones in IRFC segmentations, because any IRFC segmentation that has a large tie-zone is also unstable with respect to tiny changes in affinity values.[9]

3 Concise Path-Based Characterizations of Fuzzy Connectedness Segmentations

In Sect. 2 we defined RFC, IRFC, and MOFS segmentations as the segmentations produced by Algorithms 1–3, and also characterized each of the segmentations mathematically in terms of inductively defined sets. In this section, we give more concise path-based mathematical characterizations of these segmentations, both for their independent interest and because other properties of the segmentations can be conveniently deduced from these characterizations.

3.1 (ψ, S)-Optimal V-paths and Hereditarily (ψ, S)-Optimal V-paths

As we shall see in Theorems 3.6 and 3.8 below, RFC and IRFC segmentations can be characterized in terms of the concepts of *optimal* and *hereditarily optimal* paths which we now introduce.

Let $\psi : V \times V \to [0, 1]$ be an affinity and let $S \subseteq V$. Then we say that a V-path $p = \langle v_0, \ldots, v_l\rangle$ is (ψ, S)-*optimal* if $v_0 \in S$ and $\psi(p) = \psi^V(S, v_l)$, and we say p is *hereditarily* (ψ, S)-*optimal* if p and all nonempty proper initial segments of p are (ψ, S)-optimal. Thus a V-path $p = \langle v_0, \ldots, v_l\rangle$ is hereditarily (ψ, S)-optimal just if $v_0 \in S$ and $\psi(\langle v_0, \ldots, v_k\rangle) = \psi^V(S, v_k)$ for $1 \le k \le l$. When ψ is symmetric, a V-path is hereditarily (ψ, S)-optimal just if its reverse is what [21] calls a *nice* path in V to S.

[9] Note that this cannot be shown by considering affinity perturbations of the kind we have discussed in the above paragraphs, because IRFC segmentation uses just a single affinity.

If $S \neq \emptyset$ then for any v in V it is evident that there is at least one (ψ, S)-optimal V-path to v, and we will see from the next proposition that there is at least one hereditarily (ψ, S)-optimal V-path to v.

Proposition 3.1 *Let $\psi : V \times V \to [0, 1]$ be an affinity and let $\emptyset \neq S \subseteq V$. Then for each $v \in V$ there exists a hereditarily (ψ, S)-optimal V-path to v.*

This follows easily from Lemma 3.3 in [21] (whose proof is readily confirmed to be valid even if the affinity is not symmetric). For the convenience of readers we will give a self-contained proof of the proposition here.

Proof of Proposition. Suppose the proposition is false. Among those points $v \in V$ for which no V-path to v is hereditarily (ψ, S)-optimal pick a point v^* for which $\psi^V(S, v^*)$ is maximal, and let $\langle v_0, \ldots, v_l\rangle$ be a V-path from S to $v_l = v^*$ such that $\psi(\langle v_0, \ldots, v_l\rangle) = \psi^V(S, v_l)$. Let k be the greatest index for which $\psi(\langle v_0, \ldots, v_k\rangle) \neq \psi^V(S, v_k)$; k exists since $\langle v_0, \ldots, v_l\rangle$ cannot be hereditarily (ψ, S)-optimal. Now $\psi^V(S, v_k) > \psi(\langle v_0, \ldots, v_k\rangle) \ge \psi(\langle v_0, \ldots, v_l\rangle) = \psi^V(S, v_l)$, so it follows from our definition of $v_l = v^*$ that there exists a V-path p to v_k such that p is a hereditarily (ψ, S)-optimal V-path. For all $k' \in \{k+1, \ldots, l\}$ let $p_{k'}$ be the V-path $p \cdot \langle v_k, \ldots, v_{k'}\rangle$ from S to $v_{k'}$. Then, since $\psi(p) = \psi^V(S, v_k) > \psi(\langle v_0, \ldots, v_k\rangle)$, for all $k' \in \{k+1, \ldots, l\}$ we have that $\psi(p_{k'}) = \psi(p \cdot \langle v_k, \ldots, v_{k'}\rangle) \ge \psi(\langle v_0, \ldots, v_k\rangle \cdot \langle v_k, \ldots, v_{k'}\rangle) = \psi(\langle v_0, \ldots, v_{k'}\rangle) = \psi^V(S, v_{k'}) \ge \psi(p_{k'})$ (where the equality $\psi(\langle v_0, \ldots, v_{k'}\rangle) = \psi^V(S, v_{k'})$ follows from the definition of k), which implies $\psi(p_{k'}) = \psi^V(S, v_{k'})$. From this (and the fact that p is a hereditarily (ψ, S)-optimal V-path) it follows that the V-path p_l is a hereditarily (ψ, S)-optimal V-path to $v_l = v^*$, a contradiction. □

3.2 $\Psi_{\mathcal{S}}(p)$, Recursively $(\Psi, \mathcal{S})$-Optimal V-paths, and the Sets $O_m^{\Psi,\mathcal{S}}$

As mentioned above (and as we will see in the next subsection) RFC and IRFC segmentations can be characterized in terms of the concepts of optimal and hereditarily optimal paths, which raises the question of whether MOFS segmentations can be characterized in a similar way.

The answer is "yes," but it is not immediately obvious how this can be done because optimal and hereditarily optimal paths are defined for a single affinity ψ, whereas MOFS segmentation uses M affinities $\psi_1, \ldots, \psi_M$: In MOFS segmentation each of the M objects has its own affinity, which is used to define the strengths of paths from that object's seed set. The main purpose of this subsection is to introduce the concept of a *recursively optimal* path, which generalizes the concept of a hereditarily optimal path to this broader context.

Our definition below of recursive optimality may not seem, at first sight, to be a very natural generalization of hereditary optimality. But the definition will be justified by Theorem 3.10, which will characterize MOFS segmentations in terms of recursively optimal paths in a way that is clearly similar to our characterizations of RFC and IRFC segmentations in terms of optimal and hereditarily optimal paths. Just as importantly, we will find that it is a definition which is convenient to use in inductive arguments.

Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be a sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ a sequence of (pairwise disjoint nonempty) seed sets. For any V-path p from $\bigcup_j S_j$ we define the $(\Psi, \mathcal{S})$-*strength* of p to be $\psi_m(p)$, where m is the unique element of $\{1, \ldots, M\}$ such that p is a V-path from S_m; this value will be denoted by $\Psi_{\mathcal{S}}(p)$. Note that $\Psi_{\mathcal{S}}(p)$ is undefined if the V-path p is not a V-path from $\bigcup_j S_j$.

Now there is an easy but rather unsatisfactory way to generalize our concept of a hereditarily (ψ, S)-optimal V-path: We first define $\Psi_{\mathcal{S}}(v)$ for each $v \in V$ to be the maximum value attained by $\Psi_{\mathcal{S}}(q)$ as q ranges over all V-paths from $\bigcup_j S_j$ to v. Then we say that a V-path $\langle v_0, \ldots, v_l \rangle$ is $(\Psi, \mathcal{S})$-optimal if $v_0 \in \bigcup_j S_j$ and $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_l \rangle) = \Psi_{\mathcal{S}}(v_l)$, and say that a V-path p is *hereditarily* $(\Psi, \mathcal{S})$-*optimal* if p and all nonempty proper initial segments of p are $(\Psi, \mathcal{S})$-optimal. Unfortunately, this more general concept turns out to be less useful than the single-affinity concept of hereditary (ψ, S)-optimality, because Proposition 3.1 fails to generalize to hereditarily $(\Psi, \mathcal{S})$-optimal V-paths: It is *not* true that for each $v \in V$ there must exist a hereditarily $(\Psi, \mathcal{S})$-optimal V-path to v, as the following example illustrates:

Example 3.2 Let $V = \{s_1, s_2, c, d\}$, $M = 2$, and $\mathcal{S} = \langle S_1, S_2 \rangle$, where $S_1 = \{s_1\}$ and $S_2 = \{s_2\}$. Let $\Psi = \langle \psi_1, \psi_2 \rangle$, where ψ_1 and ψ_2 are the symmetric affinities on V defined by $\psi_1(s_1, c) = \psi_1(c, s_1) = 0.7$, $\psi_1(c, d) = \psi_1(d, c) = 0.4$, $\psi_2(s_2, c) = \psi_2(c, s_2) = \psi_2(c, d) = \psi_2(d, c) = 0.6$, and $\psi_i(x, y) = 0$ in all other cases where $i \in \{1, 2\}$ and x and y are distinct points in V. (See Fig. 4.) Then no V-path to d is hereditarily $(\Psi, \mathcal{S})$-optimal: Indeed, if p is a $(\Psi, \mathcal{S})$-optimal V-path to d then p is a V-path from s_2, so p has an initial segment that ends at c and is *not* $(\Psi, \mathcal{S})$-optimal (because $\Psi_{\mathcal{S}}(\langle s_2, c \rangle) = 0.6 < 0.7 = \Psi_{\mathcal{S}}(\langle s_1, c \rangle)$).

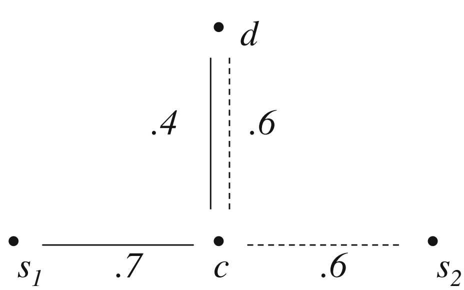

Fig. 4 Scene and affinity values for Example 3.2

It is readily confirmed (by inspection of Algorithm 3) that in this example the objects of the MOFS segmentation are $O_1 = \{s_1, c, d\}$ and $O_2 = \{s_2\}$: Thus $d \in O_1$, even though there is no $(\Psi, \mathcal{S})$-optimal V-path from S_1 to d because $\psi_1^V(S_1, d) = 0.4 < \Psi_{\mathcal{S}}(d) = \psi_2^V(S_2, d) = 0.6$. Examples such as this one strongly suggest that MOFS segmentations do not have simple characterizations in terms of $(\Psi, \mathcal{S})$-optimal and hereditarily $(\Psi, \mathcal{S})$-optimal V-paths.

We now define the class of *recursively* $(\Psi, \mathcal{S})$-*optimal* V-paths, which generalizes the class of hereditarily (ψ, S)-optimal V-paths in a more useful way. Recursively $(\Psi, \mathcal{S})$-optimal V-paths will be defined by induction on their $(\Psi, \mathcal{S})$-strength, starting with the strongest such paths (those of $(\Psi, \mathcal{S})$-strength 1).

For this purpose, let $1 = \alpha_1 > \cdots > \alpha_{|A|}$ be (as in Algorithm 3) the sequence obtained by sorting the finite set $A = \bigcup_j \psi_j[V \times V] \backslash \{0\}$ into decreasing order, and define $\alpha_{|A|+1} = 0$. (It follows that the $(\Psi, \mathcal{S})$-strength of every V-path is α_n for some $n \in \{1, \ldots, |A|, |A| + 1\}$.)

We define the recursively $(\Psi, \mathcal{S})$-optimal V-paths of $(\Psi, \mathcal{S})$-strength $\alpha_1 = 1$ to be just the V-paths from $\bigcup_j S_j$ whose $(\Psi, \mathcal{S})$-strength is 1: Every V-path from $\bigcup_j S_j$ whose $(\Psi, \mathcal{S})$-strength is 1 is recursively $(\Psi, \mathcal{S})$-optimal. Once we have defined the recursively $(\Psi, \mathcal{S})$-optimal V-paths of $(\Psi, \mathcal{S})$-strength $> \alpha_n$ for some $n \in \{2, \ldots, |A| + 1\}$, we define the recursively $(\Psi, \mathcal{S})$-optimal V-paths of $(\Psi, \mathcal{S})$-strength α_n to be the V-paths $\langle v_0, \ldots, v_l \rangle$ from $\bigcup_j S_j$ of $(\Psi, \mathcal{S})$-strength α_n that satisfy the following condition:

- For $1 \le k \le l$ there is no recursively $(\Psi, \mathcal{S})$-optimal V-path to v_k of $(\Psi, \mathcal{S})$-strength $> \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k \rangle)$.

Note that in this condition "recursively $(\Psi, \mathcal{S})$-optimal V-path to v_k of $(\Psi, \mathcal{S})$-strength $> \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k \rangle)$" has already been defined because $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k \rangle) \ge \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_l \rangle) = \alpha_n$ and we are assuming we have already defined the recursively $(\Psi, \mathcal{S})$-optimal V-paths of $(\Psi, \mathcal{S})$-strength $> \alpha_n$.

More formally, a V-path $\langle v_0, \ldots, v_l \rangle$ is *recursively* $(\Psi, \mathcal{S})$-*optimal* if (and only if) the following condition $\mathbf{RO}_\Psi^{\mathcal{S}}(\langle v_0, \ldots, v_l \rangle)$ holds:

> $\mathbf{RO}_\Psi^{\mathcal{S}}(\langle v_0, \ldots, v_l \rangle)$: $v_0 \in \bigcup_j S_j$ and, for $1 \le k \le l$, no V-path p to v_k satisfies both $\mathbf{RO}_\Psi^{\mathcal{S}}(p)$ and $\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k \rangle)$.

It is evident that any two recursively $(\Psi, \mathcal{S})$-optimal V-paths to the same point must have the same $(\Psi, \mathcal{S})$-strength. It is also evident that any nonempty initial segment of a recursively $(\Psi, \mathcal{S})$-optimal V-path is itself a recursively $(\Psi, \mathcal{S})$-optimal V-path. We will see in Corollary 3.5 that this concept is indeed a generalization of the concept of a hereditarily (ψ, S)-optimal V-path.

For V-paths p from $\bigcup_j S_j$ of $(\Psi, \mathcal{S})$-strength <1, the property of being recursively $(\Psi, \mathcal{S})$-optimal can be characterized in terms of p's *longest initial segment of greater strength*: Let $\langle v_0, \ldots, v_l \rangle$ be any V-path from $\bigcup_j S_j$ such that $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_l \rangle) = \alpha < 1$, and let $\langle v_0, \ldots, v_r \rangle$ be the *longest* initial segment of $\langle v_0, \ldots, v_l \rangle$ such that $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_r \rangle) > \alpha$ (so that $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k \rangle) = \alpha$ for each $k \in \{r+1, \ldots, l\}$). Then, readily, $\langle v_0, \ldots, v_l \rangle$ is a recursively $(\Psi, \mathcal{S})$-optimal V-path if and only if both of the following are true:

1. $\langle v_0, \ldots, v_r \rangle$ is a recursively $(\Psi, \mathcal{S})$-optimal V-path.
2. No v_k with $r < k \le l$ is the last point of a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength $> \alpha$.

As mentioned above, one weakness of the concept of a hereditarily $(\Psi, \mathcal{S})$-optimal V-path as a generalization of the concept of a hereditarily (ψ, S)-optimal V-path is that there may be points $v \in V$ for which there is no hereditarily $(\Psi, \mathcal{S})$-optimal V-path to v, as we showed in Example 3.2. But it is very easy to see that the concept of a recursively $(\Psi, \mathcal{S})$-optimal V-path does not have this drawback:

Proposition 3.3 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ any sequence of pairwise disjoint nonempty subsets of V. Then for each $v \in V$ there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path to v.*

Proof Let v_0 be any point in $\bigcup_j S_j$. Then for each $v \in V$ *either* $\mathbf{RO}^{\mathcal{S}}_{\Psi}(\langle v_0, v \rangle)$ holds *or* $\mathbf{RO}^{\mathcal{S}}_{\Psi}(p)$ holds for some V-path p to v such that $\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle v_0, v \rangle)$. □

For any sequence of affinities $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ on V, any sequence $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ of pairwise disjoint nonempty subsets of V, and every $m \in \{1, \ldots, M\}$ we define

$$O_m^{\Psi,\mathcal{S}} = \{v \in V \mid \text{there exists a recursively } (\Psi, \mathcal{S})\text{-optimal } V\text{-path from } S_m \text{ to } v\}.$$

Evidently, $S_m \subseteq O_m^{\Psi,\mathcal{S}}$, and $O_m^{\Psi,\mathcal{S}} \subseteq V \setminus \bigcup_{i \neq m} S_i$ if $S_1, \ldots, S_M$ are consistent with the affinities $\psi_1, \ldots, \psi_M$. Moreover, Proposition 3.3 implies $\bigcup_j O_j^{\Psi,\mathcal{S}} = V$. This is an important concept, because we will see later (from Theorem 3.10) that the mth object found by MOFS segmentation with affinities $\psi_1, \ldots, \psi_M$ and seed sets $S_1, \ldots, S_M$ that are consistent with the affinities is the set $\{v \in O_m^{\Psi,\mathcal{S}} \mid \psi_m^{O_m^{\Psi,\mathcal{S}}}(S_m, v) > 0\}$.

Our next theorem tells us that if $\mathcal{S}$ is consistent with Ψ, then the recursively $(\Psi, \mathcal{S})$-optimal V-paths from S_m are just the hereditarily $(\psi_m|_{O_m^{\Psi,\mathcal{S}} \times O_m^{\Psi,\mathcal{S}}}, S_m)$-optimal $O_m^{\Psi,\mathcal{S}}$-paths. Here $\psi_m|_{O_m^{\Psi,\mathcal{S}} \times O_m^{\Psi,\mathcal{S}}}$ denotes the affinity on $O_m^{\Psi,\mathcal{S}}$ that is obtained by restricting the affinity ψ_m to $O_m^{\Psi,\mathcal{S}} \times O_m^{\Psi,\mathcal{S}}$.[10]

Theorem 3.4 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ any sequence of pairwise disjoint nonempty subsets of V that are consistent with the affinities. Then, for all $m \in \{1, \ldots, M\}$:*

1. *A recursively $(\Psi, \mathcal{S})$-optimal V-path is an $O_m^{\Psi,\mathcal{S}}$-path just if it is a V-path from S_m.*
2. *An $O_m^{\Psi,\mathcal{S}}$-path is a recursively $(\Psi, \mathcal{S})$-optimal V-path just if it is a hereditarily $(\psi_m|_{O_m^{\Psi,\mathcal{S}} \times O_m^{\Psi,\mathcal{S}}}, S_m)$-optimal $O_m^{\Psi,\mathcal{S}}$-path.*
3. *For all $v \in O_m^{\Psi,\mathcal{S}}$, the $(\Psi, \mathcal{S})$-strength of every recursively $(\Psi, \mathcal{S})$-optimal V-path to v is $\psi_m^{O_m^{\Psi,\mathcal{S}}}(S_m, v)$.*

A proof of this theorem will be given in Sect. 6.2. One important implication of this result is that in the case where $\psi_1 = \cdots = \psi_M = \psi$ (i.e., the same affinity ψ is used for each of the M objects, as in (I)RFC segmentation) recursive $(\Psi, \mathcal{S})$-optimality is equivalent to hereditary $(\psi, \bigcup_j S_j)$-optimality:

Corollary 3.5 *Let $\Psi^{\psi} = \langle \psi, \ldots, \psi \rangle$ be a sequence of M occurrences of the same affinity ψ on V, and let $\langle S_1, \ldots, S_M \rangle$ be any sequence of pairwise disjoint nonempty subsets of V. Then a V-path is recursively $(\Psi^{\psi}, \langle S_1, \ldots, S_M \rangle)$-optimal just if it is hereditarily $(\psi, \bigcup_j S_j)$-optimal.*

Proof Since $\Psi^{\psi}_{\langle S_1, \ldots, S_M \rangle}(p) = \psi(p)$ for every V-path p from $\bigcup_j S_j$, we see that a V-path is recursively $(\Psi^{\psi}, \langle S_1, \ldots, S_M \rangle)$-optimal just if it is recursively $(\langle \psi \rangle, \langle \bigcup_j S_j \rangle)$-optimal. Putting $\Psi = \langle \psi \rangle$, $\mathcal{S} = \langle \bigcup_j S_j \rangle$, and $m = M = 1$, we see from Proposition 3.3 that $O_1^{\langle \psi \rangle, \langle \bigcup_j S_j \rangle} = V$ and then see from statement 2 of Theorem 3.4 that a V-path is recursively $(\langle \psi \rangle, \langle \bigcup_j S_j \rangle)$-optimal just if it is hereditarily $(\psi, \bigcup_j S_j)$-optimal. □

3.3 Path-Based Characterizations of RFC, IRFC, and MOFS Segmentations

The following theorem gives two (extremely similar) characterizations of RFC segmentations:

Theorem 3.6 *Let $M \ge 2$, and let $\langle O_1^{\mathrm{RFC}}, \ldots, O_M^{\mathrm{RFC}} \rangle$ be the segmentation of V found by Algorithm 1 for an affinity ψ on V and pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$. Suppose further that the seed sets $S_1, \ldots, S_M$ are consistent with the affinity ψ. Then the following are true for $1 \le i \le M$:*

[10] Note that an $O_m^{\Psi,\mathcal{S}}$-path to v that is $(\psi_m|_{O_m^{\Psi,\mathcal{S}} \times O_m^{\Psi,\mathcal{S}}}, S_m)$-optimal need not be a V-path to v that is (ψ_m, S_m)-optimal: It may have lower ψ_m-strength than a V-path from S_m to v that is not an $O_m^{\Psi,\mathcal{S}}$-path.

1. $v \in O_i^{\mathrm{RFC}}$ *just if every* $(\psi, \bigcup_j S_j)$*-optimal* V*-path to* v *is a* V*-path from* S_i.
2. $O_i^{\mathrm{RFC}} = \{v \in V \mid \max_{j \neq i} \psi^V(S_j, v) < \psi^V(S_i, v)\}$.

Theorem 3.6 will be proved in Sect. 6.3. Statement 2 of the theorem has been used as a definition of RFC segmentations in the (I)RFC-track literature—see, e.g., [21] or [33]. It follows that, under the hypotheses of the theorem, Algorithm 1 does indeed produce the same segmentations as are produced by RFC segmentation algorithms in the literature.

From Theorem 3.6 it is easy to deduce two basic properties of RFC objects:

Corollary 3.7 *Under the hypotheses of Theorem 3.6, the following are true for* $1 \leq i \leq M$*:*

1. *For all* v *in* O_i^{RFC}*, every* $(\psi, \bigcup_j S_j)$*-optimal* V*-path to* v *is an* O_i^{RFC}*-path.*
2. *For all* v *in* O_i^{RFC}*, we have that* $\psi^{O_i^{\mathrm{RFC}}}(S_i, v) = \psi^V(S_i, v) > \max_{j \neq i} \psi^V(S_j, v)$.

Proof Statement 2 follows from statement 1, so we need only prove statement 1. To do this, fix a v in O_i^{RFC} and an i in $\{1, \ldots, M\}$, and let $\langle v_0, \ldots, v_k \rangle$ be a $(\psi, \bigcup_j S_j)$-optimal V-path to v. We claim that $v_n \in O_i^{\mathrm{RFC}}$ for $0 \leq n \leq k$. Indeed, suppose not. Then there is an $n < k$ such that $v_n \notin O_i^{\mathrm{RFC}}$, so by statement 1 of Theorem 3.6 (applied to v_n) there exists a $(\psi, \bigcup_j S_j)$-optimal V-path p to v_n from $\bigcup_{j \neq i} S_j$. Thus $p \cdot \langle v_n, \ldots, v_k \rangle$ is a V-path from $\bigcup_{j \neq i} S_j$ to v with $\psi(p \cdot \langle v_n, \ldots, v_k \rangle) \geq \psi(\langle v_0, \ldots, v_k \rangle)$ and so is a $(\psi, \bigcup_j S_j)$-optimal V-path to v that is not from S_i, which contradicts statement 1 of Theorem 3.6. □

Our next theorem gives three characterizations of IRFC segmentations:

Theorem 3.8 *Let* $M \geq 2$*, and let* $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}} \rangle$ *be the segmentation of* V *found by Algorithm 2 for an affinity* ψ *on* V *and pairwise disjoint nonempty seed sets* $S_1, \ldots, S_M \subset V$*. Suppose further that the seed sets* $S_1, \ldots, S_M$ *are consistent with the affinity* ψ*. Then the following are true for* $1 \leq i \leq M$*:*

1. $v \in O_i^{\mathrm{IRFC}}$ *just if every hereditarily* $(\psi, \bigcup_j S_j)$*-optimal* V*-path to* v *is a* V*-path from* S_i.
2. O_i^{IRFC} *is the unique set* O *that satisfies* $O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^O(S_i, v)\}$.
3. O_i^{IRFC} *is the unique set* O *that satisfies* $O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^V(S_i, v)\}$.

Theorem 3.8 will be proved in Sect. 6.5. It follows from this theorem that, under its hypotheses, Algorithm 2 produces the same segmentations as IRFC algorithms from the (I)RFC-track literature.[11]

Corollary 3.9 below is an IRFC analog of Corollary 3.7. Its statement 1 can be deduced from statement 1 of Theorem 3.8 in very much the same way as statement 1 of Corollary 3.7 was deduced from statement 1 of Theorem 3.6. Its statement 2 follows from its statement 1 and statement 3 of Theorem 3.8.

Corollary 3.9 *Under the hypotheses of Theorem 3.8, the following are true for* $1 \leq i \leq M$*:*

1. *For all* v *in* O_i^{IRFC}*, every hereditarily* $(\psi, \bigcup_j S_j)$*-optimal* V*-path to* v *is an* O_i^{IRFC}*-path.*
2. *For all* v *in* O_i^{IRFC}*, we have that* $\psi^{O_i^{\mathrm{IRFC}}}(S_i, v) = \psi^V(S_i, v) > \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v)$.

The following theorem gives characterizations of MOFS segmentations that are similar in flavor to the first two characterizations of IRFC objects in Theorem 3.8:

Theorem 3.10 *Let* $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}} \rangle$ *be the segmentation of* V *found by Algorithm 3 for a sequence* $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ *of affinities on* V *and a sequence* $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ *of pairwise disjoint nonempty seed sets. Suppose further that the seed sets* $S_1, \ldots, S_M$ *are consistent with the affinities* $\psi_1, \ldots, \psi_M$*. Then:*

1. *For* $1 \leq i \leq M$*,* $v \in O_i^{\mathrm{MOFS}}$ *just if there is a recursively* $(\Psi, \mathcal{S})$*-optimal* V*-path of nonzero* $(\Psi, \mathcal{S})$*-strength from* S_i *to* v.
2. $O_i^{\mathrm{MOFS}} = \{v \in O_i^{\Psi, \mathcal{S}} \mid \psi_i^{O_i^{\Psi, \mathcal{S}}}(S_i, v) > 0\}$ *for every* $1 \leq i \leq M$.

[11] It is enough to verify that O_i^{IRFC} is the same set as the ith IRFC object according to [20]. To see this, let $X_i = \bigcup_{j \neq i} S_j$ $(1 \leq i \leq M)$ and let $f : \mathcal{P}(V \setminus X_i) \to \mathcal{P}(V \setminus X_i)$ be the set function defined by $f(O) = \{v \in V \mid \psi^{V \setminus O}(X_i, v) < \psi^V(S_i, v)\}$ for all $O \subseteq V \setminus X_i$. Consider the sequence $O_i^0, O_i^1, O_i^2, \ldots$ where $O_i^0 = \emptyset$ and $O_i^{k+1} = f(O_i^k)$ for $0 \leq k < \infty$. Since f is monotonic (i.e., $f(O) \subseteq f(O')$ whenever $O \subseteq O'$) and $\emptyset = O_i^0 \subseteq O_i^1$, we see that $O_i^k = f(O_i^{k-1}) \subseteq f(O_i^k) = O_i^{k+1}$ for $k = 1, 2, 3, \ldots$. Writing O_i^∞ to denote the union (i.e., the largest set) of the chain $\emptyset = O_i^0 \subseteq O_i^1 \subseteq O_i^2 \subseteq \ldots$ of subsets of the finite set $V \setminus X_i$, we have that $O_i^\infty = f(O_i^\infty)$. Equivalently, $O_i^\infty = \{v \in V \mid \psi^{V \setminus O_i^\infty}(X_i, v) < \psi^V(S_i, v)\} = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O_i^\infty}(S_j, v) < \psi^V(S_i, v)\}$ and so by statement 3 of Theorem 3.8 we have that $O_i^{\mathrm{IRFC}} = O_i^\infty$. But since the ascending chain $\emptyset = O_i^0 \subseteq O_i^1 \subseteq O_i^2 \subseteq \ldots$ satisfies $O_i^{k+1} = O_i^k \cup O_i^{k+1} = O_i^k \cup f(O_i^k) = O_i^k \cup \{v \in V \setminus O_i^k \mid \psi^{V \setminus O_i^k}(X_i, v) < \psi^V(S_i, v)\}$ for $0 \leq k < \infty$, this chain satisfies the inductive definition of the chain $\emptyset = P^0_{S_i,T_i} \subseteq P^1_{S_i,T_i} \subseteq P^2_{S_i,T_i} \subseteq \ldots$ that is given by equation (12) of [20] in the case where the affinity is ψ and C, m, and T_i are respectively equal to our V, M, and X_i. So in this case the set $P^k_{S_i,T_i}$ of [20] is equal to O_i^k for $0 \leq k < \infty$, whence the set $\bigcup_k P^k_{S_i,T_i}$, which is the ith IRFC object according to [20, Sect. 4.3], is $\bigcup_k O_i^k = O_i^\infty = O_i^{\mathrm{IRFC}}$.

3. $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}} \rangle$ *is the unique sequence of sets* $\langle O_1, \ldots, O_M \rangle$ *such that*

$$O_i = \{v \in V \mid \max_{j \neq i} \psi_j^{O_j}(S_j, v) \leq \psi_i^{O_i}(S_i, v) \neq 0\} \quad \text{for } 1 \leq i \leq M. \tag{3.1}$$

Theorem 3.10 will be proved in Sect. 6.4. We can deduce from Theorem 3.10 and Theorem 1 of [10] that, under the hypotheses of Theorem 3.10, Algorithm 3 produces the same segmentations as are produced by MOFS algorithms from the MOFS-track literature.[12]

Some easy consequences of Theorem 3.10 are stated in the following two corollaries:

Corollary 3.11 *Under the hypotheses of Theorem 3.10, the following are true for* $1 \leq i \leq M$*:*

1. *For all* $v \in O_i^{\mathrm{MOFS}}$*, every recursively* $(\Psi, \mathcal{S})$*-optimal* V*-path from* S_i *to* v *is an* O_i^{MOFS}*-path.*
2. *An* O_i^{MOFS}*-path is a recursively* $(\Psi, \mathcal{S})$*-optimal* V*-path just if it is a hereditarily* $(\psi_i|_{O_i^{\mathrm{MOFS}} \times O_i^{\mathrm{MOFS}}}, S_i)$*-optimal* O_i^{MOFS}*-path.*
3. *For all* $v \in O_i^{\mathrm{MOFS}}$*,* $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$ *is the* $(\Psi, \mathcal{S})$*-strength of any recursively* $(\Psi, \mathcal{S})$*-optimal* V*-path to* v*.*

Proof Statement 1 follows from statement 1 of Theorem 3.10, as every nonempty initial segment of a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i is itself a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i.

From statement 2 of Theorem 3.10 we see that an O_i^{MOFS}-path is an $O_i^{\Psi,\mathcal{S}}$-path, and also see that an O_i^{MOFS}-path is a $(\psi_i|_{O_i^{\mathrm{MOFS}} \times O_i^{\mathrm{MOFS}}}, S_i)$-optimal O_i^{MOFS}-path if and only if it is a $(\psi_i|_{O_i^{\Psi,\mathcal{S}} \times O_i^{\Psi,\mathcal{S}}}, S_i)$-optimal $O_i^{\Psi,\mathcal{S}}$-path (whence an O_i^{MOFS}-path is a hereditarily $(\psi_i|_{O_i^{\mathrm{MOFS}} \times O_i^{\mathrm{MOFS}}}, S_i)$-optimal O_i^{MOFS}-path if and only if it is a hereditarily $(\psi_i|_{O_i^{\Psi,\mathcal{S}} \times O_i^{\Psi,\mathcal{S}}}, S_i)$-optimal $O_i^{\Psi,\mathcal{S}}$-path). These observations and statement 2 of Theorem 3.4 together imply statement 2 of the corollary. As $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$ is the $(\Psi, \mathcal{S})$-strength of a hereditarily $(\psi_i|_{O_i^{\mathrm{MOFS}} \times O_i^{\mathrm{MOFS}}}, S_i)$-optimal O_i^{MOFS}-path to any $v \in O_i^{\mathrm{MOFS}}$, and all recursively $(\Psi, \mathcal{S})$-optimal V-paths to a point v have the same $(\Psi, \mathcal{S})$-strength, statement 3 follows from statement 2. □

Corollary 3.12 *Under the hypotheses of Theorem 3.10, if for some* $i \in \{1, \ldots, M\}$ *and* $v, w \in V$ *we have that* $v \in O_i^{\mathrm{MOFS}}$ *and* $\psi_i(v, w) > 0$*, then* $w \in \bigcup_j O_j^{\mathrm{MOFS}}$*. Equivalently, for* $1 \leq i \leq M$ *we have that* $\psi_i(v, w) = 0$ *whenever* $v \in O_i^{\mathrm{MOFS}}$ *and* $w \in V \setminus \bigcup_j O_j^{\mathrm{MOFS}}$*.*

Proof It is quite easy to see that this corollary is true just by considering the final iteration of Algorithm 3's main loop. However, we will deduce the corollary from Theorem 3.10.

Suppose $v \in O_i^{\mathrm{MOFS}}$ and $\psi_i(v, w) > 0$. Then, by Theorem 3.10, there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p from S_i to v such that $\psi_i(p) > 0$. Let p' be the V-path from S_i to w obtained by appending w to p, so that $\psi_i(p') = \min(\psi_i(p), \psi_i(v, w)) > 0$. As p is a recursively $(\Psi, \mathcal{S})$-optimal V-path, either p' is a recursively $(\Psi, \mathcal{S})$-optimal V-path, or there is a recursively $(\Psi, \mathcal{S})$-optimal V-path q from $\bigcup_j S_j$ to w such that $\Psi_{\mathcal{S}}(q) > \Psi_{\mathcal{S}}(p') = \psi_i(p') > 0$. In both cases there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of nonzero $(\Psi, \mathcal{S})$-strength from $\bigcup_j S_j$ to w, and so $w \in \bigcup_j O_j^{\mathrm{MOFS}}$ by statement 1 of Theorem 3.10. □

3.4 Instability of IRFC Segmentations That Have Large Tie-Zones

We now show that the tie-zone of any IRFC segmentation can be completely eliminated by making arbitrarily small changes in affinity values.

Let us write $\langle O_1^{\mathrm{IRFC}}(\psi, \mathcal{S}), \ldots, O_M^{\mathrm{IRFC}}(\psi, \mathcal{S}) \rangle$ to denote the IRFC segmentation given by an affinity ψ on V and a sequence $\mathcal{S}$ of M pairwise disjoint nonempty seed sets in V, write $TZ(\psi, \mathcal{S})$ to denote $V \setminus \bigcup_j O_j^{\mathrm{IRFC}}(\psi, \mathcal{S})$ (which is the segmentation's tie-zone), and write $\|\psi - \psi'\|$ to denote the value $\max_{u,v \in V} |\psi(u, v) - \psi'(u, v)|$ (for any affinities ψ and ψ' on V). Then a precise statement of our result is as follows:

(♣) For any affinity ψ on V, any sequence $\mathcal{S}$ of pairwise disjoint nonempty seed sets in V, and any $\epsilon > 0$, there exists an affinity ψ' on V such that $\|\psi' - \psi\| < \epsilon$, $TZ(\psi', \mathcal{S}) = \emptyset$, and ψ' is symmetric if ψ is.

It follows from this result that any IRFC segmentation whose tie-zone is large must be unstable with respect to tiny changes in the affinity values.

For any affinity ψ on V and any sequence $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ of pairwise disjoint nonempty seed sets in V, we define a $(\psi, \mathcal{S})$-*bottleneck point* to be a point $b \in TZ(\psi, \mathcal{S})$ for which there exists a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path to b such that no other point of the V-path

[12] Indeed, let OBJ_i denote the ith object (in the sense of this paper) of the segmentation produced by the MOFS algorithm of [10] for our affinities $\psi_1, \ldots, \psi_M$ and seed sets $S_1, \ldots, S_M$. Then what we want to verify is that $\mathrm{OBJ}_i = O_i^{\mathrm{MOFS}}$ for $1 \leq i \leq M$. In the notation of [10], $\mathrm{OBJ}_i = \{c \in V \mid \sigma_i^c \neq 0\}$. Theorem 1 of [10] uses the notation V_i to denote the seed set we refer to as S_i and uses s_i^c to denote the value $\psi_i^{\mathrm{OBJ}_i}(S_i, c)$. For $1 \leq i \leq M$ and all $c \in V$, statement (i) of that theorem implies that $\sigma_i^c \neq 0$ just if $\max_{j \neq i} s_j^c \leq s_i^c \neq 0$. Equivalently, for $1 \leq i \leq M$ we have that $c \in \mathrm{OBJ}_i$ just if $\max_{j \neq i} \psi_j^{\mathrm{OBJ}_j}(S_j, c) \leq \psi_i^{\mathrm{OBJ}_i}(S_i, c) \neq 0$, whence we see from statement 3 of Theorem 3.10 that $\mathrm{OBJ}_i = O_i^{\mathrm{MOFS}}$, as required.

lies in $TZ(\psi, \mathcal{S})$. (Here we are thinking of $TZ(\psi, \mathcal{S})$ as a multiple-necked bottle whose necks are the places where hereditarily $(\psi, \bigcup_j S_j)$-optimal V-paths enter the bottle.) If $TZ(\psi, \mathcal{S}) \neq \emptyset$ then there is at least one $(\psi, \mathcal{S})$-bottleneck point, by Proposition 3.1 and the fact that every initial segment of a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path.

The fact (♣) is a straightforward consequence of the following theorem (which is a weaker version of (♣)). This theorem will be proved in Sect. 6.6.

Theorem 3.13 *Let ψ be an affinity on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ a sequence of pairwise disjoint nonempty subsets of V such that $TZ(\psi, \mathcal{S}) \neq \emptyset$. Let b be any $(\psi, \mathcal{S})$-bottleneck point, and let δ be any positive constant. Then there exists an affinity ψ' on V such that $\|\psi' - \psi\| < \delta$, $TZ(\psi', \mathcal{S}) \subseteq TZ(\psi, \mathcal{S}) \setminus \{b\}$, and ψ' is symmetric if ψ is symmetric.*

To deduce (♣) from this theorem, let ψ, $\mathcal{S}$, and ϵ be any affinity on V, any sequence of pairwise disjoint nonempty subsets of V, and any positive value. Let $\delta_0, \delta_1, \delta_2, \ldots$ be an infinite sequence of positive values such that $\sum_{j=0}^{\infty} \delta_j < \epsilon$. Then we see from the theorem that we can define a sequence of affinities $\psi_0, \psi_1, \psi_2, \ldots$ on V in the following way: Define $\psi_0 = \psi$, and define ψ_i for $i = 1, 2, 3, \ldots$ as follows: If $TZ(\psi_i, \mathcal{S}) \neq \emptyset$ then let b_i be any $(\psi_i, \mathcal{S})$-bottleneck point and let ψ_{i+1} be any affinity on V such that $\|\psi_{i+1} - \psi_i\| < \delta_i$, $TZ(\psi_{i+1}, \mathcal{S}) \subseteq TZ(\psi_i, \mathcal{S}) \setminus \{b_i\}$, and ψ_{i+1} is symmetric if ψ_i is; but if $TZ(\psi_i, \mathcal{S}) = \emptyset$ then let $\psi_{i+1} = \psi_i$. Now (♣) follows from the fact that $\|\psi_i - \psi\| = \|\sum_{j=0}^{i-1}(\psi_{j+1} - \psi_j)\| \leq \sum_{j=0}^{i-1} \|\psi_{j+1} - \psi_j\| < \sum_{j=0}^{i-1} \delta_j < \epsilon$ for $1 \leq i < \infty$, and the fact that there must be some i for which $TZ(\psi_i, \mathcal{S}) = \emptyset$.

4 Efficient Fuzzy Connectedness Segmentation Algorithms

4.1 Efficient Computation of RFC Segmentations

The following variant of Dijkstra's well known algorithm [24] for finding shortest paths in a weighted digraph can be used to compute RFC segmentations. A very similar algorithm is described on p. 76 of [8], and the paper [25] discusses algorithms of this general kind.

For any seed set S and affinity ψ on V, Algorithm 4 computes the ψ-strength of a (ψ, S)-optimal V-path to each point v in V: It computes $\psi^V(S, v)$ for every v in V. Given seed sets $\langle S_1, \ldots, S_M \rangle$, we can apply this algorithm M times (once with $S = S_i$ for each $i \in \{1, \ldots, M\}$) to compute $\psi^V(S_i, v)$ for $1 \leq i \leq M$ and all v in V. Then the RFC objects $\langle O_1^{\text{RFC}}, \ldots, O_M^{\text{RFC}} \rangle$ for the seed sets $S_1, \ldots, S_M$ and affinity ψ can all be found by applying statement 2 of Theorem 3.6, assuming the seed sets are consistent with the affinity.

Line 6 of this algorithm assumes that we have identified a set $E \subseteq \{(v, v') \in V \times V \mid v \neq v'\}$ such that $\psi(v, v') = 0$ whenever $v \neq v'$ and $(v, v') \notin E$. Each member of E is called a *ψ-edge*. We can always take this set E of ψ-edges to be the whole of $\{(v, v') \in V \times V \mid v \neq v'\}$, but the algorithm will be more efficient if for every $u \in V$ the set of all ψ-edges (u, v) is small and the algorithm can quickly iterate over that set.

Algorithm 4: Finds, for each $v \in V$, the ψ-strength of a (ψ, S)-optimal V-path to v

Data: a finite set V, a nonempty subset S of V, and an affinity ψ on V
Result: an array $\sigma[\,]$ such that, for $v \in V$, $\sigma[v] = \psi^V(S, v)$

1 **foreach** $\mathsf{v} \in V$ **do** $\sigma[\mathsf{v}] \leftarrow 0$ /* initialization loop 1 */
2 **foreach** $\mathsf{s} \in S$ **do** $\sigma[\mathsf{s}] \leftarrow 1$ /* initialization loop 2 */
3 **create** a max-priority queue H that contains every v in V, with key $\sigma[v]$
4 **while** H is not empty **do** /* the main loop */
5   **remove** an element w of $\arg\max_{u \in \mathsf{H}} \sigma[u]$ **from** H
6   **foreach** x such that (w, x) is a ψ-edge and $\sigma[\mathsf{w}] > \sigma[\mathsf{x}]$ **do**
7         $\sigma' \leftarrow \min(\sigma[\mathsf{w}], \psi(\mathsf{w}, \mathsf{x}))$
8   **if** $\sigma' > \sigma[\mathsf{x}]$ **then** $\sigma[\mathsf{x}] \leftarrow \sigma'$
        /* $\sigma[\mathsf{x}] \leftarrow \sigma'$ involves update of H, because $\mathsf{x} \in \mathsf{H}$ */

Let us assume that, for each $v \in V$, there are only $O(1)$ points x for which (v, x) is a ψ-edge and those points can all be found in $O(1)$ time. Then, if H is represented as a binary max-heap [22, Sect. 6.1] (so line 3 takes $O(|V|)$ time and each iteration of the main loop can be executed in $O(\log |V|)$ time), the running time of Algorithm 4 is $O(|V| \log |V|)$. But we can do better if all values of ψ are (or can safely be rounded to) multiples of $1/N$ for an integer N that is $O(|V|)$, by using an array of doubly linked lists instead of a heap to represent the priority queue H: For example, we can create a doubly linked node $\mathsf{Node}[v]$ for each $v \in V$ and maintain an array Harr such that, for $n \in \{1, \ldots, N\}$, $\mathsf{Harr}[n]$ is a (possibly null) doubly linked list which contains $\mathsf{Node}[v]$ for each $v \in V$ that currently satisfies $\sigma[v] = n/N$. Priority queues are frequently implemented in this kind of way both in MOFS-track and in (I)RFC-track FC segmentation [10, 20]. Assuming this representation of H, each execution of line 8 requires only $O(1)$ time, and the $|V|$ executions of line 5 require a total of $O(|V|)$-time, so we see that the running time of Algorithm 4 is $O(|V|)$.

Standard justifications of Dijkstra's algorithm can be adapted to prove that Algorithm 4 achieves what its **Result** line promises. But we will see that this also follows from the correctness of Algorithm 5 below.

It would be easy to modify Algorithm 4 so it also creates an array pred[ ] such that, for every point $v \notin S$, when the

Algorithm 5: Finds, for each $v \in V$, all those $i \in \{1, \ldots, M\}$ for which $v \in O_i^{\mathrm{MOFS}}$; also finds the value of $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$ for each such i

Data: a finite set V and a sequence $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ of pairwise disjoint nonempty subsets of V
Data: a sequence $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ of affinities on V
Result: an array $\sigma[\,]$ and Boolean arrays $\chi^1[\,], \ldots, \chi^M[\,]$ such that, for $1 \le i \le M$ and all $v \in V$,
$\sigma[v] =$ the $(\Psi, \mathcal{S})$-strength of every recursively $(\Psi, \mathcal{S})$-optimal V-path to v
$\chi^i[v] =$ **true** if there is a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i to v and $\sigma[v] \neq 0$; $\chi^i[v] =$ **false** otherwise
Comment: At termination, assuming $\mathcal{S}$ is consistent with Ψ, Theorem 3.10 and Corollary 3.11 imply
$\{v \in V \mid \chi^i[v] = \textbf{true}\} = O_i^{\mathrm{MOFS}}$ and $\sigma[v] = \psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$ for $1 \le i \le M$ and all $v \in O_i^{\mathrm{MOFS}}$.

```
1  foreach v ∈ V do                                                /* initialization loop 1 */
2      σ[v] ← 0
3      for i ← 1 to M do χ^i[v] ← false
4  for i ← 1 to M do                                               /* initialization loop 2 */
5      foreach s ∈ S_i do
6          σ[s] ← 1
7          χ^i[s] ← true
8  create a max-priority queue H that contains every v in V, with key σ[v]
9  while H is not empty do                                         /* the main loop */
10     remove an element w of arg max_{u∈H} σ[u] from H
11     foreach x such that (w, x) is a Ψ-edge and σ[w] ≥ σ[x] do
12         foreach i ∈ {1, ..., M} such that χ^i[w] = true do
13             σ′ ← min(σ[w], ψ_i(w, x))
14             if σ′ > σ[x] then
15                 σ[x] ← σ′  /* involves update of H, because x ∈ H */
16                 for j ← 1 to M do χ^j[x] ← false
17                 χ^i[x] ← true
18             else if σ′ = σ[x] and σ′ > 0 and χ^i[x] = false then
19                 χ^i[x] ← true
20                 if x ∉ H then H ← H ∪ {x}
```

algorithm terminates pred$[v]$ is the predecessor of v on a (ψ, S)-optimal V-path to v.

While M-object RFC segmentations can be computed in the manner described above using M separate applications of Algorithm 4, they can often be computed more efficiently (e.g., by using the idea presented by Badura and Pietka in [5, Sect. 2.3]). However, we will leave this matter for future exploration.

4.2 Efficient Computation of MOFS and IRFC Segmentations

Algorithm 5 below, like Algorithm 4 above, can be regarded as a variant of Dijkstra's shortest path algorithm, but it is necessarily less simple because it needs to allow the use of a different affinity for each of the M objects. The algorithm is similar to some other MOFS segmentation algorithms, such as the algorithms in Sects. 3 and 5 of [10]. It will be seen from the **Result** lines and Theorem 3.10 that, when Algorithm 5 terminates, each object O_i^{MOFS} of the MOFS segmentation for seed sets $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ and affinities $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ is given by

$$O_i^{\mathrm{MOFS}} = \{v \in V \mid \chi^i[v] = \textbf{true}\}$$

assuming the seed sets are consistent with the affinities.

It follows from this (and Corollary 2.7) that the IRFC segmentation for an affinity ψ and seed sets $S_1, \ldots, S_M$ consistent with ψ can be found by executing Algorithm 5 with $\psi_1 = \cdots = \psi_M = \psi$: At termination, each object O_i^{IRFC} of the segmentation is given by

$$O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}} = \{v \in V \mid \chi^i[v] = \textbf{true} \text{ and } \chi^j[v] = \textbf{false} \text{ for all } j \in \{1, \ldots, M\} \setminus \{i\}\}.$$

Line 11 of this algorithm assumes that we have identified a set $E \subseteq \{(v, v') \in V \times V \mid v \neq v'\}$ such that $\psi_i(v, v') = 0$ for all $i \in \{1, \ldots, M\}$ whenever $v \neq v'$ and $(v, v') \notin E$. Each member of E is called a Ψ-*edge*. As with the ψ-edges of Algorithm 4, if for every $u \in V$ the set of all Ψ-edges (u, v) is small, and the algorithm can quickly iterate over that set, then the algorithm will run more efficiently.

In this algorithm, as in Algorithm 4, H consists of points whose labels carry potentially important new information, and when a point is removed from H its relevant information is passed on to other points (whose labels are updated accordingly). An important difference between this algorithm and Algorithm 4 is that in this algorithm labels of points that are no longer in H may still be updated (by line 19); when that happens the point in question is reinserted into H (by line 20) so that information carried by the updated label can be passed on to other points at a later iteration of the main loop.

It would be quite easy to modify Algorithm 5 so it also creates an array pred[ ][ ] such that, for each $i \in \{1, ..., M\}$ and each point $v \in O_i^{\text{MOFS}} \backslash S_i$, when the algorithm terminates pred$[i][v]$ is the predecessor of v on a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i to v.

Efficiency of Algorithm 5

We assume that, for each $v \in V$, there are only $O(1)$ points x for which (v, x) is a Ψ-edge and those points can all be found in $O(1)$ time. We also assume the priority queue H is implemented in such a way that it only takes $O(1)$ time to determine whether $\mathsf{x} \notin \mathsf{H}$ is true or false when line 20 is executed.

As explained in Sect. 4.1, if all values of the ψ_is are (or can safely be rounded to) multiples of $1/N$ for an integer N that is $O(|V|)$, then we can use an array of doubly linked lists to represent the priority queue H. This is how we would represent H in most of the applications we have in mind.

However, let us first consider the case where the values of the ψ_is need not satisfy the above condition, and H is represented as a binary max-heap [22, Sect. 6.1]. Then line 8 can be executed in $O(|V|)$ time. The main loop iterates at most $M|V|$ times; this follows from observation (viii) in Sect. 6.8. Under the above assumptions, at each iteration of the main loop it takes $O(\log |V|)$ time to execute line 10, and then the inner loop on lines 11 – 20 iterates $O(1)$ times at a time cost of $O(M(\log |V| + M))$. The latter bound follows from the fact that the "**foreach** $\mathsf{i} \in \{1, \ldots, M\}$ such that $\chi^{\mathsf{i}}[\mathsf{w}] = \textbf{true}$" loop iterates *at most* M times, and it takes $O(\log |V| + M)$ time to execute its body (i.e., lines 13–20) once: Indeed, the structure of the heap H can only be modified by lines 15 and 20, execution of which takes $O(\log |V|)$ time in each case, while the time cost of line 16 is $O(M)$ and that of the rest of lines 13 – 20 is $O(1)$. So it takes a total of $O(M|V|(\log |V| + M(\log |V| + M))) = O(M^2|V| \log |V| + M^3|V|))$ time to execute all iterations of the main loop. Since execution of lines 1–8 only takes $O(M|V|)$ time, the total running time of the algorithm is also $O(M^2|V| \log |V| + M^3|V|)$, and so is $O(|V| \log |V|)$ for any given value of M, assuming H is represented as a binary max-heap.

In the more usual case where we use an array of doubly linked lists instead of a heap to represent H, each execution of line 10 takes $O(1)$ amortized time, and each execution of line 15 or line 20 takes $O(1)$ time. So when we redo the analysis in the previous paragraph we see that the total running time is $O(M^3|V|)$. Thus the running time of the algorithm is $O(|V|)$ for any given value of M.

However, Algorithm 5 is typically even more efficient than may be suggested by a quick look at the algorithm and the above analysis, because the factors of M which appear in the analysis reflect the behavior of the algorithm in unusual worst-case scenarios or when M is unusually large. For example, even though execution of "**for** $\mathsf{j} \leftarrow 1$ **to** M **do** $\chi^{\mathsf{j}}[\mathsf{x}] \leftarrow$ **false**" (line 16) takes $\Theta(M)$ time, when M is small this loop is responsible for only a small part of the algorithm's running time. Still more importantly, in most current applications of MOFS segmentation the number of iterations of the main loop will usually be close to $|V|$ (whereas the upper bound $M|V|$ was used in the above analysis), and very few executions of the inner

foreach $\mathsf{i} \in \{1, \ldots, M\}$ such that $\chi^{\mathsf{i}}[\mathsf{w}] = \textbf{true}$

loop will consist of more than one iteration (as there will rarely be more than one i for which $\chi^{\mathsf{i}}[\mathsf{w}] = \textbf{true}$).

In particular, this will be true for all MOFS and IRFC applications in which segmentations are stable with respect to tiny changes in affinity values. To see this, note that the number of iterations of the main loop is $\mu|V|$ where μ is the average, over all points v in V, of the number of times v is removed from H. Assuming the seed sets are consistent with the affinities and the MOFS or IRFC segmentation is stable with respect to tiny changes in affinity values, the value of μ will be close to 1 regardless of the value of M: Indeed, a point v can be removed from H a total of $k \geq 2$ times only if that point lies in k different MOFS objects[13] and, as we saw in Sects. 2.5 and 3.4, this will be the case only for a small proportion of the points in V if the segmentation is stable. Similarly, if the seed sets are consistent with the affinities and the segmentation is stable, then the "**foreach** $\mathsf{i} \in \{1, \ldots, M\}$ such that $\chi^{\mathsf{i}}[\mathsf{w}] = \textbf{true}$" inner loop will rarely iterate more than once, because $\chi^{\mathsf{i}}[\mathsf{w}]$ can be **true** for more than one value of i only in rare cases where w lies in more than one MOFS object.

Moreover, when M is large and the time needed to determine just which values of i satisfy $\chi^{\mathsf{i}}[\mathsf{w}] = \textbf{true}$ and to execute "**for** $\mathsf{j} \leftarrow 1$ **to** M **do** $\chi^{\mathsf{j}}[\mathsf{x}] \leftarrow \textbf{false}$" are significant issues, the information Algorithm 5 stores in the M Boolean arrays $\chi^1[v], \ldots, \chi^M[v]$ can be represented more efficiently. For example, we might use an integer array $\chi[v]$ and an array $\chi\mathsf{list}[v]$ with the following properties: For each $v \in V$, $\chi[v]$ is the value of $\mathsf{i} \in \{1, \ldots, M\}$ for which the value of $\chi^{\mathsf{i}}[v]$ would have been **true**, provided there is just one such value

[13] This follows from observations (iv), (vi), and (vii) in Sect. 6.8, statement 2 of Proposition 6.12, and Theorem 3.10.

of i. When there is no such value of i or more than one such value, $\chi[v] = 0$ and $\chi\mathsf{list}[v]$ is a (possibly null) list of all such values of i. We may also choose to use such arrays $\chi[v]$ and $\chi\mathsf{list}[v]$ instead of the Boolean arrays $\chi^1[v], \ldots, \chi^M[v]$ if the latter would use too much memory.

Two Other Efficient Ways to Compute IRFC Objects

The $\mathrm{GC}_{\max}$ algorithm of [20] will efficiently compute any single object of an M-object IRFC segmentation. While $\mathrm{GC}_{\max}$ would need to be executed M times to compute all the objects of an M-object IRFC segmentation, it may be faster than Algorithm 5 when we need to compute just one of the M objects.[14]

When the affinity is symmetric, the TZWS by union-find method of [2], which was developed as a way to compute tie-zone IFT watershed transforms—see footnote 2—is another way to compute IRFC segmentations.[15] Like Algorithm 5 and other MOFS algorithms, the TZWS by union-find method computes all the objects of an IRFC segmentation simultaneously. To briefly describe this method (which we will do in terms of IRFC segmentation rather than tie-zone IFT watershed transform computation), let ψ be a symmetric affinity on V and let $S_1, \ldots, S_M \subset V$ be pairwise disjoint nonempty seed sets that are consistent with ψ. Let $\sim$ be the binary relation on V such that $x \sim y$ just if $\psi^V(\bigcup_j S_j, x) = \psi^V(\bigcup_j S_j, y) \leq \psi(x, y)$. Since ψ is symmetric, $\sim$ is symmetric and its transitive closure is an equivalence relation, each of whose equivalence classes is called a *flat-zone*.

It is not hard to show that each flat-zone *either* lies entirely within a single object of the IRFC segmentation generated by ψ and $\langle S_1, \ldots, S_M\rangle$ *or* lies entirely within the segmentation's tie-zone. The TZWS by union-find method is somewhat similar to Algorithm 5, but as it runs it takes full advantage of the fact stated in the previous sentence by using a union-find data structure to represent all the maximal $\sim$-connected sets (i.e., all "fragments of flat-zones") that have so far been discovered: For example, whenever a point v is found to lie in the tie-zone, v's representative in the union-find data structure (which is the representative of all points that have so far been discovered to be $\sim$-connected to v) is labeled to indicate that the entire flat-zone fragment it represents lies in the tie-zone.

Two important differences between the TZWS by union-find method and our Algorithm 5 are that TZWS by union-find's main loop will never iterate more times than there are points in the image, and that TZWS by union-find does not have any analog of Algorithm 5's "**foreach** $\mathsf{i} \in \{1, \ldots, M\}$ such that $\chi^{\mathsf{i}}[\mathsf{w}] = $ **true**" inner loop. For these reasons, it will outperform Algorithm 5 in some cases. But in applications where the average number of MOFS objects that contain a point is close to 1 (which includes most of the applications we have in mind, in which tie-zones are typically small) the time saved as a result of the above two differences may be outweighed by the time cost of maintaining and accessing the union-find data structure. This is because when Algorithm 5 is used in such applications the inner loop "**foreach** $\mathsf{i} \in \{1, \ldots, M\}$ such that $\chi^{\mathsf{i}}[\mathsf{w}] = $ **true**" will rarely iterate more than once (i.e., when this loop is executed there will rarely be more than one i for which $\chi^{\mathsf{i}}[\mathsf{w}] = $ **true**), few points will be placed into the priority queue more than once, and the total number of iterations of Algorithm 5's main loop will be close to $|V|$.

5 Robustness of MOFS Segmentation with Respect to Small Changes in Seed Sets

RFC and IRFC segmentation are known to be robust with respect to small changes in the seeds [21, Sect. 2.4]. In this section we show that MOFS segmentations are also robust in this sense.

Let $\Psi = \langle \psi_1, \ldots, \psi_M\rangle$ be any sequence of affinities on V and let $\mathcal{S} = \langle S_1, \ldots, S_M\rangle$ be any sequence of M pairwise disjoint nonempty subsets of V. Then we write $\langle O_1^{\mathrm{MOFS}}(\Psi, \mathcal{S}), \ldots, O_M^{\mathrm{MOFS}}(\Psi, \mathcal{S})\rangle$ to denote the MOFS segmentation $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}}\rangle$ that is produced when either Algorithm 3 or Algorithm 5 is carried out with affinities $\psi_1, \ldots, \psi_M$ and seed sets $S_1, \ldots, S_M$.

One might expect that, if $\mathcal{S} = \langle S_1, \ldots, S_M\rangle$ and $\mathcal{S}^* = \langle S_1^*, \ldots, S_M^*\rangle$ are such that each seed set S_i^* is "sufficiently close" to the seed set S_i, then the segmentation $\langle O_1^{\mathrm{MOFS}}(\Psi, \mathcal{S}), \ldots, O_M^{\mathrm{MOFS}}(\Psi, \mathcal{S})\rangle$ will be exactly the same as the segmentation $\langle O_1^{\mathrm{MOFS}}(\Psi, \mathcal{S}^*), \ldots, O_M^{\mathrm{MOFS}}(\Psi, \mathcal{S}^*)\rangle$. We will see from Corollaries 5.5 and 5.6, and Remark 5.7, that MOFS segmentation is indeed robust in this sense. These robustness properties of MOFS will be shown to follow from

[14] Given an affinity on V and pairwise disjoint nonempty seed sets $S_1, \ldots, S_M \subset V$, we can compute the IRFC object O_i^{IRFC} associated with the seed set S_i by executing the $\mathrm{GC}_{\max}$ algorithm of [20, p. 386] with $W = \bigcup_j S_j$ and a priority map λ such that $\lambda(c) = 0$ if $c \in S_i$ and $\lambda(c) = 1$ if $c \in \bigcup_{j \neq i} S_j$. This will compute a forest in which the nodes of the trees rooted at points in S_i are exactly the points of O_i^{IRFC}. To compute the entire IRFC segmentation $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}}\rangle$ we do this M times, with $i = 1, \ldots, M$. A modified version of $\mathrm{GC}_{\max}$, which uses a priority map λ that satisfies $\lambda(c) = j - 1$ for all $c \in S_j$ $(1 \leq j \leq M)$, will compute a forest such that, for $1 \leq j \leq M$, the nodes of the trees rooted at points in S_j include all points of O_j^{IRFC} but possibly also some points of the tie-zone if $j \neq 1$. When the tie-zone is very small, a good approximation to the entire IRFC segmentation $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}}\rangle$ can be obtained by executing such a modified version of $\mathrm{GC}_{\max}$ just once.

[15] We are grateful to a referee for bringing this method to our attention. We think the method is sound and quite clever, but it seems to us that the corresponding pseudocode [2, Pseudocode 1] does not anticipate every situation that might arise in an image and may in certain cases fail to identify some tie-zone points, though we think it would not be difficult to modify the pseudocode to correctly handle such cases.

Theorem 5.4, a special case of Corollary 5.5 that is the main result of this section. Theorem 5.4 and Corollary 5.5 can be viewed as MOFS analogs of known robustness results for IRFC segmentation—specifically, as MOFS analogs of the first assertions of [21, Thm. 2.5] and [21, Cor. 2.6].

Our results will be stated in terms of sets $\text{Core}_1^{\Psi,\mathcal{S}}, \ldots, \text{Core}_M^{\Psi,\mathcal{S}}$ which we now define:

Definition 5.1 Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ any sequence of M pairwise disjoint nonempty subsets of V that are consistent with the affinities. Then, for $1 \le i \le M$, we define $\mathcal{P}_i(\Psi, \mathcal{S})$ to be the collection of all subsets P of V that satisfy both of the following conditions:

(a) $P \subseteq O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \ne i} O_j^{\text{MOFS}}(\Psi, \mathcal{S})$.

(b) $\psi_i^{O_i^{\text{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \ge \psi_i^{\emptyset}(v, V \setminus P)$ for every $v \in P$.

The *core* of $O_i^{\text{MOFS}}(\Psi, \mathcal{S})$, denoted by $\text{Core}_i^{\Psi,\mathcal{S}}$, is defined as the union of all the sets in $\mathcal{P}_i(\Psi, \mathcal{S})$.

Since $S_i \subseteq O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \ne i} O_j^{\text{MOFS}}(\Psi, \mathcal{S})$ (as $S_i \subseteq O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \subseteq S_i \cup (V \setminus \bigcup_j S_j)$ for $1 \le i \le M$, by Theorem 2.6), we see that $S_i \in \mathcal{P}_i(\Psi, \mathcal{S})$ and therefore $S_i \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$.

It is also clear that $\text{Core}_i^{\Psi,\mathcal{S}} \subseteq O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \ne i} O_j^{\text{MOFS}}(\Psi, \mathcal{S})$ for $1 \le i \le M$. This implies that the cores of distinct MOFS objects are always disjoint: $\text{Core}_i^{\Psi,\mathcal{S}} \cap \text{Core}_j^{\Psi,\mathcal{S}} = \emptyset$ whenever $i \ne j$.

Moreover, the union of two sets in $\mathcal{P}_i(\Psi, \mathcal{S})$ is a set in $\mathcal{P}_i(\Psi, \mathcal{S})$. (Indeed, if $P_1, P_2 \in \mathcal{P}_i(\Psi, \mathcal{S})$ then $P = P_1 \cup P_2$ clearly satisfies (a), and also satisfies (b) as $\psi_i^{\emptyset}(v, V \setminus P_j) \ge \psi_i^{\emptyset}(v, V \setminus P)$ for $j = 1, 2$.) So $\text{Core}_i^{\Psi,\mathcal{S}} \in \mathcal{P}_i(\Psi, \mathcal{S})$: $\text{Core}_i^{\Psi,\mathcal{S}}$ is the *largest* member of $\mathcal{P}_i(\Psi, \mathcal{S})$ and contains all other members of $\mathcal{P}_i(\Psi, \mathcal{S})$.

Significantly, for typical affinities and seed sets we will have that $\{v \in V \mid \psi_i^V(S_i, v) \ge \alpha\} \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$ for any α that is large enough. We can express this more precisely. Given affinities $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ and seed sets $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$, let us say (for any $i \in \{1, \ldots, M\}$) that a value $\alpha \in (0, 1]$ is $(\Psi, \mathcal{S}, i)$*-large* if

$$\{v \in V \mid \psi_i^V(S_i, v) \ge \alpha\} \cap \bigcup_{j \ne i} \{v \in V \mid \psi_j^V(S_j, v) \ge \alpha\} = \emptyset.$$

Then the following is true:

Proposition 5.2 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be a sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ a sequence of pairwise disjoint nonempty subsets of V that are consistent with Ψ. Let $i \in \{1, \ldots, M\}$ and let $\alpha \in (0, 1]$ be any $(\Psi, \mathcal{S}, i)$-large value. Then $\{v \in V \mid \psi_i^V(S_i, v) \ge \alpha\} \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$.*

Proof We claim that, for each v in $\{v \in V \mid \psi_i^V(S_i, v) \ge \alpha\}$, every hereditarily (ψ_i, S_i)-optimal V-path to v is recursively $(\Psi, \mathcal{S})$-optimal. Indeed, suppose not. Then there exists a hereditarily (ψ_i, S_i)-optimal V-path $p = \langle v_0, \ldots, v_l \rangle$ such that $\psi_i^V(S_i, v_l) \ge \alpha$ and p is not recursively $(\Psi, \mathcal{S})$-optimal. As p is not recursively $(\Psi, \mathcal{S})$-optimal, there is some $k \in \{1, \ldots, l\}$ and some recursively $(\Psi, \mathcal{S})$-optimal V-path q to v_k such that $\Psi_{\mathcal{S}}(q) > \psi_i(\langle v_0, \ldots, v_k \rangle)$. Since $\langle v_0, \ldots, v_k \rangle$ is hereditarily (ψ_i, S_i)-optimal, q cannot be a V-path from S_i and so q is a V-path from $\bigcup_{j \ne i} S_j$. Now $\psi_i(\langle v_0, \ldots, v_k \rangle) \ge \psi_i(p) \ge \alpha$, so $v_k \in \{v \in V \mid \psi_i^V(S_i, v) \ge \alpha\}$. But q is a V-path from $\bigcup_{j \ne i} S_j$ to v_k such that $\Psi_{\mathcal{S}}(q) > \psi_i(\langle v_0, \ldots, v_k \rangle) \ge \alpha$, so $v_k \in \bigcup_{j \ne i} \{v \in V \mid \psi_j^V(S_j, v) \ge \alpha\}$. This is a contradiction (as α is $(\Psi, \mathcal{S}, i)$-large), and so we have proved our claim.

Now let $P = \{v \in V \mid \psi_i^V(S_i, v) \ge \alpha\}$. Then it follows from the claim that, for each $v \in P$, there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength $\ge \alpha$ from S_i to v, whence there is no recursively $(\Psi, \mathcal{S})$-optimal V-path from $\bigcup_{j \ne i} S_j$ to v (as any such V-path from $\bigcup_{j \ne i} S_j$ to v would also have $(\Psi, \mathcal{S})$-strength $\ge \alpha$ and so could not exist, since α is $(\Psi, \mathcal{S}, i)$-large). From this, Theorem 3.10, and Corollary 3.11 we see that $P \subseteq O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \ne i} O_j^{\text{MOFS}}(\Psi, \mathcal{S})$ and $\psi_i^{O_i^{\text{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \ge \alpha$ for all $v \in P$. We also see from the definition of P that $\psi_i^{\emptyset}(v, V \setminus P) < \alpha$ for all $v \in P$. So $P \in \mathcal{P}_i(\Psi, \mathcal{S})$ and therefore $P \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$. □

The characterizations of MOFS segmentations given by Theorem 3.10 assume the seed sets are consistent with the affinities. The next proposition tells us that consistency is preserved when seed sets are replaced with new sets contained in the cores of the objects of the original segmentation:

Proposition 5.3 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be a sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ a sequence of pairwise disjoint nonempty seed sets consistent with the affinities. Let $\mathcal{S}^* = \langle S_1^*, \ldots, S_M^* \rangle$ be any sequence of nonempty sets such that $S_i^* \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$. Then $\mathcal{S}^*$ is consistent with the affinities.*

Proof Suppose $\mathcal{S}^*$ is not consistent with the affinities. Then for some distinct i and j there is a V-path $\langle v_{0,}, \ldots, v_l \rangle$ from $S_i^* \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$ to $S_j^* \subseteq \text{Core}_j^{\Psi,\mathcal{S}} \subseteq V \setminus \text{Core}_i^{\Psi,\mathcal{S}}$ such that $\psi(\langle v_{0,}, \ldots, v_l \rangle) = 1$. Let k be an index such that $v_k \in \text{Core}_i^{\Psi,\mathcal{S}}$ but $v_{k+1} \in V \setminus \text{Core}_i^{\Psi,\mathcal{S}}$. Then $\psi_i^{O_i^{\text{MOFS}}(\Psi,\mathcal{S})}(S_i, v_k) \ge \psi_i^{\emptyset}(v_k, V \setminus \text{Core}_i^{\Psi,\mathcal{S}}) \ge \psi(v_k, v_{k+1}) \ge \psi(\langle v_{0,}, \ldots, v_l \rangle) = 1$ (since $\text{Core}_i^{\Psi,\mathcal{S}} \in \mathcal{P}_i(\Psi, \mathcal{S})$), whence there is a V-path p from S_i to v_k such that $\psi_i(p) = 1$. Now $p \cdot \langle v_k, \ldots, v_l \rangle$ is a V-path from S_i to v_l, and since $\psi_i(p \cdot \langle v_k, \ldots, v_l \rangle) = \min(\psi_i(p), \psi_i(\langle v_k, \ldots, v_l \rangle)) = 1$ this V-path is recursively $(\Psi, \mathcal{S})$-optimal. So $v_l \in O_i^{\text{MOFS}}(\Psi, \mathcal{S})$, by Theorem 3.10.

But this is impossible since $v_l \in S_j^* \subseteq \mathrm{Core}_j^{\Psi,\mathcal{S}} \subseteq O_j^{\mathrm{MOFS}}(\Psi,\mathcal{S})\setminus\bigcup_{m\neq j} O_m^{\mathrm{MOFS}}(\Psi,\mathcal{S})$. □

We now state our main robustness theorem, which specifies an extent to which the seed sets $S_1,\ldots,S_M$ can be enlarged without changing the MOFS objects. Note that the theorem assumes the affinities remain unchanged when we change the seed sets $\langle S_1,\ldots,S_M\rangle$ to $\langle R_1,\ldots,R_M\rangle$, so it does not apply to seed-set-dependent affinities (such as those used in the experiments presented in [10]). Object-feature-based affinities of the kind discussed in [18, Sect. 2.2] are examples of affinities to which this theorem and Corollaries 5.5 and 5.6 would apply (even if different affinities are used for different objects).

Theorem 5.4 *Let $\Psi = \langle\psi_1,\ldots,\psi_M\rangle$ be a sequence of affinities on V and $\mathcal{S} = \langle S_1,\ldots,S_M\rangle$ a sequence of pairwise disjoint nonempty seed sets consistent with the affinities. Let $\mathcal{R} = \langle R_1,\ldots,R_M\rangle$ be such that $S_i \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$. Then $O_i^{\mathrm{MOFS}}(\Psi,\mathcal{R}) = O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})$ for $1 \le i \le M$.*

This theorem will be proved in Sect. 6.7. It implies a far more general robustness result:

Corollary 5.5 *Let $\Psi = \langle\psi_1,\ldots,\psi_M\rangle$ be a sequence of affinities on V, and let $\mathcal{S} = \langle S_1,\ldots,S_M\rangle$ be a sequence of pairwise disjoint nonempty subsets of V consistent with the affinities. Let $\mathcal{S}^* = \langle S_1^*,\ldots,S_M^*\rangle$ be any sequence of nonempty sets that satisfy the conditions $S_i^* \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ and $S_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}^*}$ for $1 \le i \le M$. Then we have that $O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S}^*) = O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})$ for $1 \le i \le M$.*

Proof Define $R_i = \mathrm{Core}_i^{\Psi,\mathcal{S}} \cap \mathrm{Core}_i^{\Psi,\mathcal{S}^*}$ for $1 \le i \le M$, so $S_i \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ and $S_i^* \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}^*}$. Then, by Proposition 5.3 and Theorem 5.4, $O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S}) = O_i^{\mathrm{MOFS}}(\Psi,\mathcal{R}) = O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S}^*)$ for $1 \le i \le M$. □

The following is a notable special case of Corollary 5.5:

Corollary 5.6 *Let $\Psi = \langle\psi_1,\ldots,\psi_M\rangle$ be a sequence of affinities on V, let $\mathcal{S} = \langle S_1,\ldots,S_M\rangle$ be a sequence of pairwise disjoint nonempty subsets of V consistent with the affinities, and for $1 \le i \le M$ let a_i be a value in $(0,1]$ that is $(\Psi,\mathcal{S},i)$-large. Let $\mathcal{S}^* = \langle S_1^*,\ldots,S_M^*\rangle$ be any sequence of nonempty sets such that*

$$\forall i \in \{1,\ldots,M\}:\ S_i^* \subseteq \{v \in V \mid \psi_i^V(S_i,v) \ge a_i\} \\ \text{and } S_i \subseteq \{v \in V \mid \psi_i^V(S_i^*,v) \ge a_i\}. \tag{5.1}$$

Then we have that $O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S}^*) = O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})$ for $1 \le i \le M$.

Proof We claim a_i is $(\Psi,\mathcal{S}^*,i)$-large for $1 \le i \le M$. Indeed, let $i,j \in \{1,\ldots,M\}$ and let $a = \min(a_i,a_j)$. Then $S_i^* \subseteq \{v \in V \mid \psi_i^V(S_i,v) \ge a\}$ and so $\{v \in V \mid \psi_i^V(S_i^*,v) \ge a\} \subseteq \{v \in V \mid \psi_i^V(S_i,v) \ge a\}$. Similarly, $\{v \in V \mid \psi_j^V(S_j^*,v) \ge a\} \subseteq \{v \in V \mid \psi_j^V(S_j,v) \ge a\}$. Therefore

$$\{v \in V \mid \psi_i^V(S_i^*,v) \ge a_i\} \cap \{v \in V \mid \psi_j^V(S_j^*,v) \ge a_i\} \\ \subseteq \{v \in V \mid \psi_i^V(S_i,v) \ge a\} \cap \{v \in V \mid \psi_j^V(S_j,v) \ge a\}.$$

As $a = \min(a_i,a_j)$, the intersection on the right is $\{v \in V \mid \psi_i^V(S_i,v) \ge a_i\} \cap \{v \in V \mid \psi_j^V(S_j,v) \ge a_i\}$ or is $\{v \in V \mid \psi_i^V(S_i,v) \ge a_j\} \cap \{v \in V \mid \psi_j^V(S_j,v) \ge a_j\}$, and these intersections are empty if $i \neq j$ since a_i is $(\Psi,\mathcal{S},i)$-large and a_j is $(\Psi,\mathcal{S},j)$-large. Thus $\{v \in V \mid \psi_i^V(S_i^*,v) \ge a_i\} \cap \{v \in V \mid \psi_j^V(S_j^*,v) \ge a_i\} = \emptyset$ if $i \neq j$. This justifies our claim. The corollary follows from this claim, Propositions 5.2 and 5.3, and Corollary 5.5. □

Remark 5.7 If the affinities $\psi_1,\ldots,\psi_M$ in Corollary 5.6 are symmetric, then the corollary can be understood as saying that the segmentations $\langle O_1^{\mathrm{MOFS}}(\Psi,\mathcal{S}^*),\ldots,O_M^{\mathrm{MOFS}}(\Psi,\mathcal{S}^*)\rangle$ and $\langle O_1^{\mathrm{MOFS}}(\Psi,\mathcal{S}),\ldots,O_M^{\mathrm{MOFS}}(\Psi,\mathcal{S})\rangle$ are identical whenever the sets S_i^* satisfy

$$\forall i \in \{1,\ldots,M\}:\ S_i^* = \bigcup_{s\in S_i} P(s) \tag{5.2}$$

for some sets $P(s)$ such that

$$\forall i \in \{1,\ldots,M\}, \forall s \in S_i: \\ \emptyset \neq P(s) \subseteq \{v \in V \mid \psi_i^V(s,v) \ge a_i\}. \tag{5.3}$$

(Indeed, assuming $\psi_1,\ldots,\psi_M$ are symmetric, conditions (5.2) and (5.3) imply (5.1), and conversely (5.1) implies that (5.2) and (5.3) hold when $P(s) = \{v \in S_i^* \mid \psi_i^V(s,v) \ge a_i\}$.) Here the points in $P(s)$ may be thought of as resulting from "perturbations" of the point s of S_i within the region $\{v \in V \mid \psi_i^V(s,v) \ge a_i\}$.

The next proposition tells us that the cores of MOFS objects are robust (with respect to small changes in seed sets) in the same way that the MOFS objects themselves are robust:

Proposition 5.8 *The hypotheses of Theorem 5.4 imply $\mathrm{Core}_i^{\Psi,\mathcal{R}} = \mathrm{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$. Similarly, the hypotheses of Corollary 5.5 imply $\mathrm{Core}_i^{\Psi,\mathcal{S}^*} = \mathrm{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$.*

Proof Let $\Psi = \langle\psi_1,\ldots,\psi_M\rangle$ be any sequence of affinities on V and $\mathcal{S} = \langle S_1,\ldots,S_M\rangle$ any sequence of pairwise disjoint nonempty subsets of V. We first establish that

$$\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i,w) = \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(R_i,w) \\ \text{whenever } S_i \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}} \text{ and } w \in V\setminus\mathrm{Core}_i^{\Psi,\mathcal{S}}. \tag{5.4}$$

Let $S_i \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ and $w \in V \backslash \mathrm{Core}_i^{\Psi,\mathcal{S}}$. As $S_i \subseteq R_i$, $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, w) \leq \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(R_i, w)$. To show the values are equal, let $\langle x_0, \ldots, x_l \rangle$ be an $(O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S}) \cup \{w\})$-path from $x_0 \in R_i$ to $x_l = w$ such that $\psi_i(\langle x_0, \ldots, x_l \rangle) = \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(R_i, w)$, let k be the least index for which $x_{k+1} \notin \mathrm{Core}_i^{\Psi,\mathcal{S}}$ (so that $x_k \in \mathrm{Core}_i^{\Psi,\mathcal{S}} \subseteq O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$), and let p be an $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$-path from S_i to x_k such that $\psi_i(p) = \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, x_k)$. As $x_k \in \mathrm{Core}_i^{\Psi,\mathcal{S}} \in \mathcal{P}_i(\Psi, \mathcal{S})$ and $x_{k+1} \in V \backslash \mathrm{Core}_i^{\Psi,\mathcal{S}}$, condition (b) of Definition 5.1 implies $\psi_i(p) = \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, x_k) \geq \psi_i(x_k, x_{k+1}) \geq \psi_i(\langle x_k, \ldots, x_l \rangle)$, and so (since $p \cdot \langle x_k, \ldots, x_l \rangle$ is an $(O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S}) \cup \{w\})$-path from S_i to w) we must have that $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, w) \geq \psi_i(p \cdot \langle x_k, \ldots, x_l \rangle) = \psi_i(\langle x_k, \ldots, x_l \rangle) \geq \psi_i(\langle x_0, \ldots, x_l \rangle) = \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(R_i, w) \geq \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, w)$. This establishes (5.4).

Next, we prove the first assertion of the proposition. Suppose the hypotheses of Theorem 5.4 are satisfied and $i \in \{1, \ldots, M\}$. We first show $\mathrm{Core}_i^{\Psi,\mathcal{S}}$ is not a *proper* subset of $\mathrm{Core}_i^{\Psi,\mathcal{R}}$. Indeed, suppose otherwise and let $P = \mathrm{Core}_i^{\Psi,\mathcal{R}}$, so that $\mathrm{Core}_i^{\Psi,\mathcal{S}} \subsetneq P$ and hence $V \backslash \mathrm{Core}_i^{\Psi,\mathcal{S}} \supsetneq V \backslash P$. As $P \in \mathcal{P}_i(\Psi, \mathcal{R})$, we have that $P \subseteq O_i^{\mathrm{MOFS}}(\Psi, \mathcal{R}) \backslash \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}(\Psi, \mathcal{R}) = O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S}) \backslash \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}(\Psi, \mathcal{S})$, where the equality follows from Theorem 5.4. As $\mathrm{Core}_i^{\Psi,\mathcal{S}} \in \mathcal{P}_i(\Psi, \mathcal{S})$ and $V \backslash \mathrm{Core}_i^{\Psi,\mathcal{S}} \supsetneq V \backslash P$, we have that $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \geq \psi_i^{\emptyset}(v, V \backslash \mathrm{Core}_i^{\Psi,\mathcal{S}}) \geq \psi_i^{\emptyset}(v, V \backslash P)$ for all $v \in \mathrm{Core}_i^{\Psi,\mathcal{S}}$. Similarly, $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, v) = \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(R_i, v) \geq \psi_i^{\emptyset}(v, V \backslash P)$ for all $v \in P \backslash \mathrm{Core}_i^{\Psi,\mathcal{S}}$ because of (5.4), the fact that $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{R}) = O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$, and the fact that $P \in \mathcal{P}_i(\Psi, \mathcal{R})$. Hence we have that $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \geq \psi_i^{\emptyset}(v, V \backslash P)$ for all $v \in P$. Therefore $P \in \mathcal{P}_i(\Psi, \mathcal{S})$, whence $P \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$. This contradiction establishes that $\mathrm{Core}_i^{\Psi,\mathcal{S}}$ is not a *proper* subset of $\mathrm{Core}_i^{\Psi,\mathcal{R}}$. On the other hand, since $S_i \subseteq R_i$ and $O_j^{\mathrm{MOFS}}(\Psi, \mathcal{R}) = O_j^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ for $1 \leq j \leq M$ (which imply $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{R})}(R_i, v) \geq \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, v)$ for all $v \in V$), we see from Definition 5.1 that $\mathcal{P}_i(\Psi, \mathcal{S}) \subseteq \mathcal{P}_i(\Psi, \mathcal{R})$, whence $\mathrm{Core}_i^{\Psi,\mathcal{S}} \subseteq \mathrm{Core}_i^{\Psi,\mathcal{R}}$. Thus $\mathrm{Core}_i^{\Psi,\mathcal{S}} = \mathrm{Core}_i^{\Psi,\mathcal{R}}$. This proves the first assertion.

To prove the second assertion, suppose the hypotheses of Corollary 5.5 are satisfied. For $1 \leq i \leq M$ define $R_i = \mathrm{Core}_i^{\Psi,\mathcal{S}} \cap \mathrm{Core}_i^{\Psi,\mathcal{S}^*}$. Then Ψ, $\mathcal{S}$, and $\mathcal{R} = \langle R_1, \ldots, R_M \rangle$ satisfy the hypotheses of Theorem 5.4. So we see from the first assertion that $\mathrm{Core}_i^{\Psi,\mathcal{S}} = \mathrm{Core}_i^{\Psi,\mathcal{R}}$ $(1 \leq i \leq M)$. Symmetrically, $\mathrm{Core}_i^{\Psi,\mathcal{S}^*} = \mathrm{Core}_i^{\Psi,\mathcal{R}}$. □

The following two propositions will give the reader a sense of how big the sets $\mathrm{Core}_i^{\Psi,\mathcal{S}}$ are. In particular, the next proposition defines a set $Q_i^{\Psi,\mathcal{S}}$ that will typically be much larger than S_i, but which is still a member of $\mathcal{P}_i(\Psi, \mathcal{S})$ and therefore a subset of $\mathrm{Core}_i^{\Psi,\mathcal{S}}$. When the affinity ψ_i is symmetric, $\mathrm{Core}_i^{\Psi,\mathcal{S}}$ is exactly $Q_i^{\Psi,\mathcal{S}}$.

Proposition 5.9 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be a sequence of affinities on V, and let $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ be a sequence of pairwise disjoint nonempty subsets of V that are consistent with the affinities. For $1 \leq i \leq M$ let $Q_i^{\Psi,\mathcal{S}} = \{v \in O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S}) \backslash \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}(\Psi, \mathcal{S}) \mid \psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \geq \psi_i^V(v, \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}(\Psi, \mathcal{S}))\}$. Then:*

1. *$Q_i^{\Psi,\mathcal{S}} \in \mathcal{P}_i(\Psi, \mathcal{S})$, and hence $\mathrm{Core}_i^{\Psi,\mathcal{S}} \supseteq Q_i^{\Psi,\mathcal{S}}$, for each $i \in \{1, \ldots, M\}$.*
2. *$\mathrm{Core}_i^{\Psi,\mathcal{S}} = Q_i^{\Psi,\mathcal{S}}$ for each $i \in \{1, \ldots, M\}$ such that ψ_i is symmetric.*

Proof For brevity we will write O_j^{MOFS} for $O_j^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ in this proof.

Let $i \in \{1, \ldots, M\}$. First we will establish statement 1 by verifying that, when $P = Q_i^{\Psi,\mathcal{S}}$, condition (b) of Definition 5.1 holds. (It is evident that (a) holds.) Let $v \in P = Q_i^{\Psi,\mathcal{S}}$ and $w \in V \backslash P = V \setminus Q_i^{\Psi,\mathcal{S}}$. What we need to show is that $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v) \geq \psi_i(v, w)$.

Now either $w \in \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$, or $w \in V \backslash \bigcup_j O_j^{\mathrm{MOFS}}$, or $w \in O_i^{\mathrm{MOFS}} \backslash \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$. If $w \in \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$, then $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v) \geq \psi_i(v, w)$ holds because $v \in Q_i^{\Psi,\mathcal{S}}$. If $w \in V \backslash \bigcup_j O_j^{\mathrm{MOFS}}$, then $\psi_i(v, w) = 0$ by Corollary 3.12 (as $v \in Q_i^{\Psi,\mathcal{S}} \subseteq O_i^{\mathrm{MOFS}}$) so we certainly have that $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v) \geq \psi_i(v, w)$. Suppose finally that $w \in O_i^{\mathrm{MOFS}} \backslash \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$. Then $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v) < \psi_i(v, w)$ is again impossible: It would imply $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, w) \geq \psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$ (because $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, w) \geq \min(\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v), \psi_i(v, w))$), and since $\psi_i^V(w, \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}) > \psi_i^{O_i^{\mathrm{MOFS}}}(S_i, w)$ (as $w \notin Q_i^{\Psi,\mathcal{S}}$) we would then have that $\psi_i^V(w, \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}) > \psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$, whence $\psi_i^V(v, \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}) \geq \min(\psi_i(v, w), \psi_i^V(w, \bigcup_{j \neq i} O_j^{\mathrm{MOFS}})) > \psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$, which would contradict $v \in Q_i^{\Psi,\mathcal{S}}$. So statement 1 holds.

To establish statement 2, we assume ψ_i is symmetric. We have shown that $\mathrm{Core}_i^{\Psi,\mathcal{S}} \supseteq Q_i^{\Psi,\mathcal{S}}$, but must now prove that $\mathrm{Core}_i^{\Psi,\mathcal{S}} \subseteq Q_i^{\Psi,\mathcal{S}}$. For this, fix a $v \in \mathrm{Core}_i^{\Psi,\mathcal{S}}$. We need to show that $v \in Q_i^{\Psi,\mathcal{S}}$.

Suppose not. Then $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v) < \psi_i^V(v, \bigcup_{j \neq i} O_j^{\mathrm{MOFS}})$ and so there exists a V-path $\langle v_0, \ldots, v_l \rangle$ from $v = v_0$ to $\bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$ with $\psi_i(\langle v_0, \ldots, v_l \rangle) > \psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$. Let k be the greatest index for which $\langle v_0, \ldots, v_k \rangle$ is a $\mathrm{Core}_i^{\Psi,\mathcal{S}}$-

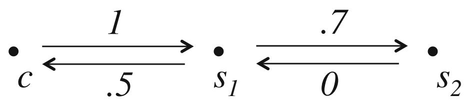

Fig. 5 Affinity values for Example 5.10

path. Then $k < l$ (since $v_l \in \bigcup_{j \neq i} O_j^{\text{MOFS}} \subseteq V \setminus \text{Core}_i^{\Psi,\mathcal{S}}$) and $\langle v_0, \ldots, v_k \rangle$ is an O_i^{MOFS}-path.

As $\psi_i(v_k, v_{k+1}) \geq \psi_i(\langle v_0, \ldots, v_l \rangle) > \psi_i^{O_i^{\text{MOFS}}}(S_i, v)$, it is impossible that $\psi_i^{O_i^{\text{MOFS}}}(S_i, v) \geq \psi_i^{O_i^{\text{MOFS}}}(S_i, v_k)$, because we would then have that $\psi_i(v_k, v_{k+1}) > \psi_i^{O_i^{\text{MOFS}}}(S_i, v_k)$ and so condition (b) of Definition 5.1 would be violated when $P = \text{Core}_i^{\Psi,\mathcal{S}}$ (since $v_k \in \text{Core}_i^{\Psi,\mathcal{S}} \in$ but $v_{k+1} \notin \text{Core}_i^{\Psi,\mathcal{S}}$). We therefore have that $\psi_i^{O_i^{\text{MOFS}}}(S_i, v_k) > \psi_i^{O_i^{\text{MOFS}}}(S_i, v)$, whence there exists an O_i^{MOFS}-path p from S_i to v_k for which $\psi_i(p) > \psi_i^{O_i^{\text{MOFS}}}(S_i, v)$. But now $p \cdot \langle v_k, \ldots, v_0 \rangle$ is an O_i^{MOFS}-path from S_i to v, and (since ψ_i is symmetric) we have that

$$\begin{aligned}\psi_i(p \cdot \langle v_k, \ldots, v_0 \rangle) &= \min(\psi_i(p), \psi_i(\langle v_k, \ldots, v_0 \rangle)) \\ &= \min(\psi_i(p), \psi_i(\langle v_0, \ldots, v_k \rangle)) \\ &\geq \min(\psi_i(p), \psi_i(\langle v_0, \ldots, v_l \rangle)) \\ &> \psi_i^{O_i^{\text{MOFS}}}(S_i, v),\end{aligned}$$

a contradiction. □

Regarding statement 2 of this proposition, we now give an example which shows that $Q_i^{\Psi,\mathcal{S}} = \text{Core}_i^{\Psi,\mathcal{S}}$ need not hold even if all the affinities ψ_i are equal to the same affinity ψ, if the affinity ψ is not symmetric:

Example 5.10 Suppose $V = \{c, s_1, s_2\}$, $\mathcal{S} = \langle S_1, S_2 \rangle$, and $\Psi = \langle \psi, \psi \rangle$, where $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, $\psi(s_1, c) = 0.5$, $\psi(c, s_1) = 1$, $\psi(s_1, s_2) = 0.7$, and $\psi(u, v) = 0$ in all other cases where $u, v \in V$ are distinct (e.g., $\psi(c, s_2) = 0$): See Figure 5. Then $\text{Core}_1^{\Psi,\mathcal{S}} = \{s_1, c\}$—i.e., $\text{Core}_1^{\Psi,\mathcal{S}} = O_1^{\text{MOFS}}$. But $c \notin Q_1^{\Psi,\mathcal{S}}$ because the ψ-strength of a ψ-strongest V-path from c to S_2 is 0.7, which exceeds the ψ-strength of a ψ-strongest V-path from S_1 to c.

In the important case where the affinities $\psi_1, \ldots, \psi_M$ are all equal to the same affinity ψ and this affinity ψ is symmetric, the next proposition tells us that $\text{Core}_i^{\Psi,\mathcal{S}} = Q_i^{\Psi,\mathcal{S}}$ is exactly the ith object of the IRFC segmentation for affinity ψ and seed sets $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$, which we denote by $O_i^{\text{IRFC}}(\psi, \mathcal{S})$ as in Sect. 3.3.

Proposition 5.11 *Let ψ be a symmetric affinity on V, let Ψ be a sequence $\langle \psi, \ldots, \psi \rangle$ of M occurrences of ψ, let $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ be any sequence of pairwise disjoint nonempty subsets of V that are consistent with ψ. Then for $1 \leq i \leq M$ we have that* $\text{Core}_i^{\Psi,\mathcal{S}} = O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}}(\Psi, \mathcal{S}) = O_i^{\text{IRFC}}(\psi, \mathcal{S})$.

Proof For brevity we will write O_j^{MOFS} for $O_j^{\text{MOFS}}(\Psi, \mathcal{S})$ in this proof. For $1 \leq k \leq M$, Corollary 3.5 and Theorem 3.10 imply that $x \in O_k^{\text{MOFS}}$ just if there is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path of nonzero ψ-strength from S_k to x, whence (by Corollaries 3.5 and 3.11) if p is any hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path from S_k to $x \in O_k^{\text{MOFS}}$ then $0 < \psi(p) = \psi^V(\bigcup_j S_j, x) = \psi^{O_k^{\text{MOFS}}}(S_k, x)$.

Let $i \in \{1, \ldots, M\}$. We see from Definition 5.1 that $\text{Core}_i^{\Psi,\mathcal{S}} \subseteq O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}}$, and from Corollary 2.7 that $O_i^{\text{IRFC}}(\psi, \mathcal{S}) = O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}}$. It remains only to show $O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}} \subseteq \text{Core}_i^{\Psi,\mathcal{S}}$. For this purpose it is enough to show $O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}} \in \mathcal{P}_i(\Psi, \mathcal{S})$. So it is enough to verify that condition (b) of Definition 5.1 holds when $P = O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}}$ (since condition (a) obviously holds). To do this, let v be any point in $O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}}$ and let w be any point in $V \setminus (O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}})$. What we need to show is that $\psi^{O_i^{\text{MOFS}}}(S_i, v) \geq \psi(v, w)$.

Suppose not. Then $\psi(v, w) > \psi^{O_i^{\text{MOFS}}}(S_i, v)$. So $w \in \bigcup_j O_j^{\text{MOFS}}$ by Corollary 3.12, and therefore $w \in \bigcup_{j \neq i} O_j^{\text{MOFS}}$ (as $w \in V \setminus (O_i^{\text{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}})$). Since $v \in O_i^{\text{MOFS}}$, we see from the remarks in the first paragraph that there is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path p_v from S_i to v such that $0 < \psi(p_v) = \psi^V(\bigcup_j S_j, v) = \psi^{O_i^{\text{MOFS}}}(S_i, v) < \psi(v, w)$. Similarly, since $w \in \bigcup_{j \neq i} O_j^{\text{MOFS}}$, there is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path p_w from $\bigcup_{j \neq i} S_j$ to w that satisfies $\psi(p_w) = \psi^V(\bigcup_j S_j, w) \geq \psi(p_v \cdot \langle v, w \rangle) = \min(\psi(p_v), \psi(v, w)) = \psi(p_v) > 0$ and therefore satisfies $\psi(p_w \cdot \langle w, v \rangle) = \min(\psi(p_w), \psi(w, v)) = \min(\psi(p_w), \psi(v, w)) \geq \min(\psi(p_v), \psi(v, w)) = \psi(p_v) = \psi^V(\bigcup_j S_j, v)$. This and the hereditary $(\psi, \bigcup_j S_j)$-optimality of p_w imply that $p_w \cdot \langle w, v \rangle$ is hereditarily $(\psi, \bigcup_j S_j)$-optimal. But, since $v \notin \bigcup_{j \neq i} O_j^{\text{MOFS}}$, the remarks in the first paragraph imply there is no hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path of nonzero ψ-strength from $\bigcup_{j \neq i} S_j$ to v, a contradiction. □

When the seeds sets are consistent with the affinity, the IRFC robustness results mentioned earlier (i.e., the first assertions of [21, Theorem 2.5] and [21, Corollary 2.6]) can be deduced from this proposition, Theorem 5.4, and Corollary 5.5.

We end this section with an example which shows that Proposition 5.11 is not true (i.e., $\text{Core}_i^{\Psi,\mathcal{S}}$ need not be the whole of $O_i^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \neq i} O_j^{\text{MOFS}}(\Psi, \mathcal{S})$) if we drop the hypothesis that the affinities $\psi_1, \ldots, \psi_M$ are all equal, even if we assume every affinity ψ_j is symmetric:

Example 5.12 Let $V = \{s_1, s_2, c, d, e\}$, $M = 2$, $\mathcal{S} = \langle S_1, S_2 \rangle = \langle \{s_1\}, \{s_2\} \rangle$, and $\Psi = \langle \psi_1, \psi_2 \rangle$, where ψ_1 and

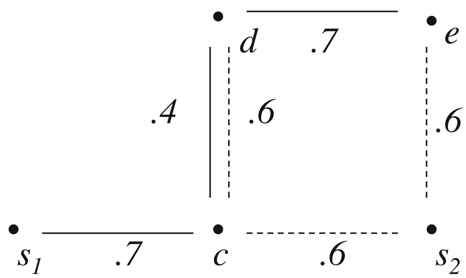

Fig. 6 Affinity values for Example 5.12

ψ_2 are the symmetric affinities on V such that $\psi_1(s_1, c) = \psi_1(c, s_1) = \psi_1(d, e) = \psi_1(e, d) = 0.7$, $\psi_1(c, d) = \psi_1(d, c) = 0.4$, $\psi_2(s_2, c) = \psi_2(c, s_2) = \psi_2(s_2, e) = \psi_2(e, s_2) = \psi_2(c, d) = \psi_2(d, c) = 0.6$, and $\psi_i(u, v) = 0$ in all other cases where u and v are distinct points in V and $i \in \{1, 2\}$. (See Fig. 6.) Then it is readily confirmed that $O_1^{\text{MOFS}}(\Psi, \mathcal{S}) = \{s_1, c, d\}$ and $O_2^{\text{MOFS}}(\Psi, \mathcal{S}) = \{s_2, e\}$, and so we have that $O_1^{\text{MOFS}}(\Psi, \mathcal{S}) \setminus \bigcup_{j \neq 1} O_j^{\text{MOFS}}(\Psi, \mathcal{S}) = O_1^{\text{MOFS}}(\Psi, \mathcal{S}) = \{s_1, c, d\}$. But the set $\text{Core}_1^{\Psi,\mathcal{S}} = Q_1^{\Psi,\mathcal{S}}$ is smaller: $\text{Core}_1^{\Psi,\mathcal{S}} = Q_1^{\Psi,\mathcal{S}} = \{s_1, c\}$. Moreover, if we define $R_1 = O_1^{\text{MOFS}}(\Psi, \mathcal{S}) \not\subset \text{Core}_1^{\Psi,\mathcal{S}}$ and $R_2 = S_2$, then the conclusion of Theorem 5.4 would not hold any more, since $O_1^{\text{MOFS}}(\Psi, \langle R_1, R_2 \rangle) = \{s_1, c, d, e\} \neq O_1^{\text{MOFS}}(\Psi, \mathcal{S})$.

6 Proofs of Theorems and Justification of Algorithm 5

6.1 Proofs of Theorems 2.4, 2.5, and 2.6

We claim that, when Algorithm 1, 2, or 3 is executed, those of the conditions (6.1)–(6.10) below which apply to that algorithm will hold *immediately before* execution of line 5 at the nth iteration of the algorithm's main loop (for all $n \in \{1, \dots, |A|\}$ and $i \in \{1, \dots, M\}$). The three theorems will be deduced from this claim.

Although Algorithm 2 uses just a single affinity ψ, whereas Algorithm 3 uses M affinities $\psi_1, \dots, \psi_M$, these two algorithms share some important properties. To avoid having to state these shared properties twice, we adopt the following convention: When we are considering Algorithm 2, the notation ψ_i will mean the single affinity ψ of Algorithm 2, regardless of the value of the subscript i (which should be ignored). This convention is used, for example, in (6.6)–(6.10) below and in the statements of Propositions 6.3 and 6.4.

for Algorithms 1, 2, and 3:
$$T_i \subseteq newT_i \subseteq T_i \cup (V \setminus \textstyle\bigcup_j T_j) \tag{6.1}$$

for Algorithm 1:
$$newT_i = T_i \cup \{v \in V \setminus \textstyle\bigcup_j T_j \mid \psi^V(T_i, v) \geq \alpha_n\} \tag{6.2}$$
$$= T_i \cup \{v \in V \setminus \textstyle\bigcup_j T_j \mid \psi^V(T_i, v) = \alpha_n\} \tag{6.3}$$
$$\psi^V(u, v) < \alpha_n \text{ if } u \in newT_i \text{ and } v \in (V \setminus \textstyle\bigcup_j T_j) \setminus newT_i \tag{6.4}$$
$$\psi^V(u, v) \leq \alpha_n \text{ if } u \in T_i \text{ and } v \in V \setminus \textstyle\bigcup_j T_j \tag{6.5}$$

for Algorithms 2 and 3:
$$newT_i = T_i \cup \left\{v \in V \setminus \textstyle\bigcup_j T_j \mid \psi_i^{V \setminus \bigcup_j T_j}(T_i, v) \geq \alpha_n\right\} \tag{6.6}$$
$$= T_i \cup \left\{v \in V \setminus \textstyle\bigcup_j T_j \mid \psi_i^{V \setminus \bigcup_j T_j}(T_i, v) = \alpha_n\right\} \tag{6.7}$$
$$\psi_i^{V \setminus \bigcup_j T_j}(u, v) < \alpha_n \text{ if } u \in newT_i \text{ and } v \in (V \setminus \textstyle\bigcup_j T_j) \setminus newT_i \tag{6.8}$$
$$\psi_i^{V \setminus \bigcup_j T_j}(u, v) \leq \alpha_n \text{ if } u \in T_i \text{ and } v \in V \setminus \textstyle\bigcup_j T_j \tag{6.9}$$
$$\psi_i^{newT_i}(T_i, v) = \alpha_n. \quad \text{whenever } v \in newT_i \setminus T_i \tag{6.10}$$

Now we justify our claim. It is readily confirmed by inspection of each algorithm that our claim is valid in the cases of (6.1), (6.2), and (6.6). It is also easy to see that (6.2) implies (6.4): Indeed, if we suppose (6.2) holds and $u \in newT_i$ (so that $\psi^V(T_i, u) \geq \alpha_n$), then if $v \in V \setminus \bigcup_j T_j$ satisfied $\psi^V(u, v) \geq \alpha_n$ we would have that $\psi^V(T_i, v) \geq \alpha_n$ (by Proposition 2.2) and hence that $v \in newT_i$. Similarly, (6.6) implies (6.8).

Since $\alpha_1 = 1$, (6.5) and (6.9) cannot be false when $n = 1$. Now suppose $n > 1$. Let us write T_j^{cur} and $newT_i^{\text{cur}}$ for the values of T_j and $newT_i$ immediately before execution of line 5 at the nth iteration of the main loop, and write T_j^{prev} and $newT_i^{\text{prev}}$ for the values of T_j and $newT_i$ at the same stage of the $n-1$st iteration. Then we see from (6.4) and (6.8) that

$$\begin{aligned}&\psi^V(u, v) < \alpha_{n-1} \text{ if}\\ &\quad u \in newT_i^{\text{prev}} \text{ and } v \in (V \setminus \textstyle\bigcup_j T_j^{\text{prev}}) \setminus newT_i^{\text{prev}}\\ &\qquad \text{for Algorithm 1}\end{aligned} \tag{6.11}$$

$$\begin{aligned}&\psi_i^{V \setminus \bigcup_j T_j^{\text{prev}}}(u, v) < \alpha_{n-1} \text{ if}\\ &\quad u \in newT_i^{\text{prev}} \text{ and } v \in (V \setminus \textstyle\bigcup_j T_j^{\text{prev}}) \setminus newT_i^{\text{prev}}\\ &\qquad \text{for Algorithms 2, 3}\end{aligned} \tag{6.12}$$

Since $T_i^{\text{cur}} = newT_i^{\text{prev}} \supseteq T_i^{\text{prev}}$, we have that $V \setminus \bigcup_j T_j^{\text{prev}} \supseteq V \setminus \bigcup_j T_j^{\text{cur}}$ and therefore also have that $(V \setminus \bigcup_j T_j^{\text{prev}}) \setminus newT_i^{\text{prev}} \supseteq (V \setminus \bigcup_j T_j^{\text{cur}}) \setminus newT_i^{\text{prev}} = (V \setminus \bigcup_j T_j^{\text{cur}}) \setminus T_i^{\text{cur}} = V \setminus \bigcup_j T_j^{\text{cur}}$. Moreover, an affinity value is $< \alpha_{n-1}$ just if it is $\leq \alpha_n$. So we deduce from (6.11) and (6.12) that

$$\begin{aligned}&\psi^V(u, v) \leq \alpha_n \text{ if } u \in T_i^{\text{cur}} \text{ and } v \in V \setminus \textstyle\bigcup_j T_j^{\text{cur}}\\ &\quad \text{for Algorithm 1}\\ &\psi_i^{V \setminus \bigcup_j T_j^{\text{cur}}}(u, v) \leq \alpha_n \text{ if } u \in T_i^{\text{cur}} \text{ and } v \in V \setminus \textstyle\bigcup_j T_j^{\text{cur}}\\ &\quad \text{for Algorithms 2, 3}\end{aligned}$$

Thus we have shown that (6.5) and (6.9) are true immediately before execution of line 5 at the nth iteration of the main loop. Evidently, (6.3) and (6.7) follow from (6.2), (6.5), (6.6), and (6.9).

To see that our claim is valid in the case of (6.10), let $i \in \{1, \ldots, M\}$ and consider the values of T_i and $newT_i$ immediately before execution of line 5 at the nth iteration of the main loop of Algorithm 2 or 3. Let v be any point in $newT_i \setminus T_i$. Then, by (6.7), we have that $v \in V \setminus \bigcup_j T_j$ and that $\psi_i^{V \setminus \bigcup_j T_j}(T_i, v) = \alpha_n$, whence there is a $(T_i \cup (V \setminus \bigcup_j T_j))$-path $\langle v_0, \ldots, v_l \rangle$ from T_i to $v_l = v$ such that $\psi_i(\langle v_0, \ldots, v_l \rangle) = \alpha_n$. Now for each point v_k of this $(T_i \cup (V \setminus \bigcup_j T_j))$-path we have that $\psi_i(\langle v_0, \ldots, v_k \rangle) \geq \psi_i(\langle v_0, \ldots, v_l \rangle) = \alpha_n$, and so $v_k \in T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi_i^{V \setminus \bigcup_j T_j}(T_i, v) \geq \alpha_n\}$, whence $v_k \in newT_i$ by (6.6). This shows that $\langle v_0, \ldots, v_l \rangle$ is a $newT_i$-path from T_i to v, so that $\psi_i^{newT_i}(T_i, v) \geq \psi_i(\langle v_0, \ldots, v_l \rangle) = \alpha_n$. However, since (6.1) implies $newT_i \subseteq T_i \cup (V \setminus \bigcup_j T_j)$ we also have that $\psi_i^{newT_i}(T_i, v) \leq \psi_i^{V \setminus \bigcup_j T_j}(T_i, v) = \alpha_n$ (by Proposition 2.1). This confirms that our claim is valid in the case of (6.10) and completes the justification of our claim.

Deduction of Theorems 2.4, 2.5, and 2.6 from (6.1)–(6.10)

As in Theorems 2.4 – 2.6, we will write T_i^n ($1 \leq i \leq M$, $0 \leq n < |A|$) to denote the value of the variable T_i at the beginning of the $n+1$st iteration of the main loop when Algorithm 1, 2, or 3 is executed, and write $T_i^{|A|}$ to denote the value of T_i at the end of the $|A|$th iteration (i.e., the final value of T_i).

In each of Theorems 2.4–2.6, inspection of Algorithm 1, 2, or 3 will convince the reader of the truth of statement 1(b), the first part of statement 1(a), and the claim that the ith object of the segmentation contains the seed set S_i. The other parts of these theorems will then follow from propositions that are proved below: In all three theorems, the claim that the ith object of the segmentation is contained in $S_i \cup (V \setminus \bigcup_j S_j)$ will follow from (the case $n = 1$ of statement 1 of) Proposition 6.1. In the case of Theorem 2.4, statement 2 and the second part of statement 1(a) will follow from Proposition 6.2 and (statement 2 of) Proposition 6.1. In the cases of Theorems 2.5 and 2.6, statement 2 and the second part of statement 1(a) will follow from Propositions 6.1 and 6.4; statement 3 of each of those theorems will follow from statement 2 of Proposition 6.4 and Propositions 6.1 and 2.1 (since Proposition 6.1 implies $T_i^n \subseteq T_i^k \subseteq T_i^{n-1} \cup (V \setminus \bigcup_j T_j^{n-1})$ for $1 \leq n \leq k \leq |A|$).

Proposition 6.1 *When Algorithm 1, 2, or 3 is executed, the following are true for* $1 \leq i \leq M$ *and* $1 \leq n \leq |A|$*:*

1. $T_i^{n-1} \subseteq T_i^n \subseteq T_i^{n-1} \cup (V \setminus \bigcup_j T_j^{n-1})$
2. $T_i^{n-1} \cup (V \setminus \bigcup_j T_j^{n-1}) \supseteq T_i^n \cup (V \setminus \bigcup_j T_j^n)$ $\supseteq T_i^{|A|} \cup (V \setminus \bigcup_j T_j^{|A|}) \supseteq T_i^{|A|}$

Proof Statement 1 holds since (6.1) holds immediately before execution of line 5 of the nth iteration of the main loop (and T_i^{n-1} and T_i^n are the values of T_i and $newT_i$ at that time). Readily, statement 1 implies $T_i^{n-1} \cup (V \setminus \bigcup_j T_j^{n-1}) \supseteq T_i^n \cup (V \setminus \bigcup_j T_j^n)$ for $1 \leq n \leq |A|$, whence $T_i^n \cup (V \setminus \bigcup_j T_j^n) \supseteq T_i^{|A|} \cup (V \setminus \bigcup_j T_j^{|A|})$. □

Proposition 6.2 *When Algorithm 1 is executed, the following hold for* $1 \leq i \leq M$ *and* $1 \leq n \leq |A|$*:*

1. $\{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) > \alpha_n\} = \emptyset$
2. $T_i^n \backslash T_i^{n-1} = \{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) = \alpha_n\}$

Proof Let $1 \leq i \leq M$ and $1 \leq n \leq |A|$. Then $\{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi_i^V(T_i^{n-1}, v) > \alpha_n\} = \emptyset$, by (6.5). Statement 1 follows from this, Proposition 6.1 (which implies $S_i = T_i^0 \subseteq T_i^{n-1}$), and Proposition 2.1.

Moreover, it follows from (6.3) and (6.4) that

$$\text{for } 1 \leq i \leq M, 1 \leq n \leq |A| : T_i^n \backslash T_i^{n-1} = \left\{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V\left(T_i^{n-1}, v\right) = \alpha_n\right\} \tag{6.13}$$

$$\text{for } 1 \leq i \leq M, 1 \leq n \leq |A| : \psi^V(u, v) < \alpha_n \text{ whenever } u \in T_i^n \text{ and } v \in \left(V \backslash \bigcup_j T_j^{n-1}\right) \backslash T_i^n \tag{6.14}$$

We will deduce from (6.13) that

$$\text{for } 1 \leq i \leq M, 1 \leq n \leq |A| : \psi^V(S_i, v) \geq \alpha_n \text{ whenever } v \in T_i^n \tag{6.15}$$

Let $v \in T_i^n$. Then we can define points $v_0, \ldots, v_n$ such that $v_0 = v$ and such that, for $1 \leq k \leq n$, v_k is a point in T_i^{n-k} that is defined in terms of v_{k-1} in the following way: If $v_{k-1} \in T_i^{n-k}$ then we define $v_k = v_{k-1}$, and if $v_{k-1} \notin T_i^{n-k}$ then we define v_k to be an arbitrary point in T_i^{n-k} such that $\psi^V(v_k, v_{k-1}) = \alpha_{n-(k-1)}$. (Note that in the latter case $v_{k-1} \in T_i^{n-(k-1)} \backslash T_i^{n-k}$, and so (6.13) implies the existence of a point v_k in T_i^{n-k} such that $\psi^V(v_k, v_{k-1}) = \alpha_{n-(k-1)}$.) Now $\langle v_n, \ldots, v_0\rangle$ is a V-path from $T_i^0 = S_i$ to v, and $\psi\langle v_n, \ldots, v_0\rangle \geq \alpha_n$ since $\psi(v_k, v_{k-1}) \geq \alpha_n$ for $1 \leq k \leq n$. Hence $\psi^V(S_i, v) \geq \alpha_n$. This establishes (6.15).

Now $T_i^n \backslash T_i^{n-1} \subseteq \{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) \geq \alpha_n\}$ by (6.13) and (6.15). On the other hand, we must have that $\{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) \geq \alpha_n\} \subseteq T_i^n \backslash T_i^{n-1}$. Indeed, since $\psi^V(T_i^n, v) \geq \psi^V(S_i, v)$ (as $S_i = T_i^0 \subseteq T_i^n$) we see that $\{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) \geq \alpha_n\} \backslash (T_i^n \setminus T_i^{n-1})$ is a subset of $\{v \in (V \backslash \bigcup_j T_j^{n-1}) \backslash T_i^n \mid \psi^V(T_i^n, v) \geq \alpha_n\}$, but the latter set is empty because of (6.14). This shows that $T_i^n \backslash T_i^{n-1} = \{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) \geq \alpha_n\}$. Statement 2 follows from this and statement 1. □

Proposition 6.3 *When Algorithm 2 or 3 is executed, statements 1 – 3 below are true for* $1 \leq i \leq M$ *and* $1 \leq n \leq |A|$:

1. $\psi_i^{V \backslash \bigcup_j T_j^{n-1}}(u, v) < \alpha_n$ *whenever* $u \in T_i^n$ *and* $v \in (V \backslash \bigcup_j T_j^{n-1}) \backslash T_i^n$.
2. $\psi_i^{T_i^n}(S_i, v) = \psi_i^{T_i^{|A|}}(S_i, v) = \psi_i^{T_i^{n-1} \cup (V \backslash \bigcup_j T_j^{n-1})}(S_i, v) = \alpha_n$ *for all* $v \in T_i^n \backslash T_i^{n-1}$.
3. $\psi_i^{T_i^n}(S_i, v) = \psi_i^{T_i^{|A|}}(S_i, v) = \psi_i^{T_i^{n-1} \cup (V \backslash \bigcup_j T_j^{n-1})}(S_i, v) \geq \alpha_n$ *for all* $v \in T_i^n$.

Proof Statement 1 follows from the fact that (6.8) holds immediately before execution of line 5 of the nth iteration of the main loop. Similarly, it follows from (6.7) that

$$\text{for } 1 \leq i \leq M, 1 \leq n \leq |A| : T_i^n \backslash T_i^{n-1} = \{v \in V \backslash \bigcup_j T_j^{n-1} \mid \psi_i^{V \backslash \bigcup_j T_j^{n-1}}(T_i^{n-1}, v) = \alpha_n\} \tag{6.16}$$

We will use (6.10) and (6.16) to establish statement 2. The first step is to observe that

$$\text{for } 1 \leq i \leq M, 1 \leq n \leq |A| : \psi_i^{T_i^n}(T_i^{n-1}, v) = \alpha_n \text{ whenever } v \in T_i^n \backslash T_i^{n-1} \tag{6.17}$$

because (6.10) holds immediately before execution of line 5 of the nth iteration of the loop.

The next step will be to deduce from (6.17) that

$$\text{for } 1 \leq i \leq M, 1 \leq n \leq |A| : \quad \psi_i^{T_i^n}(S_i, v) \geq \alpha_n \text{ whenever } v \in T_i^n \tag{6.18}$$

Suppose as an induction hypothesis that (6.18) holds in the case $n = k - 1$ for some $k \in \{1, \ldots, |A|\}$, so that $\psi_i^{T_i^k}(S_i, x) \geq \psi_i^{T_i^{k-1}}(S_i, x) \geq \alpha_{k-1} > \alpha_k$ whenever $x \in T_i^{k-1}$. We will show this implies (6.18) also holds in the case $n = k$. To do this, let v be any point in T_i^k; now what we must verify is that $\psi_i^{T_i^k}(S_i, v) \geq \alpha_k$. This is certainly true if $v \in T_i^{k-1}$ (as we have just seen), so we may assume $v \in T_i^k \backslash T_i^{k-1}$. Then, on putting $n = k$ in (6.17), we see there is some $x \in T_i^{k-1}$ such that $\psi_i^{T_i^k}(x, v) = \alpha_k$, and since $\psi_i^{T_i^k}(S_i, x) > \alpha_k$ whenever $x \in T_i^{k-1}$ (as we have seen) we see from Proposition 2.2 that $\psi_i^{T_i^k}(S_i, v) \geq \alpha_k$, as required. Moreover, (6.18) holds when $n = 1$ because (6.17) holds. So (6.18) holds in all cases.

For $v \in T_i^n \backslash T_i^{n-1}$ it follows from (6.18), Propositions 2.1 and 6.1, and (6.16) that

$$\begin{aligned}\alpha_n &\leq \psi_i^{T_i^n}(S_i, v) \leq \psi_i^{T_i^{|A|}}(S_i, v) \\ &\leq \psi_i^{T_i^{n-1} \cup (V \backslash \bigcup_j T_j^{n-1})}(S_i, v) \\ &\leq \psi_i^{V \backslash \bigcup_j T_j^{n-1}}(T_i^{n-1}, v) = \alpha_n\end{aligned}$$

which implies statement 2.

Statement 3 is plainly valid if $v \in T_i^0 = S_i$. To prove that statement 3 holds in all other cases, let $v \in T_i^n \backslash T_i^0$ and let k be the least m such that $v \in T_i^m$, so that $1 \leq k \leq n$ and $v \in T_i^k \backslash T_i^{k-1}$. It now follows from statement 2 that $\psi_i^{T_i^k}(S_i, v)$, $\psi_i^{T_i^{|A|}}(S_i, v)$, and $\psi_i^{T_i^{k-1} \cup (V \backslash \bigcup_j T_j^{k-1})}(S_i, v)$ are all equal to α_k. We deduce from this that $\psi_i^{T_i^n}(S_i, v)$ and $\psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v)$ must also be equal to α_k, because we have that $\psi_i^{T_i^k}(S_i, v) \leq \psi_i^{T_i^n}(S_i, v) \leq \psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v) \leq \psi_i^{T_i^{k-1}\cup(V\setminus\bigcup_j T_j^{k-1})}(S_i, v)$ by Propositions 2.1 and 6.1. Thus $\psi_i^{T_i^n}(S_i, v) = \psi_i^{T_i^{|A|}}(S_i, v) = \psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v) = \alpha_k \geq \alpha_n$ and so statement 3 holds. □

Proposition 6.4 *When Algorithm 2 or 3 is executed, the following are true for $1 \leq i \leq M$ and $1 \leq n \leq |A|$:*

1. $\{v \in V\setminus\bigcup_j T_j^{n-1} \mid \psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v) > \alpha_n\} = \emptyset$
2. $T_i^n\setminus T_i^{n-1} = \{v \in V\setminus\bigcup_j T_j^{n-1} \mid \psi_i^{T_i^{n-1}}(S_i, v) = \alpha_n\}$ $= \{v \in V\setminus\bigcup_j T_j^{n-1} \mid \psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v) = \alpha_n\}$

Proof Let $1 \leq i \leq M$ and $1 \leq n \leq |A|$. Then (6.9) implies:

$$\{v \in V\setminus\bigcup_j T_j^{n-1} \mid \psi_i^{V\setminus\bigcup_j T_j^{n-1}}(T_i^{n-1}, v) > \alpha_n\} = \emptyset.$$

Statement 1 follows from this and Propositions 2.1 and 6.1. To prove statement 2, it is now enough to verify:

$$T_i^n\setminus T_i^{n-1} \supseteq \{v \in V\setminus\bigcup_j T_j^{n-1} \mid \psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v) \geq \alpha_n\} \quad (6.19)$$

$$\supseteq \{v \in V\setminus\bigcup_j T_j^{n-1} \mid \psi_i^{T_i^n}(S_i, v) \geq \alpha_n\} \quad (6.20)$$

$$\supseteq T_i^n\setminus T_i^{n-1}. \quad (6.21)$$

Suppose (6.19) is false. Then $\psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v_0) \geq \alpha_n$ for some $v_0 \in V\setminus\bigcup_j T_j^{n-1}$ such that $v_0 \notin T_i^n$, whence $\psi_i^{V\setminus\bigcup_j T_j^{n-1}}(T_i^n, v_0) \geq \psi_i^{T_i^{n-1}\cup(V\setminus\bigcup_j T_j^{n-1})}(S_i, v_0) \geq \alpha_n$ (as $S_i = T_i^0 \subseteq T_i^n$). But statement 1 of Proposition 6.3 implies $\psi_i^{V\setminus\bigcup_j T_j^{n-1}}(T_i^n, v) < \alpha_n$ for all $v \in (V\setminus\bigcup_j T_j^{n-1})\setminus T_i^n$. This contradiction establishes (6.19). The inclusion (6.20) follows from Propositions 2.1 and 6.1. As $T_i^n\setminus T_i^{n-1} \subseteq V\setminus\bigcup_j T_j^{n-1}$ (by statement 1 of Proposition 6.1) the inclusion (6.21) follows from statement 3 of Proposition 6.3. □

6.2 Proof of Theorem 3.4

Let m be any element of $\{1, \ldots, M\}$. To see that statement 1 is true, we first recall that every nonempty initial segment of a recursively $(\Psi, \mathcal{S})$-optimal V-path is recursively $(\Psi, \mathcal{S})$-optimal. It follows from this and the definition of $O_m^{\Psi,\mathcal{S}}$ that if a recursively $(\Psi, \mathcal{S})$-optimal V-path p is a V-path from S_m, then p is an $O_m^{\Psi,\mathcal{S}}$-path. The converse is also true, as $\mathcal{S}$ is consistent with Ψ and so no point in $\bigcup_{j\neq m} S_j$ lies in $O_m^{\Psi,\mathcal{S}}$.

For brevity in proving statement 2, let us write ψ_{O_m} for $\psi_m|_{O_m^{\Psi,\mathcal{S}}\times O_m^{\Psi,\mathcal{S}}}$. To show that every hereditarily (ψ_{O_m}, S_m)-optimal $O_m^{\Psi,\mathcal{S}}$-path is a recursively $(\Psi, \mathcal{S})$-optimal V-path, suppose $\langle v_0, \ldots, v_l\rangle$ is an $O_m^{\Psi,\mathcal{S}}$-path that is hereditarily (ψ_{O_m}, S_m)-optimal but is not a recursively $(\Psi, \mathcal{S})$-optimal V-path. Then $v_0 \in S_m$ and there is some $j \in \{1, \ldots, l\}$ and some V-path p to v_j such that $\mathbf{RO}_\Psi^{\mathcal{S}}(p)$ and

$$\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_j\rangle) = \psi_{O_m}(\langle v_0, \ldots, v_j\rangle). \quad (6.22)$$

As $\langle v_0, \ldots, v_l\rangle$ is an $O_m^{\Psi,\mathcal{S}}$-path, we have that $v_j \in O_m^{\Psi,\mathcal{S}}$ and so there also exists a V-path p' from S_m to v_j that satisfies $\mathbf{RO}_\Psi^{\mathcal{S}}(p')$; note that p' is an $O_m^{\Psi,\mathcal{S}}$-path, by statement 1. Now $\mathbf{RO}_\Psi^{\mathcal{S}}(p)$ and $\mathbf{RO}_\Psi^{\mathcal{S}}(p')$ imply $\Psi_{\mathcal{S}}(p') = \Psi_{\mathcal{S}}(p)$. This and (6.22) imply $\psi_{O_m}(p') = \Psi_{\mathcal{S}}(p') > \psi_{O_m}(\langle v_0, \ldots, v_j\rangle)$, which contradicts the hereditary (ψ_{O_m}, S_m)-optimality of $\langle v_0, \ldots, v_l\rangle$ since p' is an $O_m^{\Psi,\mathcal{S}}$-path from S_m to v_j. This contradiction establishes that an $O_m^{\Psi,\mathcal{S}}$-path is a recursively $(\Psi, \mathcal{S})$-optimal V-path if it is hereditarily (ψ_{O_m}, S_m)-optimal.

To establish the converse, suppose $\langle v_0, \ldots, v_l\rangle$ is an $O_m^{\Psi,\mathcal{S}}$-path that is not hereditarily (ψ_{O_m}, S_m)-optimal but is a recursively $(\Psi, \mathcal{S})$-optimal V-path. Then $v_0 \in S_m$ (by statement 1) and, since $\langle v_0, \ldots, v_l\rangle$ is an $O_m^{\Psi,\mathcal{S}}$-path from S_m that is not hereditarily (ψ_{O_m}, S_m)-optimal, there is some $j \in \{1, \ldots, l\}$ such that $\langle v_0, \ldots, v_j\rangle$ is not (ψ_{O_m}, S_m)-optimal. Let p be a hereditarily (ψ_{O_m}, S_m)-optimal $O_m^{\Psi,\mathcal{S}}$-path to v_j. (The existence of p follows from Proposition 3.1, applied with $O_m^{\Psi,\mathcal{S}}$ in place of V, when we put $(\psi, S) = (\psi_{O_m}, S_m)$.) Then $\Psi_{\mathcal{S}}(p) = \psi_{O_m}(p) > \psi_{O_m}(\langle v_0, \ldots, v_j\rangle) = \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_j\rangle)$, since p is (ψ_{O_m}, S_m)-optimal but $\langle v_0, \ldots, v_j\rangle$ is not. But p is a recursively $(\Psi, \mathcal{S})$-optimal V-path (as we showed above that any hereditarily (ψ_{O_m}, S_m)-optimal $O_m^{\Psi,\mathcal{S}}$-path is a recursively $(\Psi, \mathcal{S})$-optimal V-path) and $\langle v_0, \ldots, v_l\rangle$ is also a recursively $(\Psi, \mathcal{S})$-optimal V-path, so it is in fact impossible that $\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_j\rangle)$. This contradiction establishes that an $O_m^{\Psi,\mathcal{S}}$-path is hereditarily (ψ_{O_m}, S_m)-optimal if it is a recursively $(\Psi, \mathcal{S})$-optimal V-path. So statement 2 is proved.

To prove statement 3, let $v \in O_m^{\Psi,\mathcal{S}}$, so that there is a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_m to v. Any such V-path p is a hereditarily $(\psi_m|_{O_m^{\Psi,\mathcal{S}}\times O_m^{\Psi,\mathcal{S}}}, S_m)$-optimal $O_m^{\Psi,\mathcal{S}}$-path from S_m to v (by statements 1 and 2) and so must satisfy $\Psi_{\mathcal{S}}(p) = \psi_m(p) = \psi_m^{O_m^{\Psi,\mathcal{S}}}(S_m, v)$. □

6.3 Proof of Theorem 3.6

We will use the notation of Theorem 2.4. Recall that $T_i^0 = S_i$ for $1 \leq i \leq M$ and

$$T_i^n = T_i^{n-1} \cup \left\{ v \in V \setminus \bigcup_j T_j^{n-1} \mid \psi^V(S_i, v) = \alpha_n \right\}$$
$$\text{for } 1 \le i \le M \text{ and } 1 \le n \le |A| \tag{6.23}$$

by statement 1(a) of Theorem 2.4. We now prove by induction that the following is true for $n = 1, \ldots, |A|$:

- $T_i^n = Z_i^n$ for $1 \le i \le M$, where $Z_i^n \stackrel{\text{def}}{=} \{v \in V \mid \psi^V(S_i, v) = \psi^V(\bigcup_j S_j, v) \ge \alpha_n\}$.

The property • holds for $n = 1$: We see that $T_i^1 \subseteq Z_i^1$ because if $v \in T_i^1$ then, by (6.23), $1 \ge \psi^V(\bigcup_j S_j, v) \ge \psi^V(S_i, v) = \alpha_1 = 1$ and so $v \in Z_i^1$. To show that $Z_i^1 \subseteq T_i^1$, let $v \in Z_i^1$. Then $\psi(S_i, v) = \alpha_1 = 1$. Now if $v \in V \setminus \bigcup_j T_j^0$ then $v \in \{v \in V \setminus \bigcup_j T_j^0 \mid \psi^V(S_i, v) = \alpha_1\} \subseteq T_i^1$ by (6.23). If on the other hand $v \in \bigcup_j T_j^0 = \bigcup_j S_j$ then, since $\psi(S_i, v) = 1$ and the seed sets are consistent with the affinity, $v \in S_i = T_i^0 \subseteq T_i^1$. So • holds for $n = 1$.

Next, assume as an induction hypothesis that, for some $n \in \{2, \ldots, |A|\}$, $T_i^{n-1} = Z_i^{n-1}$ for $1 \le i \le M$. To complete the proof of •, we will deduce from the induction hypothesis that $T_i^n = Z_i^n$ for $1 \le i \le M$. For this purpose, we now fix an i in $\{1, \ldots, M\}$, show that $T_i^n \subseteq Z_i^n$, and then show that $Z_i^n \subseteq T_i^n$.

To see that $T_i^n \subseteq Z_i^n$, let v be an arbitrary point in T_i^n. We need to show that $v \in Z_i^n$. If $v \in T_i^{n-1}$ then, by the inductive assumption, $v \in Z_i^{n-1} \subseteq Z_i^n$ as required. Now suppose $v \notin T_i^{n-1}$, so that $v \in T_i^n \setminus T_i^{n-1}$. Then, by (6.23), $\psi^V(\bigcup_j S_j, v) \ge \psi^V(S_i, v) = \alpha_n$ and $v \in V \setminus \bigcup_j T_j^{n-1}$. Moreover, it is *not* possible that $\psi^V(\bigcup_j S_j, v) > \psi^V(S_i, v)$, since this would imply that, if k is any index for which $\psi^V(\bigcup_j S_j, v) = \psi^V(S_k, v)$, then $\psi^V(S_k, v) = \psi^V(\bigcup_j S_j, v) > \psi^V(S_i, v) = \alpha_n$, so that $\psi^V(S_k, v) = \psi^V(\bigcup_j S_j, v) \ge \alpha_{n-1}$ (i.e., $v \in Z_k^{n-1}$) which, by the inductive assumption, would mean that $v \in T_k^{n-1}$, contradicting $v \in V \setminus \bigcup_j T_j^{n-1}$. Hence $\psi^V(\bigcup_j S_j, v) = \psi^V(S_i, v) = \alpha_n$ and we again have that $v \in Z_i^n$. So, indeed, $T_i^n \subseteq Z_i^n$.

To see that $Z_i^n \subseteq T_i^n$, let v be an arbitrary point in Z_i^n. We need to show that $v \in T_i^n$. As $v \in Z_i^n$ we either have that $\psi^V(S_i, v) = \psi^V(\bigcup_j S_j, v) = \alpha_n$ or have that $\psi^V(S_i, v) = \psi^V(\bigcup_j S_j, v) \ge \alpha_{n-1}$. In the former case $v \in V \setminus \bigcup_j Z_j^{n-1}$ and so $v \in V \setminus \bigcup_j T_j^{n-1}$ by the induction hypothesis, whence $v \in T_i^n$ by (6.23) since $\psi^V(S_i, v) = \alpha_n$. In the latter case $v \in Z_i^{n-1} = T_i^{n-1} \subseteq T_i^n$ by the induction hypothesis. This completes the proof of •.

For $1 \le i \le M$ we have that $O_i^{\text{RFC}} = T_i^{|A|} \setminus \bigcup_{j \ne i} T_j^{|A|} = Z_i^{|A|} \setminus \bigcup_{j \ne i} Z_j^{|A|}$ by statement 1(b) of Theorem 2.4 and •, and we see from the definition of Z_i^n that $Z_i^{|A|} = \{v \in V \mid \psi^V(S_i, v) = \max_j \psi^V(S_j, v) > 0\}$ because $\psi^V(\bigcup_j S_j, v) = \max_j \psi^V(S_j, v)$. Hence $O_i^{\text{RFC}} = Z_i^{|A|} \setminus \bigcup_{j \ne i} Z_j^{|A|} = \{v \in V \mid \psi^V(S_i, v) > \max_{j \ne i} \psi^V(S_j, v)\}$ for $1 \le i \le M$, which proves statement 2 of Theorem 3.6 and readily implies statement 1 of the theorem.

6.4 Proof of Theorem 3.10

The equivalence of statements 1 and 2 of the theorem follows from statement 3 of Theorem 3.4 and the definition of $O_i^{\Psi,\mathcal{S}}$. Lemmas 6.6 and 6.7 below will show the equivalence of statements 1 and 2 to statement 3. The following result, which may be of some independent interest, will be used in the proof of Lemma 6.6:

Proposition 6.5 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V, let $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ be any sequence of pairwise disjoint nonempty subsets of V, and let $\langle v_0, \ldots, v_k \rangle$ and $\langle w_0, \ldots, w_{k'} \rangle$ be any recursively $(\Psi, \mathcal{S})$-optimal V-paths. Then $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k, w_{k'} \rangle) \le \Psi_{\mathcal{S}}(\langle w_0, \ldots, w_{k'} \rangle)$, and these two values are equal if and only if $\langle v_0, \ldots, v_k, w_{k'} \rangle$ is also recursively $(\Psi, \mathcal{S})$-optimal.*

Proof We have that $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k, w_{k'} \rangle) = \Psi_{\mathcal{S}}(\langle w_0, \ldots, w_{k'} \rangle)$ if $\langle v_0, \ldots, v_k, w_{k'} \rangle$ is recursively $(\Psi, \mathcal{S})$-optimal because two recursively $(\Psi, \mathcal{S})$-optimal V-paths to the same point must have the same $(\Psi, \mathcal{S})$-strength. There is nothing else to prove unless

$$\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k, w_{k'} \rangle) \ge \Psi_{\mathcal{S}}(\langle w_0, \ldots, w_{k'} \rangle). \tag{6.24}$$

We now assume (6.24) and complete the proof by deducing that $\langle v_0, \ldots, v_k, w_{k'} \rangle$ is recursively $(\Psi, \mathcal{S})$-optimal (whence we actually have that $\Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k, w_{k'} \rangle) = \Psi_{\mathcal{S}}(\langle w_0, \ldots, w_{k'} \rangle)$). As $\mathbf{RO}_\Psi^{\mathcal{S}}(\langle w_0, \ldots, w_{k'} \rangle)$ holds, there is no V-path p to $w_{k'}$ such that $\mathbf{RO}_\Psi^{\mathcal{S}}(p)$ and $\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle w_0, \ldots, w_{k'} \rangle)$. This and (6.24) imply there is no V-path p to $w_{k'}$ such that $\mathbf{RO}_\Psi^{\mathcal{S}}(p)$ and $\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_k, w_{k'} \rangle)$. Moreover, since $\mathbf{RO}_\Psi^{\mathcal{S}}(\langle v_0, \ldots, v_k \rangle)$ holds, for $1 \le j \le k$, there is no V-path p to v_j such that $\mathbf{RO}_\Psi^{\mathcal{S}}(p)$ and $\Psi_{\mathcal{S}}(p) > \Psi_{\mathcal{S}}(\langle v_0, \ldots, v_j \rangle)$. Hence $\mathbf{RO}_\Psi^{\mathcal{S}}(\langle v_0, \ldots, v_k, w_{k'} \rangle)$ holds. □

In Lemmas 6.6 and 6.7, $O_1^*(\Psi, \mathcal{S}), \ldots, O_M^*(\Psi, \mathcal{S})$ will denote the sets for which statements 1 and 2 of Theorem 3.10 would be true if we replaced O_i^{MOFS} in each those statements with $O_i^*(\Psi, \mathcal{S})$. In other words, $O_1^*(\Psi, \mathcal{S}), \ldots, O_M^*(\Psi, \mathcal{S})$ are the sets that satisfy the following (equivalent) conditions:

(a) For $1 \le i \le M$, $v \in O_i^*(\Psi, \mathcal{S})$ just if there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of nonzero $(\Psi, \mathcal{S})$-strength from S_i to v.

(b) $O_i^*(\Psi, \mathcal{S}) = \{v \in O_i^{\Psi,\mathcal{S}} \mid \psi_i^{O_i^{\Psi,\mathcal{S}}}(S_i, v) > 0\}$ for every $1 \le i \le M$.

Lemma 6.6 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ any sequence of pairwise disjoint nonempty subsets of V that are consistent with the affinities. Then (3.1) holds when $\langle O_1, \ldots, O_M \rangle = \langle O_1^*(\Psi, \mathcal{S}), \ldots, O_M^*(\Psi, \mathcal{S}) \rangle$.*

Proof Readily, (b) implies

$$\psi_i^{O_i^*(\Psi,\mathcal{S})}(S_i, v) = \psi_i^{O_i^{\Psi,\mathcal{S}}}(S_i, v) \text{ for all } i \in \{1, \ldots, M\} \text{ and all } v \in V. \tag{6.25}$$

Let $\langle O_1, \ldots, O_M \rangle = \langle O_1^*(\Psi, \mathcal{S}), \ldots, O_M^*(\Psi, \mathcal{S}) \rangle$, let $v \in V$, and let $i \in \{1, \ldots, M\}$. To prove that (3.1) holds, it is enough to show that

(i) If $v \in O_i$, then $\max_{j \neq i} \psi_j^{O_j}(S_j, v) \leq \psi_i^{O_i}(S_i, v) \neq 0$.
(ii) If $v \notin O_i$, then either $\max_{j \neq i} \psi_j^{O_j}(S_j, v) > \psi_i^{O_i}(S_i, v)$ or $\psi_i^{O_i}(S_i, v) = 0$.

For $1 \leq j \leq M$, let p_j, p'_j, p''_j, and p'''_j be $(O_j \cup \{v\})$-paths with the following properties:

- p_j is a shortest $(O_j \cup \{v\})$-path from S_j to v such that $\psi_j(p_j) = \psi_j^{O_j}(S_j, v)$.
- p'_j is the O_j-path obtained from p_j by omitting its last point v.
- p''_j is a recursively $(\Psi, \mathcal{S})$-optimal O_j-path from S_j to the last point of p'_j. (Since $O_j = O_j^*(\Psi, \mathcal{S})$, the existence of p''_j is ensured by (a); moreover, $\psi_j(p'_j) \leq \psi_j(p''_j)$, by statement 2 of Theorem 3.4.)
- p'''_j is the $(O_j \cup \{v\})$-path from S_j to v obtained by appending v to p''_j.

Then $\psi_j(p_j) = \psi_j^{O_j}(S_j, v) \geq \psi_j(p'''_j)$. But since p'''_j is obtained by appending v to p''_j, whereas p_j can be obtained by appending v to p'_j, and since $\psi_j(p''_j) \geq \psi_j(p'_j)$, we also have that $\psi_j(p'''_j) \geq \psi_j(p_j)$. Thus:

$$\psi_j^{O_j}(S_j, v) = \psi_j(p'''_j) \quad \text{for } 1 \leq j \leq M. \tag{6.26}$$

To establish (i), suppose $v \in O_i = O_i^*(\Psi, \mathcal{S})$, so that there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p of nonzero $(\Psi, \mathcal{S})$-strength from S_i to v (by (a)), and $\Psi_{\mathcal{S}}(p) = \psi_i^{O_i^{\Psi,\mathcal{S}}}(S_i, v) = \psi_i^{O_i}(S_i, v) \neq 0$ (by Theorem 3.4 and (6.25)). Moreover, for all $j \in \{1, \ldots, M\} \backslash \{i\}$, on applying Proposition 6.5 to the recursively $(\Psi, \mathcal{S})$-optimal V-paths p''_j and p we deduce that $\psi_j(p'''_j) = \Psi_{\mathcal{S}}(p'''_j) \leq \Psi_{\mathcal{S}}(p) = \psi_i^{O_i}(S_i, v) \neq 0$. This and (6.26) imply $\psi_j^{O_j}(S_j, v) \leq \psi_i^{O_i}(S_i, v) \neq 0$ for all $j \in \{1, \ldots, M\} \backslash \{i\}$, whence $\max_{j \neq i} \psi_j^{O_j}(S_j, v) \leq \psi_i^{O_i}(S_i, v) \neq 0$.

To establish (ii), suppose $v \notin O_i = O_i^*(\Psi, \mathcal{S})$. By Proposition 3.3, there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p from S_k to v, for some $k \in \{1, \ldots, M\}$. We know from Theorem 3.4 and (6.25) that $\Psi_{\mathcal{S}}(p) = \psi_k^{O_k^{\Psi,\mathcal{S}}}(S_k, v) = \psi_k^{O_k}(S_k, v)$. So, on applying Proposition 6.5 to the recursively $(\Psi, \mathcal{S})$-optimal V-paths p''_i and p, we deduce that *either* $\psi_i(p'''_i) = \Psi_{\mathcal{S}}(p'''_i) < \Psi_{\mathcal{S}}(p) = \psi_k^{O_k}(S_k, v)$ *or* p'''_i is recursively $(\Psi, \mathcal{S})$-optimal. In the former case (6.26) implies $\psi_i^{O_i}(S_i, v) < \psi_k^{O_k}(S_k, v)$, so that $\max_{j \neq i} \psi_j^{O_j}(S_j, v) > \psi_i^{O_i}(S_i, v)$. In the latter case p'''_i is a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i to v, and so (since $v \notin O_i = O_i^*(\Psi, \mathcal{S})$) we see from (a) that $\Psi_{\mathcal{S}}(p'''_i) = 0$; this and Theorem 3.4 imply $\psi_i^{O_i}(S_i, v) = 0$. □

Lemma 6.7 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V and $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ any sequence of pairwise disjoint nonempty subsets of V that are consistent with the affinities. Let $O_1, \ldots, O_M$ be subsets of V that satisfy (3.1). Then $\langle O_1, \ldots, O_M \rangle = \langle O_1^*(\Psi, \mathcal{S}), \ldots, O_M^*(\Psi, \mathcal{S}) \rangle$.*

Proof We will deduce the lemma from the following claim:

Claim: For all $m \in \{1, \ldots, M\}$ and $v \in V$, a V-path p from S_m to v such that $\psi_m(p) \neq 0$ is a recursively $(\Psi, \mathcal{S})$-optimal V-path if and only if p is a hereditarily $(\psi_m|_{O_m \times O_m}, S_m)$-optimal O_m-path.

As (3.1) implies $S_m \subseteq O_m$, this claim is valid if the length of p is 0: In that case $p = \langle v \rangle$, $v \in S_m$, $\Psi_{\mathcal{S}}(p) = 1$, and $\psi_m^{O_m}(S_m, v) = 1$, so p is both a recursively $(\Psi, \mathcal{S})$-optimal V-path and a hereditarily (ψ_{O_m}, S_m)-optimal O_m-path. We now assume as an induction hypothesis that, for some $\alpha \in (0, 1]$ and some integer $l > 0$:

- The claim is valid whenever $\Psi_{\mathcal{S}}(p) > \alpha$.
- The claim is valid whenever $\Psi_{\mathcal{S}}(p) = \alpha$ and the length of p is less than l.

We will deduce from this induction hypothesis that the claim is valid whenever $\Psi_{\mathcal{S}}(p) = \alpha$ and the length of p is l. This is enough to prove the claim, since there are only finitely many possible values of $\Psi_{\mathcal{S}}(p)$.

So let $m \in \{1, \ldots, M\}$ and $v \in V$. Let p be a V-path from S_m to v such that $\Psi_{\mathcal{S}}(p) = \alpha > 0$ and the length of p is $l > 0$. For $1 \leq j \leq M$ let us write ψ_{O_j} for $\psi_j|_{O_j \times O_j}$. The cases we must rule out are as follows:

(i) p is a recursively $(\Psi, \mathcal{S})$-optimal V-path but is not a hereditarily (ψ_{O_m}, S_m)-optimal O_m-path.
(ii) p is not a recursively $(\Psi, \mathcal{S})$-optimal V-path but is a hereditarily (ψ_{O_m}, S_m)-optimal O_m-path.

To derive a contradiction in case (i), let p' be the V-path of length $l - 1$ obtained from p by omitting its last point, v. Since p is a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_m, so is p'. Moreover, $\Psi_{\mathcal{S}}(p') \geq \Psi_{\mathcal{S}}(p) = \alpha$. So it follows from our induction hypothesis that p' is a hereditarily (ψ_{O_m}, S_m)-optimal O_m-path. Therefore, since p itself is not a hereditarily (ψ_{O_m}, S_m)-optimal O_m-path, either (1) $v \in O_m$ but $\psi_m^{O_m}(S_m, v) > \psi_m(p)$, or (2) $v \notin O_m$. If (2) applies, then (since $\psi_m^{O_m}(S_m, v) \geq \psi_m(p) > 0$) we see from (3.1) that there is some $j \in \{1, \dots, M\} \backslash \{m\}$ such that $\psi_j^{O_j}(S_j, v) > \psi_m^{O_m}(S_m, v) \geq \psi_m(p)$. So, regardless of whether (1) or (2) applies, $\max_j \psi_j^{O_j}(S_j, v) > \psi_m(p)$. Let $k \in \{1, \dots, M\}$ be such that $\psi_k^{O_k}(S_k, v) = \max_j \psi_j^{O_j}(S_j, v) > \psi_m(p)$. Then $v \in O_k$ by (3.1), and so (by Proposition 3.1) there exists a hereditarily (ψ_{O_k}, S_k)-optimal O_k-path p'' to v. Now $\psi_k(p'') = \psi_k^{O_k}(S_k, v) > \psi_m(p)$, so the induction hypothesis implies p'' is a recursively $(\Psi, \mathcal{S})$-optimal V-path to v. But this is a contradiction, since p is also a recursively $(\Psi, \mathcal{S})$-optimal V-path to v and $\psi_k(p'') > \psi_m(p)$.

To derive a contradiction in case (ii), let $p = \langle v_0, \dots, v_l \rangle$, so that $v_0 \in S_m$ and $v_l = v$. Since p is not a recursively $(\Psi, \mathcal{S})$-optimal V-path, there is some $i \in \{1, \dots, l\}$ for which there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path p''' to v_i such that $\Psi_{\mathcal{S}}(p''') > \Psi_{\mathcal{S}}(\langle v_0, \dots, v_i \rangle) \geq \Psi_{\mathcal{S}}(p) = \alpha$. By the induction hypothesis, p''' is a hereditarily (ψ_{O_k}, S_k)-optimal O_k-path for some $k \in \{1, \dots, M\}$. So $\psi_k^{O_k}(S_k, v_i) = \Psi_{\mathcal{S}}(p''') > \Psi_{\mathcal{S}}(\langle v_0, \dots, v_i \rangle) = \psi_m(\langle v_0, \dots, v_i \rangle) = \psi_m^{O_m}(S_m, v_i)$, where the last equality follows from the hypothesis that p is a hereditarily (ψ_{O_m}, S_m)-optimal O_m-path. Thus $\psi_k^{O_k}(S_k, v_i) > \psi_m^{O_m}(S_m, v_i)$, which contradicts (3.1) because p is an O_m-path and so $v_i \in O_m$.

We have now justified the claim. The lemma follows from the claim and the following two facts. Firstly, for all $m \in \{1, \dots, M\}$ we see from (a) that $v \in O_m^*(\Psi, \mathcal{S})$ if and only if there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p from S_m to v such that $\psi_m(p) \neq 0$. Secondly, for all $m \in \{1, \dots, M\}$ we have that $v \in O_m$ if and only if there is a hereditarily $(\psi_m|_{O_m \times O_m}, S_m)$-optimal O_m-path p to v such that $\psi_m(p) \neq 0$: The "if" part is trivial (as all points of an O_m-path lie in O_m); the "only if" part follows from Proposition 3.1, since (3.1) implies $\psi_m^{O_m}(S_m, v) \neq 0$ for all $v \in O_m$. $\square$

Completion of the Proof of Theorem 3.10

Theorem 3.4 and Lemmas 6.6 and 6.7 imply the equivalence of statements 1, 2, and 3, so it is now enough to prove statement 1. In fact we will make a more general claim, which uses the notation of Theorem 2.6:

Claim For $1 \leq n \leq |A|$, $1 \leq i \leq M$, and $v \in V$, we have that $v \in T_i^n$ if and only if there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength $\geq \alpha_n$ from S_i to v.

As statement 1 is just the case $n = |A|$ of this claim, Theorem 3.10 will be proved if we can justify the claim.

We see from statement 1(a) of Theorem 2.6 that the claim is valid when $n = 1$: The "only if" part is true because if $v \in T_i^1$ then there is a V-path of ψ_i-strength $\alpha_1 = 1$ from $T_i^0 = S_i$ to v, and any such V-path is recursively $(\Psi, \mathcal{S})$-optimal. The "if" part is true because the seed sets $S_1, \dots, S_M$ are consistent with the affinities $\psi_1, \dots, \psi_M$, whence a V-path from S_i to v of $(\Psi, \mathcal{S})$-strength $\alpha_1 = 1$ must be an $(S_i \cup (V \backslash \bigcup_j S_j))$-path (i.e., a $(T_i^0 \cup (V \backslash \bigcup_j T_j^0))$-path) and so the existence of such a V-path would imply $v \in T_i^1$.

We now assume as an induction hypothesis that the claim is valid for $n = k-1$ (where k is some integer in $\{2, \dots, |A|\}$) and complete the proof by deducing that the claim is also valid when $n = k$.

To establish the "if" part of the claim in the case $n = k$, let $i \in \{1, \dots, M\}$ and $v \in V$, and suppose there is a recursively $(\Psi, \mathcal{S})$-optimal V-path $\langle v_0, \dots, v_l \rangle$ from $v_0 \in S_i$ to $v_l = v$ such that $\psi_i(\langle v_0, \dots, v_l \rangle) \geq \alpha_k$. What we need to show is that $v \in T_i^k$. If $\psi_i(\langle v_0, \dots, v_l \rangle) \geq \alpha_{k-1}$, then $v \in T_i^{k-1} \subseteq T_i^k$ by the induction hypothesis. So let us assume $\psi_i(\langle v_0, \dots, v_l \rangle) = \alpha_k$. Let m be the least index in $\{1, \dots, l\}$ such that $\psi_i(\langle v_0, \dots, v_m \rangle) = \alpha_k$, so that $\psi_i(\langle v_0, \dots, v_r \rangle) = \alpha_k$ for $m \leq r \leq l$. Then for $m \leq r \leq l$ we must have that $v_r \in V \backslash \bigcup_j T_j^{k-1}$, for if $v_r \in \bigcup_j T_j^{k-1}$ the induction hypothesis would imply there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength $\geq \alpha_{k-1} > \Psi_{\mathcal{S}}(\langle v_0, \dots, v_r \rangle)$ from $\bigcup_j S_j$ to v_r, which is impossible since $\langle v_0, \dots, v_r \rangle$ is recursively $(\Psi, \mathcal{S})$-optimal. Moreover, for $0 \leq r < m$, $\langle v_0, \dots, v_r \rangle$ is a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i to v_r such that $\psi_i(\langle v_0, \dots, v_r \rangle) \geq \alpha_{k-1}$, and so the induction hypothesis implies $v_r \in T_i^{k-1}$. It follows that $\langle v_0, \dots, v_l \rangle$ is a $(T_i^{k-1} \cup (V \backslash \bigcup_j T_j^{k-1}))$-path from $v_0 \in S_i$ to $v = v_l$, and that $v = v_l \in V \backslash \bigcup_j T_j^{k-1}$. Since $\psi_i(\langle v_0, \dots, v_l \rangle) = \alpha_k$, we see from statement 1(a) of Theorem 2.6 that $v \in T_i^k$, as required.

To establish the "only if" part of the claim in the case $n = k$, let $i \in \{1, \dots, M\}$ and $v \in T_i^k$. If $v \in T_i^{k-1}$ then the claim is valid (by the induction hypothesis) so let us assume $v \in T_i^k \backslash T_i^{k-1}$. Now we see from statement 1(a) of Theorem 2.6 that $v \in V \backslash \bigcup_j T_j^{k-1}$ and $\psi_i^{T_i^{k-1} \cup (V \backslash \bigcup_j T_j^{k-1})}(S_i, v) = \alpha_k$. Let $\langle v_0, \dots, v_l \rangle$ be a $(T_i^{k-1} \cup (V \backslash \bigcup_j T_j^{k-1}))$-path from $v_0 \in S_i$ to $v = v_l$ such that $\psi_i(\langle v_0, \dots, v_l \rangle) = \alpha_k$, and let m be the greatest index in $\{0, \dots, l\}$ such that $v_m \in T_i^{k-1}$, so that $v_r \in V \backslash \bigcup_j T_j^{k-1}$ for $m+1 \leq r \leq l$. As $v_m \in T_i^{k-1}$, the induction hypothesis implies there is a recursively $(\Psi, \mathcal{S})$-optimal V-path $\langle u_0, \dots, u_s \rangle$ from $u_0 \in S_i$ to $u_s = v_m$ such that $\psi_i(\langle u_0, \dots, u_s \rangle) \geq \alpha_{k-1}$. Now consider the V-path $\langle u_0, \dots, u_s = v_m, v_{m+1}, \dots, v_l \rangle$. For $m+1 \leq r \leq l$ we have that

$$\begin{aligned}&\psi_i(\langle u_0,\ldots,u_s=v_m,v_{m+1},\ldots,v_r\rangle)\\&\quad=\min(\psi_i(\langle u_0,\ldots,u_s\rangle),\psi_i(\langle v_m,\ldots,v_r\rangle))\\&\quad\geq\min(\psi_i(\langle u_0,\ldots,u_s\rangle),\psi_i(\langle v_0,\ldots,v_l\rangle))=\alpha_k\end{aligned}$$

so it follows from the induction hypothesis (and the fact that $v_r \notin \bigcup_j T_j^{k-1}$) that *no* recursively $(\Psi, \mathcal{S})$-optimal V-path q to v_r satisfies $\Psi_{\mathcal{S}}(q) > \Psi_{\mathcal{S}}(\langle u_0, \ldots, u_s = v_m, v_{m+1}, \ldots, v_r\rangle)$. This and the recursive $(\Psi, \mathcal{S})$-optimality of $\langle u_0, \ldots, u_s\rangle$ imply that $\langle u_0, \ldots, u_s = v_m, v_{m+1}, \ldots, v_l\rangle$ is a recursively $(\Psi, \mathcal{S})$-optimal V-path, and so we have shown that there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength $\geq \alpha_k$ from S_i to $v_l = v$, as required.

6.5 Proof of Theorem 3.8

In this proof we assume the hypotheses of Theorem 3.8 are satisfied. In addition, we will use the notation

$$X_i \stackrel{\text{def}}{=} \bigcup_{j\neq i} S_j$$

for $1 \leq i \leq M$. For every $Q \subseteq V$ we see from the definition of ψ^Q that $\max_{j\neq i} \psi^Q(S_j, v) = \psi^Q(X_i, v)$. So a set $O \subseteq V$ satisfies the condition of statement 2 of Theorem 3.8 just if $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^O(S_i, v)\}$, and satisfies the condition of statement 3 of the theorem just if $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^V(S_i, v)\}$. (Note that each of these conditions implies $X_i \subseteq V\setminus O$.) We will deduce Theorem 3.8 from these simple observations and three lemmas.

Lemma 6.8 *Let $i \in \{1, \ldots, M\}$ and let $O \subseteq V$ be such that $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^V(S_i, v)\}$. Then $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^O(S_i, v)\}$.*

Proof We first claim that, for any $v \in O$, every hereditarily (ψ, S_i)-optimal V-path to v is an O-path. To justify this claim, let $p = \langle v_0, \ldots, v_l\rangle$ be a hereditarily (ψ, S_i)-optimal V-path to a point in O and, by way of contradiction, suppose p is not an O-path. Let $k \in \{0, \ldots, l-1\}$ be the *largest* index such that $v_k \notin O$. As $v_k \notin O$, $\psi^{V\setminus O}(X_i, v_k) \not< \psi^V(S_i, v_k) = \psi(\langle v_0, \ldots, v_k\rangle)$ and so there is a $(V\setminus O)$-path q from X_i to v_k such that $\psi(q) = \psi^{V\setminus O}(X_i, v_k) \not< \psi(\langle v_0, \ldots, v_k\rangle)$. But now v_{k+1} is the only point of $q \cdot \langle v_k, v_{k+1}\rangle$ that lies in O, and so $\psi^{V\setminus O}(X_i, v_{k+1}) \not< \psi(q \cdot \langle v_k, v_{k+1}\rangle) \not< \psi(\langle v_0, \ldots, v_{k+1}\rangle) = \psi^V(S_i, v_{k+1})$ as $\langle v_0, \ldots, v_l\rangle$ is hereditarily (ψ, S_i)-optimal. As this contradicts the fact that $v_{k+1} \in O$, the claim is established. The claim implies that $\psi^V(S_i, v) = \psi^O(S_i, v)$ for all $v \in O$. So if $v \in O$ then we have that $\psi^{V\setminus O}(X_i, v) < \psi^O(S_i, v)$; the converse is evidently true because $\psi^O(S_i, v) \leq \psi^V(S_i, v)$ for all $v \in V$. □

Lemma 6.9 *Let $i \in \{1, \ldots, M\}$ and let $O \subseteq V$ be such that $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^O(S_i, v)\}$. Then $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^V(S_i, v)\}$.*

Proof Let $O' = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^V(S_i, v)\}$. Clearly $O \subseteq O'$, as $\psi^O(S_i, v) \leq \psi^V(S_i, v)$. To see the other inclusion, let $v \in O'$ and, by way of contradiction, suppose $v \notin O$.

Let $\langle v_0, \ldots, v_l\rangle$ be a hereditarily (ψ, S_i)-optimal V-path from $v_0 \in S_i$ to $v_l = v \in O'\setminus O$, and let k be the *least* index such that $v_k \notin O$. Then, since $v_k \notin O$, $\psi^{V\setminus O}(X_i, v_k) \geq \psi^O(S_i, v_k)$ and so there exists a $(V\setminus O)$-path q from X_i to v_k such that $\psi(q) \geq \psi^O(S_i, v_k)$. Moreover, since $v_j \in O$ for $0 \leq j < k$, we have that $\psi^O(S_i, v_k) \geq \psi(\langle v_0, \ldots, v_k\rangle)$. Hence $\psi(q) \geq \psi(\langle v_0, \ldots, v_k\rangle)$. Now let $k+m$ be the *least* index greater than or equal to k such that $v_{k+m} \in O'$. Then v_{k+m} is the only point of $q \cdot \langle v_k, \ldots, v_{k+m}\rangle$ that might lie in $O \subseteq O'$, and so $\psi^{V\setminus O}(X_i, v_{k+m}) \geq \psi(q \cdot \langle v_k, \ldots, v_{k+m}\rangle) \geq \psi(\langle v_0, \ldots, v_{k+m}\rangle) = \psi^V(S_i, v_{k+m})$ as $\langle v_0, \ldots, v_l\rangle$ is hereditarily (ψ, S_i)-optimal. But this contradicts the fact that $v_{k+m} \in O'$. □

Lemma 6.10 *Let $i \in \{1, \ldots, M\}$, and let O and O' be subsets of V that, respectively, satisfy the conditions $O = \{v \in V \mid \psi^{V\setminus O}(X_i, v) < \psi^V(S_i, v)\}$ and $O' = \{v \in V \mid \psi^{V\setminus O'}(X_i, v) < \psi^V(S_i, v)\}$. Then $O = O'$.*

Proof As the statement of the lemma is symmetric with respect to O and O', it is enough to show that $O' \subseteq O$. To do this, let v be any point in $V\setminus O$. Then what we need to show is that $v \in V\setminus O'$.

Suppose not. Then $v \in O'$. But, since $v \in V\setminus O$ and $X_i \subseteq V\setminus O$, there must exist a hereditarily $(\psi|_{(V\setminus O)\times(V\setminus O)}, X_i)$-optimal $(V\setminus O)$-path $\langle v_0, \ldots, v_l\rangle$ from $v_0 \in X_i$ to $v_l = v \in O'$ (by Proposition 3.1). Let k be the least index such that $v_k \in O'$. Then $\langle v_0, \ldots, v_k\rangle$ is a $((V\setminus O') \cup \{v_k\})$-path from X_i to v_k, so $\psi(\langle v_0, \ldots, v_k\rangle) \leq \psi^{V\setminus O'}(X_i, v_k) < \psi^V(S_i, v_k)$; the second inequality holds because $v_k \in O'$. But $\langle v_0, \ldots, v_l\rangle$ is a hereditarily $(\psi|_{(V\setminus O)\times(V\setminus O)}, X_i)$-optimal $(V\setminus O)$-path, so we have that $\psi(\langle v_0, \ldots, v_k\rangle) = \psi^{V\setminus O}(X_i, v_k) \geq \psi^V(S_i, v_k)$; the inequality holds since $v_k \in V\setminus O$. This contradiction proves the lemma. □

Completion of the Proof of Theorem 3.8

Let $i \in \{1, \ldots, M\}$. Lemma 6.10 implies that if there is a set O which satisfies the condition of statement 3 then it is the unique set with that property. Lemmas 6.8 and 6.9 imply that the condition on O in statement 2 is equivalent to the condition on O in statement 3. Thus the proof of Theorem 3.8 will be complete if we can show that O_i^{IRFC} satisfies statement 1 and also prove that O_i^{IRFC} is a set O that satisfies the condition of statement 2. For this purpose, let $\langle O_1^{\text{MOFS}}, \ldots, O_M^{\text{MOFS}}\rangle$

be the MOFS segmentation of V found by Algorithm 3 for seed sets $\langle S_1, \dots, S_M \rangle$ and affinities $\langle \psi, \dots, \psi \rangle$.

To see that O_i^{IRFC} satisfies statement 1, fix a $v \in V$. By Corollary 2.7, $O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$. Hence, by statement 1 of Theorem 3.10 and Corollary 3.5, $v \in O_i^{\mathrm{IRFC}}$ if, and only if, i is the unique index $k \in \{1, \dots, M\}$ for which there exists a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path from S_k to v. (Here we are also using the fact that if i is the *unique* index $k \in \{1, \dots, M\}$ for which there exists a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path from S_k to v, then such a V-path has nonzero ψ-strength: For if its ψ-strength were 0 then $\psi^V(\bigcup_j S_j, v) = 0$ and so a V-path of length 1 from any S_k to v would be hereditarily $(\psi, \bigcup_j S_j)$-optimal.) So we have proved that O_i^{IRFC} satisfies statement 1.

It remains to prove that O_i^{IRFC} is a set O which satisfies the condition of statement 2. To do this, fix an $i \in \{1, \dots, M\}$ and let $O^* = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v) < \psi^{O_i^{\mathrm{IRFC}}}(S_i, v)\}$. Then what we need to show is that $O_i^{\mathrm{IRFC}} = O^*$. To show this, we first observe that statement 3 of Theorem 3.10 readily implies:

$$O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}} = \left\{ v \in V \mid \max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v) < \psi^{O_i^{\mathrm{MOFS}}}(S_i, v) \right\}. \tag{6.27}$$

Since $V \setminus O_i^{\mathrm{IRFC}} \supseteq O_j^{\mathrm{MOFS}}$ whenever $j \neq i$, and since $O_i^{\mathrm{IRFC}} \subseteq O_i^{\mathrm{MOFS}}$, for every $v \in O^*$ we have that $\max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v) \le \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v) < \psi^{O_i^{\mathrm{IRFC}}}(S_i, v) \le \psi^{O_i^{\mathrm{MOFS}}}(S_i, v)$, whence $v \in O_i^{\mathrm{IRFC}}$ by (6.27). Thus $O^* \subseteq O_i^{\mathrm{IRFC}}$. The reverse inclusion $O_i^{\mathrm{IRFC}} \subseteq O^*$ is a consequence of (6.27) and the following two facts, which we establish below:

$$\text{If } v \in O_i^{\mathrm{IRFC}}, \text{ then } \max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v) = \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v). \tag{6.28}$$

$$\text{If } v \in O_i^{\mathrm{IRFC}}, \text{ then } \psi^{O_i^{\mathrm{MOFS}}}(S_i, v) = \psi^{O_i^{\mathrm{IRFC}}}(S_i, v) = \psi^V(\textstyle\bigcup_j S_j, v). \tag{6.29}$$

To establish (6.28) and (6.29), fix a $v \in O_i^{\mathrm{IRFC}}$. The first step in justifying (6.28) is to observe that $\max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v) \ge \max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v)$ because $V \setminus O_i^{\mathrm{IRFC}} \supseteq O_j^{\mathrm{MOFS}}$ for all $j \in \{1, \dots, M\} \setminus \{i\}$. But we must show $\max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v) \ge \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v)$. This is plainly true if $v \in \bigcup_{j \neq i} S_j$ or $\max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v) = 0$, so we assume $v \notin \bigcup_{j \neq i} S_j$ and $\max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v) > 0$. Let n be any element of $\{1, \dots, M\} \setminus \{i\}$ such that $\psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_n, v) = \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v) > 0$ and let $p = \langle v_0, \dots, v_l \rangle$ be a shortest $((V \setminus O_i^{\mathrm{IRFC}}) \cup \{v\})$-path from S_n to v with $\psi(p) = \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_n, v) > 0$. Then $v_0 \in S_n$ and $v_l = v \notin S_n$, so $l > 0$. Also, $v_0, \dots, v_{l-1} \in V \setminus O_i^{\mathrm{IRFC}}$ and so $v_0, \dots, v_{l-1} \in \bigcup_j O_j^{\mathrm{MOFS}} \setminus O_i^{\mathrm{IRFC}} = \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$ by Corollary 3.12, since $\psi(p) > 0$. Hence $v_{l-1} \in O_k^{\mathrm{MOFS}}$ for some $k \neq i$. For this k it follows from Theorem 3.10 and Corollary 3.5 that there exists a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path q from S_k to v_{l-1}, and by Corollary 3.11 q is an O_k^{MOFS}-path. Now $\psi(q) \ge \psi(\langle v_0, \dots, v_{l-1} \rangle)$ as q is $(\psi, \bigcup_j S_j)$-optimal. So $q \cdot \langle v_{l-1}, v_l \rangle$ is a $(O_k^{\mathrm{MOFS}} \cup \{v\})$-path from S_k to v with $\psi(q \cdot \langle v_{l-1}, v_l \rangle) \ge \psi(\langle v_0, \dots, v_l \rangle) = \psi(p)$. We therefore have that

$$\begin{aligned} \max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v) &\ge \psi^{O_k^{\mathrm{MOFS}}}(S_k, v) \\ &\ge \psi(q \cdot \langle v_{l-1}, v_l \rangle) \ge \psi(p) \\ &= \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_n, v) \\ &= \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v). \end{aligned}$$

So $\max_{j \neq i} \psi^{O_j^{\mathrm{MOFS}}}(S_j, v) \ge \max_{j \neq i} \psi^{V \setminus O_i^{\mathrm{IRFC}}}(S_j, v)$, and (6.28) is proved.

To establish (6.29), let $p = \langle v_0, \dots, v_l \rangle$ be a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path to $v \in O_i^{\mathrm{IRFC}}$. Then $\psi(p) = \psi^V(\bigcup_j S_j, v)$, and $v_0 \in S_i$ by statement 1 (of this theorem). We claim p is an O_i^{IRFC}-path. To see this, fix a $k \in \{0, \dots, l\}$ and, by way of contradiction, assume $v_k \notin O_i^{\mathrm{IRFC}}$. Then, by statement 1, there exists a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path q from S_j to v_k for some $j \neq i$. This readily implies $q \cdot \langle v_k, \dots, v_l \rangle$ is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path from S_j to v, which contradicts statement 1 as $j \neq i$ and $v \in O_i^{\mathrm{IRFC}}$. Thus p is indeed an O_i^{IRFC}-path (from $v_0 \in S_i$ to v), and so since $O_i^{\mathrm{MOFS}} \supseteq O_i^{\mathrm{IRFC}}$ we have that $\psi^{O_i^{\mathrm{MOFS}}}(S_i, v) \ge \psi^{O_i^{\mathrm{IRFC}}}(S_i, v) \ge \psi(p) = \psi^V(\bigcup_j S_j, v) \ge \psi^{O_i^{\mathrm{MOFS}}}(S_i, v)$, which implies (6.29).

6.6 Proof of Theorem 3.13

Let us assume for the moment that $\psi(u, v) < 1$ for all distinct u and v in V. After we prove the theorem under this hypothesis (which implies $\mathcal{S}$ is consistent with ψ), we will deduce that the theorem is true even if this hypothesis is not satisfied.

As b is a $(\psi, \mathcal{S})$-bottleneck point, there is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path $\langle v_0, \dots, v_l \rangle$ such that $v_l = b$ and none of $v_0, \dots, v_{l-1}$ lies in $TZ(\psi, \mathcal{S})$, so that $v_k \in \bigcup_j O_j^{\mathrm{IRFC}}(\psi, \mathcal{S})$ for $0 \le k \le l - 1$. Let i be the element of $\{1, \dots, M\}$ such that $v_0 \in S_i$. Then for $0 \le k \le l - 1$ we see from statement 1 of Theorem 3.8 that $v_k \notin \bigcup_{j \neq i} O_j^{\mathrm{IRFC}}(\psi, \mathcal{S})$, because the V-path $\langle v_0, \dots, v_k \rangle$ is a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path from S_i to v_k. So, since $v_k \in \bigcup_j O_j^{\mathrm{IRFC}}(\psi, \mathcal{S})$ for $0 \le k \le l - 1$, we have that $v_k \in O_i^{\mathrm{IRFC}}(\psi, \mathcal{S})$ for $0 \le k \le l - 1$.

Now let ϵ be a positive constant such that $\epsilon < \delta$ and such that

1. $\psi(u, v) + \epsilon < 1$ for all distinct points u and v in V.
2. $\epsilon < |x - y|$ for every pair (x, y) of distinct values in the range of ψ.

Let ψ' be the affinity on V such that $\psi'(v_i, v_{i+1}) = \psi(v_i, v_{i+1}) + \epsilon$ and $\psi'(v_{i+1}, v_i) = \psi(v_{i+1}, v_i) + \epsilon$ for $0 \leq i < l$, and $\psi'(u, v) = \psi(u, v)$ for all other pairs (u, v) in $V \times V$, so that $\|\psi' - \psi\| = \epsilon < \delta$ and ψ' is symmetric if ψ is symmetric. Note that $\psi'(u, v) \leq \psi(u, v) + \epsilon < 1$ for all distinct u and v in V, by property 1. Hence $\mathcal{S}$ is consistent with ψ'.

For all V-paths p, it follows from property 2 that the value of $\psi'(p)$ is either $\psi(p)$ or $\psi(p) + \epsilon$. From this and property 2 we deduce that $\psi'(p) < \psi'(q)$ for all V-paths p and q such that $\psi(p) < \psi(q)$, and hence that every hereditarily $(\psi', \bigcup_j S_j)$-optimal V-path is also a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path. This and statement 1 of Theorem 3.8 imply that $O_j^{\text{IRFC}}(\psi, \mathcal{S}) \subseteq O_j^{\text{IRFC}}(\psi', \mathcal{S})$ for $1 \leq j \leq M$, whence $TZ(\psi', \mathcal{S}) \subseteq TZ(\psi, \mathcal{S})$.

We claim there is no hereditarily $(\psi', \bigcup_j S_j)$-optimal V-path from $\bigcup_{j \neq i} S_j$ to b. Indeed, suppose p is such a V-path (so that each initial segment of p is also such a V-path). Then no point of $O_i^{\text{IRFC}}(\psi, \mathcal{S})$ is a point of p (by statement 1 of Theorem 3.8), whence none of $v_0, \ldots, v_{l-1}$ is a point of p and so $\psi'(p) = \psi(p)$. Moreover, since p is a hereditarily $(\psi', \bigcup_j S_j)$-optimal V-path to b, p is also a hereditarily $(\psi, \bigcup_j S_j)$-optimal V-path to b (just as $\langle v_0, \ldots, v_l \rangle$ is) and therefore $\psi(p) = \psi(\langle v_0, \ldots, v_l \rangle) = \psi'(\langle v_0, \ldots, v_l \rangle) - \epsilon$. Hence $\psi'(p) = \psi'(\langle v_0, \ldots, v_l \rangle) - \epsilon$, which contradicts the $(\psi', \bigcup_j S_j)$-optimality of p, as $\langle v_0, \ldots, v_l \rangle$ and p are both V-paths from $\bigcup_j S_j$ to b. So our claim is valid.

It follows from the claim (and statement 1 of Theorem 3.8) that $b \in O_i^{\text{IRFC}}(\psi', \mathcal{S})$, whence $b \notin TZ(\psi', \mathcal{S})$ and so (since $TZ(\psi', \mathcal{S}) \subseteq TZ(\psi, \mathcal{S})$) we have that $TZ(\psi', \mathcal{S}) \subseteq TZ(\psi, \mathcal{S}) \backslash \{b\}$. This completes the proof of the theorem under the additional hypothesis that $\psi(u, v) < 1$ for all distinct u and v in V.

To prove that the theorem holds even without this extra hypothesis, let ψ^* be the affinity on V such that $\psi^*(u, v) = \lambda\psi(u, v)$ for all distinct u and v in V, where λ is a positive constant in the open interval $(1 - \delta/2, 1)$. Then $\|\psi^* - \psi\| \leq 1 - \lambda < \delta/2$ and $\psi^*(u, v) < 1$ for all distinct points u and v in V. Readily, $\langle O_1^{\text{IRFC}}(\psi^*, \mathcal{S}), \ldots, O_M^{\text{IRFC}}(\psi^*, \mathcal{S}) \rangle = \langle O_1^{\text{IRFC}}(\psi, \mathcal{S}), \ldots, O_M^{\text{IRFC}}(\psi, \mathcal{S}) \rangle$ and hence $TZ(\psi^*, \mathcal{S}) = TZ(\psi, \mathcal{S}) \neq \emptyset$. Moreover, a V-path is hereditarily $(\psi^*, \bigcup_j S_j)$-optimal just if it is hereditarily $(\psi, \bigcup_j S_j)$-optimal, and so a point is a $(\psi^*, \mathcal{S})$-bottleneck point just if it is a $(\psi, \mathcal{S})$-bottleneck point. As we have already shown that the theorem is true if $\psi(u, v) < 1$ for all distinct u and v in V, we know the theorem is true with ψ^* and $\delta/2$ in place of ψ and δ. Moreover, ψ^* is symmetric if ψ is symmetric. Hence there is an affinity ψ' on V such that $\|\psi' - \psi^*\| < \delta/2$ (which implies $\|\psi' - \psi\| < \delta$), $TZ(\psi', \mathcal{S}) \subseteq TZ(\psi^*, \mathcal{S}) \backslash \{b\} = TZ(\psi, \mathcal{S}) \backslash \{b\}$, and ψ' is symmetric if ψ is symmetric. This proves the theorem.

6.7 Proof of Theorem 5.4

We will deduce the theorem from the following lemma:

Lemma 6.11 *Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be any sequence of affinities on V, and let $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$, $\mathcal{R} = \langle R_1, \ldots, R_M \rangle$, and $\mathcal{P} = \langle P_1, \ldots, P_M \rangle$ be three sequences of pairwise disjoint nonempty subsets of V that have the following properties for $1 \leq i \leq M$:*

(i) $S_i \subseteq R_i \subseteq P_i$.
(ii) For each $v \in P_i$ there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v from S_i to v such that $\psi_i^{\emptyset}(v, V \backslash P_i) \leq \psi_i(p_v)$.
(iii) There is no recursively $(\Psi, \mathcal{S})$-optimal V-path from $\bigcup_{j \neq i} S_j$ to P_i.

Then for $1 \leq i \leq M$ we have that

1. *If $\langle w_0, \ldots, w_k \rangle$ is any V-path in which $w_0 \in P_i$ and $w_1 \in V \backslash P_i$, then:*

   $$\psi_i(p_{w_0} \cdot \langle w_0, \ldots, w_k \rangle) = \psi_i(\langle w_0, \ldots, w_k \rangle).$$

2. *For all $v \in V \backslash P_i$ and $\xi \in [0, 1]$, there is a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength ξ from S_i to v if and only if there is a recursively $(\Psi, \mathcal{R})$-optimal V-path of $(\Psi, \mathcal{R})$-strength ξ from R_i to v.*
3. *For all $v \in P_i$ and $\xi \in [0, 1]$, if $\psi_i(p_v) = \xi$ then there is a recursively $(\Psi, \mathcal{R})$-optimal V-path from R_i to v whose $(\Psi, \mathcal{R})$-strength is $\geq \xi$.*

Proof Let $i \in \{1, \ldots, M\}$. Then under the hypotheses of statement 1 it follows from property (ii) that $\psi_i(p_{w_0}) \geq \psi_i^{\emptyset}(w_0, V \backslash P_i) \geq \psi_i(w_0, w_1)$. Hence,

$$\begin{aligned}
&\psi_i(p_{w_0} \cdot \langle w_0, \ldots, w_k \rangle) \\
&\quad = \min(\psi_i(p_{w_0}), \psi_i(w_0, w_1), \psi_i(\langle w_1, \ldots, w_k \rangle)) \\
&\quad = \min(\psi_i(w_0, w_1), \psi_i(\langle w_1, \ldots, w_k \rangle)) \\
&\quad = \psi_i(\langle w_0, \ldots, v_k \rangle).
\end{aligned}$$

This proves statement 1.

Now we prove statements 2 and 3. As before, let $i \in \{1, \ldots, M\}$. Bearing in mind that every V-path from S_i is a V-path from R_i (since $S_i \subseteq R_i$) and that every V-path from R_i of ψ_i-strength 1 is recursively $(\Psi, \mathcal{R})$-optimal, we see that statement 3 and the "only if" part of statement 2 are true when $\xi = 1$. To see that the "if" part of statement 2 is true when $\xi = 1$, suppose there is a V-path p of $(\Psi, \mathcal{R})$-strength

1 from R_i to $v \in V\backslash P_i$. Then there are two consecutive points a and b of p such that $a \in P_i$ and $b \in V\backslash P_i$, and it follows from property (ii) that $\psi_i(p_a) \geq \psi_i^{\emptyset}(a, V\backslash P_i) \geq \psi_i(a, b) \geq \psi_i(p) = 1$, whence $\psi_i(p_a) = 1$. Concatenation of p_a with the part of p that consists of a and all subsequent points produces a V-path from S_i to v whose $(\Psi, \mathcal{S})$-strength is 1, and which is therefore recursively $(\Psi, \mathcal{S})$-optimal.

Having verified that statements 2 and 3 hold when $\xi = 1$, we now assume as an induction hypothesis that (for some $\alpha \in [0, 1)$) statements 2 and 3 hold whenever $\xi > \alpha$. We will complete the proofs of statements 2 and 3 by deducing from this induction hypothesis that statements 2 and 3 hold when $\xi = \alpha$. (This proof method depends on the fact that if statement 2 or 3 were false then there would have to be a *greatest* value of ξ for which that statement is false, since there are only finitely many values of ξ for which there exists a V-path whose $(\Psi, \mathcal{S})$- or $(\Psi, \mathcal{R})$-strength is ξ.)

We first deduce from the induction hypothesis that statement 3 and the "only if" part of statement 2 are true when $\xi = \alpha$. Suppose there is a recursively $(\Psi, \mathcal{S})$-optimal V-path $p = \langle v_0, \ldots, v_l \rangle$ from S_i to $v = v_l$ such that $\Psi_{\mathcal{S}}(p) = \psi_i(p) = \alpha$. We will deduce from the existence of p that

(a) There exists a recursively $(\Psi, \mathcal{R})$-optimal V-path q from R_i to $v_l = v$ such that $\Psi_{\mathcal{R}}(q) \geq \alpha$.
(b) If $v_l = v \in V\backslash P_i$, then this recursively $(\Psi, \mathcal{R})$-optimal V-path q satisfies $\Psi_{\mathcal{R}}(q) = \alpha$.

Note that if we can establish (a) and (b), then we will have deduced that both statement 3 and the "only if" part of statement 2 are true when $\xi = \alpha$.

Now (a) and (b) are certainly true if p itself is recursively $(\Psi, \mathcal{R})$-optimal, so let us assume p is not recursively $(\Psi, \mathcal{R})$-optimal. Then there is a greatest index m in $\{1, \ldots, l\}$ for which there exists a recursively $(\Psi, \mathcal{R})$-optimal V-path p' to v_m such that $\Psi_{\mathcal{R}}(p') > \Psi_{\mathcal{R}}(\langle v_0, \ldots, v_m \rangle) = \psi_i(\langle v_0, \ldots, v_m \rangle) \geq \psi_i(p) = \alpha$. Since $\langle v_0, \ldots, v_m \rangle$ is a recursively $(\Psi, \mathcal{S})$-optimal V-path from S_i to v_m, it follows from property (iii) that $v_m \in V\backslash \bigcup_{j \neq i} P_j$. If p' were a V-path from R_j for some $j \neq i$, then (since $v_m \in V\backslash P_j$) the induction hypothesis would imply there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p'' from S_j to v_m such that $\Psi_{\mathcal{S}}(p'') = \Psi_{\mathcal{R}}(p') > \psi_i(\langle v_0, \ldots, v_m \rangle)$, contrary to the recursive $(\Psi, \mathcal{S})$-optimality of p.

Hence p' is a V-path from R_i, so that $\psi_i(p') = \Psi_{\mathcal{R}}(p') > \psi_i(\langle v_0, \ldots, v_m \rangle)$. Let $q = p' \cdot \langle v_m, \ldots, v_l \rangle$. Then q is recursively $(\Psi, \mathcal{R})$-optimal: Otherwise, since p' is recursively $(\Psi, \mathcal{R})$-optimal but $q = p' \cdot \langle v_m, \ldots, v_l \rangle$ is not, there would be some k in $\{m + 1, \ldots, l\}$ for which there exists a recursively $(\Psi, \mathcal{R})$-optimal V-path q' to v_k such that $\Psi_{\mathcal{R}}(q') > \Psi_{\mathcal{R}}(p' \cdot \langle v_m, \ldots, v_k \rangle) = \psi_i(p' \cdot \langle v_m, \ldots, v_k \rangle) \geq \psi_i(\langle v_0, \ldots, v_k \rangle)$ (where the $\geq$ holds because $\psi_i(p') > \psi_i(\langle v_0, \ldots, v_m \rangle)$), contrary to our definition of m. Moreover, $\Psi_{\mathcal{R}}(q) = \psi_i(q) \geq \psi_i(\langle v_0, \ldots, v_l \rangle) = \psi_i(p) = \alpha$ because $\psi_i(p') > \psi_i(\langle v_0, \ldots, v_m \rangle)$. This establishes (a).

If $v_l = v \in V\backslash P_i$, then we cannot have that $\Psi_{\mathcal{R}}(q) = \psi_i(q) > \alpha$, for in that case it would follow from the induction hypothesis that there is a recursively $(\Psi, \mathcal{S})$-optimal V-path q'' from S_i to v such that $\Psi_{\mathcal{S}}(q'') = \Psi_{\mathcal{R}}(q) > \alpha = \Psi_{\mathcal{S}}(p)$, which would contradict the recursive $(\Psi, \mathcal{S})$-optimality of p. This establishes (b). Thus we have shown that if statements 2 and 3 hold when $\xi > \alpha$, then statement 3 and the "only if" part of statement 2 hold when $\xi = \alpha$.

It remains to deduce from the induction hypothesis that the "if" part of statement 2 is true when $\xi = \alpha$. Suppose $p = \langle u_0, \ldots, u_l \rangle$ is a recursively $(\Psi, \mathcal{R})$-optimal V-path from R_i to $v \in V\backslash P_i$ such that $\Psi_{\mathcal{R}}(p) = \alpha$. (Thus $u_0 \in R_i$ and $u_l = v$.) We need to deduce that there is also a recursively $(\Psi, \mathcal{S})$-optimal V-path of $(\Psi, \mathcal{S})$-strength α from S_i to $u_l = v$.

Let t be the greatest index in $\{0, \ldots, l\}$ such that $u_t \in P_i$, and let q be the V-path $p_{u_t} \cdot \langle u_t, \ldots, u_l \rangle$ from S_i to $u_l = v$. By statement 1, $\psi_i(q) = \psi_i(\langle u_t, \ldots, u_l \rangle) \geq \psi_i(p)$. On the other hand, since $\langle u_0, \ldots, u_t \rangle$ is a recursively $(\Psi, \mathcal{R})$-optimal V-path from R_i to $u_t \in P_i$ such that $\psi_i(\langle u_0, \ldots, u_t \rangle) \geq \psi_i(p) = \alpha$, and statement 3 holds when $\xi > \alpha$ (by the induction hypothesis), we cannot have that $\psi_i(p_{u_t}) > \psi_i(\langle u_0, \ldots, u_t \rangle)$. Hence $\psi_i(p_{u_t}) \leq \psi_i(\langle u_0, \ldots, u_t \rangle)$ and so $\psi_i(q) = \psi_i(p_{u_t} \cdot \langle u_t, \ldots, u_l \rangle) \leq \psi_i(\langle u_0, \ldots, u_l \rangle) = \psi_i(p) = \alpha$. Therefore $\psi_i(q) = \psi_i(p) = \alpha$.

We claim that q is recursively $(\Psi, \mathcal{S})$-optimal. Suppose not. Then, since p_{u_t} is recursively $(\Psi, \mathcal{S})$-optimal but $q = p_{u_t} \cdot \langle u_t, \ldots, u_l \rangle$ is not, there is some $k \in \{t + 1, \ldots, l\}$ for which there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path q' to u_k such that $\Psi_{\mathcal{S}}(q') > \Psi_{\mathcal{S}}(p_{u_t} \cdot \langle u_t, \ldots, u_k \rangle) = \psi_i(\langle u_t, \ldots, u_k \rangle) \geq \Psi_{\mathcal{R}}(p) = \alpha$, where the first equality follows from statement 1. As $\Psi_{\mathcal{S}}(q') > \alpha$, it follows from the induction hypothesis that there is a recursively $(\Psi, \mathcal{R})$-optimal V-path q'' to u_k such that $\Psi_{\mathcal{R}}(q'') \geq \Psi_{\mathcal{S}}(q') > \psi_i(\langle u_t, \ldots, u_k \rangle) \geq \Psi_{\mathcal{R}}(\langle u_0, \ldots, u_k \rangle)$, contrary to the recursive $(\Psi, \mathcal{R})$-optimality of $p = \langle u_0, \ldots, u_l \rangle$. This contradiction justifies our claim. Thus we have shown that if statements 2 and 3 hold when $\xi > \alpha$, then the "if" part of statement 2 holds whenever $\xi = \alpha$. This completes the proof of the lemma. □

Deduction of Theorem 5.4 from Lemma 6.11

We claim that, for $1 \leq i \leq M$ and all points $v \in V$, there exists a recursively $(\Psi, \mathcal{R})$-optimal V-path of nonzero $(\Psi, \mathcal{R})$-strength from R_i to v if and only if there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path of nonzero $(\Psi, \mathcal{S})$-strength from S_i to v. As this claim, Proposition 5.3, and Theorem 3.10 imply $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S}) = O_i^{\mathrm{MOFS}}(\Psi, \mathcal{R})$ for $1 \leq i \leq M$, it remains only to justify the claim.

Define $P_j = \mathrm{Core}_j^{\Psi,\mathcal{S}}$ for $1 \le j \le M$. Then we see from Definition 5.1, Theorem 3.10, and Corollary 3.11 that the sets $P_1, \ldots, P_M$ and $R_1, \ldots, R_M$ have the properties (i), (ii), and (iii) of Lemma 6.11. Now let $i \in \{1, \ldots, M\}$. Then it follows from statement 2 of the lemma that the claim is correct if $v \in V \backslash P_i$. Next, suppose $v \in P_i$, so that $v \in O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$. Then there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path of nonzero $(\Psi, \mathcal{S})$-strength from S_i to v (by Theorem 3.10), and so it follows from statement 3 of the lemma that there also exists a recursively $(\Psi, \mathcal{R})$-optimal V-path of nonzero $(\Psi, \mathcal{R})$-strength from R_i to v. Thus the claim is correct if $v \in P_i$. We have now shown that the claim is correct for all $v \in V$, and so the theorem is proved.

6.8 Justification of Algorithms 4 and 5

We now show that Algorithm 5 achieves what is promised by its **Result** lines. This will be deduced from:

Proposition 6.12 *The following statements are true for every $\alpha \in (0, 1]$, $v \in V$, and $i \in \{1, \ldots, M\}$ at the end of each iteration of the main loop of Algorithm 5:*

1. *If $\sigma[v] = \alpha$, then there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v from $\bigcup_j S_j$ to v satisfying $\Psi_{\mathcal{S}}(p_v) \ge \alpha$.*
2. *If $\sigma[v] = \alpha$ and $\chi^i[v] =$ **true**, and H no longer contains any point u such that $\sigma[u] > \alpha$, then there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v^i from S_i to v satisfying $\Psi_{\mathcal{S}}(p_v^i) = \psi_i(p_v^i) = \alpha$.*
3. *If there is a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v^i from S_i to v satisfying $\Psi_{\mathcal{S}}(p_v^i) = \psi_i(p_v^i) = \alpha$, and H no longer contains any point u such that $\sigma[u] \ge \alpha$, then $\sigma[v] = \alpha$ and $\chi^i[v] =$ **true**.*

As a first step in proving this proposition, we make some simple observations regarding the algorithm:

(i) Immediately after execution of line 10, the value of $\max_{u \in \mathsf{H}} \sigma[u]$ is $\le$ the value of $\sigma[\mathsf{w}]$. During execution of lines 11 – 20, the value of $\max_{u \in \mathsf{H}} \sigma[u]$ can change only when line 15 or line 20 is executed, and when $\max_{u \in \mathsf{H}} \sigma[u]$ changes the new value of $\max_{u \in \mathsf{H}} \sigma[u]$ is also $\le$ the value of $\sigma[\mathsf{w}]$.

(ii) It follows from (i) that the value of $\max_{u \in \mathsf{H}} \sigma[u]$ immediately after an iteration of the main loop is $\le$ the value of $\sigma[\mathsf{w}]$ at that iteration (which is just the value of $\max_{u \in \mathsf{H}} \sigma[u]$ immediately before the iteration in question). Consequently, the value of $\sigma[\mathsf{w}]$ at the next iteration (if $\mathsf{H} \ne \emptyset$) is $\le$ the value of $\sigma[\mathsf{w}]$ at this iteration.

(iii) For each point $v \in V$, the value of $\sigma[v]$ never decreases during execution of the main loop. Moreover, the value of $\sigma[v]$ cannot change at an iteration of the main loop at which the point w that is removed from H satisfies $\sigma[\mathsf{w}] \le \sigma[v]$.

(iv) For each point $v \in V$, once v has been removed from H, the value of $\sigma[v]$ will never change again and $\sigma[v] \ge \sigma[\mathsf{w}] \ge \max_{u \in \mathsf{H}} \sigma[u]$ will always hold (even if v is subsequently reinserted into H by line 20 one or more times). This follows from (i) and the second sentences of (ii) and (iii), since $\sigma[v] = \sigma[\mathsf{w}]$ (as $\mathsf{w} = v$) when v is removed from H.

(v) For each point $v \in V$ and $i \in \{1, \ldots, M\}$, the value of $\chi^i[v]$ can change from **true** to **false** during execution of the main loop only when line 16 is executed after $\sigma[v]$ has been increased by execution of line 15, and we see from (iv) that this can never happen once v has been removed from H, even if v is subsequently reinserted into H.

(vi) At the end of each iteration of the main loop, for each point $v \in V$ such that $\sigma[v] > 0$ there is some $j \in \{1, \ldots, M\}$ such that $\chi^j[v] =$ **true**. (This follows from (v), as one of the values $\chi^1[v], \ldots, \chi^M[v]$ is set to **true** by line 7 or line 17 whenever the value of $\sigma[v]$ changes.)

(vii) For each point $v \in V$, it follows from (v) that the number of **true** values in the set $\{\chi^1[v], \ldots, \chi^M[v]\}$ will never decrease once v has been removed from H, even if v is subsequently reinserted into H. On the other hand, each time v is reinserted into H the number of **true** values in the set $\{\chi^1[v], \ldots, \chi^M[v]\}$ must have just been increased by 1 (by line 19).

(viii) It follows from (vi) and (vii) (and the fact that no point x such that $\sigma[x] = 0$ is ever reinserted into H) that each point $v \in V$ can be removed and reinserted into H at most $M - 1$ times, so that v can be removed at most M times. Since just one of the $|V|$ points of V is removed from H at each iteration of the main loop, the loop iterates *at most* $M|V|$ times before the algorithm terminates.

We now establish two lemmas that will be used to prove Proposition 6.12.

Lemma 6.13 *Let α^* be any element of $\bigcup_j \psi_j[V \times V] \backslash \{0\}$. Suppose further that whenever $\alpha > \alpha^*$ statement 3 of Proposition 6.12 holds for every $v \in V$ and $i \in \{1, \ldots, M\}$ at the end of each iteration of the main loop (of Algorithm 5). Then, for $\alpha = \alpha^*$, statements 1 and 2 of Proposition 6.12 hold for every $v \in V$ and $i \in \{1, \ldots, M\}$ at the end of each iteration of the main loop.*

Proof Let $v \in V$, and suppose we are at the end of an iteration of the main loop. Suppose further that $\sigma[v] = \alpha^*$ at this time. By observation (vi) there must be some $j \in \{1, \ldots, M\}$ such that $\chi^j[v] =$ **true**. Let $\chi^i[v] =$ **true**. To prove the lemma, what we need to show is that:

A. There exists a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v from $\bigcup_j S_j$ to v satisfying $\Psi_{\mathcal{S}}(p_v) \geq \alpha^*$.
B. If H no longer contains any point u such that $\sigma[u] > \alpha^*$, then there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v^i from S_i to v satisfying $\Psi_{\mathcal{S}}(p_v^i) = \psi_i(p_v^i) = \alpha^*$.

Let $x \in V$ be any point such that $\sigma[x] \geq \alpha^*$ and $\chi^i[x] =$ **true**. We now define a V-path from S_i to x that we will call the χ^i-*chain* for x. Roughly speaking, the χ^i-chain for x is the V-path from S_i to x such that the χ^i value of the $k+1$st point (for each $k \geq 0$ which does not exceed the V-path's length) was last set to **true** when w was equal to its kth point. We will now give a more precise definition of this V-path.

If $x \in S_i$, then we define the χ^i-*chain* for x to be the V-path $\langle x \rangle$. Now suppose $x \notin S_i$. Then either the current iteration or some earlier iteration of the main loop must have set $\chi^i[x]$ to **true** by executing line 17 or line 19 when the variable x was equal to the point x and the variable i was equal to i. We write $\omega(x)$ to denote the point w that was removed from H by line 10 at the most recent iteration of the main loop that set $\chi^i[x]$ to **true**. When $\chi^i[x]$ was set to **true** at that iteration, $\chi^i[\omega(x)] = \chi^i[\mathsf{w}]$ was **true** (since lines 13–20 execute only when $\chi^i[\mathsf{w}] =$ **true**) and the value of $\sigma[x]$ satisfied $\sigma[x] = \sigma' = \min(\sigma[\mathsf{w}], \psi_i(\mathsf{w}, \mathsf{x})) = \min(\sigma[\omega(x)], \psi_i(\omega(x), x))$ (as we see by inspection of lines 13–19). Note that $\sigma[x]$ cannot have changed since that time—otherwise that iteration would not be the *most recent* iteration that set $\chi^i[x]$ to **true**, since $\chi^i[x]$ is set to **false** by line 16 each time the value of $\sigma[x]$ changes during execution of the main loop. By observations (iv) and (v), the values of $\sigma[\omega(x)]$ and $\chi^i[\omega(x)]$ also cannot have changed since that time. Hence we still have that

$$\chi^i[\omega(x)] = \textbf{true} \text{ and } \sigma[x] = \min(\sigma[\omega(x)], \psi_i(\omega(x), x)) \tag{6.30}$$

which implies $\sigma[\omega(x)] \geq \sigma[x] \geq \alpha^*$. Moreover, the most recent iteration at which $\chi^i[\omega(x)]$ was set to **true** must be an earlier iteration than the most recent iteration at which $\chi^i[x]$ was set to **true**. So, if $\omega(x) \notin S_i$, then the point $\omega(\omega(x))$ exists and is not equal to x or $\omega(x)$, and, if $\omega(\omega(x)) \notin S_i$, then $\omega(\omega(\omega(x)))$ exists and is not equal to x, $\omega(x)$, or $\omega(\omega(x))$, and so on. Thus, we can construct a V-path $\langle x, \omega(x), \omega(\omega(x)), \ldots \rangle$ from x to S_i. We define the χ^i-*chain* for x to be the reverse of this V-path.

If the χ^i-chain for x is $\langle x_0, \ldots, x_l \rangle$, then $x_l = x$, $\sigma[x_0] = 1$ (because of initialization loop 2), and for $0 \leq k \leq l-1$ we have that $\sigma[x_{k+1}] = \min(\sigma[x_k], \psi_i(x_k, x_{k+1}))$ by (6.30), whence $\sigma[x_k] = \psi_i(\langle x_0, \ldots, x_k \rangle)$ for $0 \leq k \leq l$. Hence, the ψ_i-strength of the χ^i-chain for x is $\sigma[x]$, and if y is the last point of a proper initial segment of the χ^i-chain for x, then the ψ_i-strength of that initial segment is $\sigma[y] \geq \sigma[x] \geq \alpha^*$.

For any such y there is no recursively $(\Psi, \mathcal{S})$-optimal V-path p_y to y such that $\Psi_{\mathcal{S}}(p_y) > \sigma[y]$: Indeed, if such a recursively $(\Psi, \mathcal{S})$-optimal V-path p_y^j from S_j to y exists and $a = \Psi_{\mathcal{S}}(p_y^j) > \sigma[y]$, then we see from observation (iv) that H no longer contains any point u such that $\sigma[u] \geq a > \sigma[y]$ (since $y \neq x$ lies on the χ^i-chain for x and must therefore have already been removed from H at least once) and so statement 3 does *not* hold when $\alpha = a > \alpha^*$, $v = y$, and $i = j$ (as $\sigma[y] \neq a$), a contradiction. Hence, every proper initial segment of the χ^i-chain for x is recursively $(\Psi, \mathcal{S})$-optimal, and so one of the following holds:

Case I: The χ^i-chain for x (whose $(\Psi, \mathcal{S})$-strength is $\sigma[x]$) is recursively $(\Psi, \mathcal{S})$-optimal.
Case II: There is a recursively $(\Psi, \mathcal{S})$-optimal V-path p_x to x such that $\Psi_{\mathcal{S}}(p_x) > \sigma[x]$.

We conclude from this that A holds, since $\sigma[x] \geq \alpha^*$ and we can choose the point v of A as our point x.

Now suppose H no longer contains any point u such that $\sigma[u] > \alpha^*$. Then Case II is impossible: Indeed, if $a > \sigma[x]$, then $a > \alpha^*$ and so H no longer contains any point u such that $\sigma[u] \geq a$, whence the existence of a recursively $(\Psi, \mathcal{S})$-optimal V-path p_x^j from S_j to x such that $\Psi_{\mathcal{S}}(p_x^j) = a$ would imply that statement 3 does not hold when $\alpha = a$, $v = x$, and $i = j$, a contradiction. So, Case I is the only possibility, and we conclude that B holds (since we can choose the point v of B as our point x, and the ψ_i-strength of the χ^i-chain for that point v is $\sigma[v] = \alpha^*$). □

Lemma 6.14 *Let α^* be any element of $\bigcup_j \psi_j[V \times V] \setminus \{0\}$. Suppose further that whenever $\alpha > \alpha^*$ statement 1 of Proposition 6.12 holds for every $v \in V$ and $i \in \{1, \ldots, M\}$ at the end of each iteration of the main loop (of Algorithm 5). Then, when $\alpha = \alpha^*$, statement 3 of Proposition 6.12 holds for every $v \in V$ and $i \in \{1, \ldots, M\}$ at the end of each iteration of the main loop.*

Proof Let $v \in V$ and $i \in \{1, \ldots, M\}$, and suppose there exists a recursively $(\Psi, \mathcal{S})$-optimal V-path p_v^i from S_i to v satisfying $\Psi_{\mathcal{S}}(p_v^i) = \psi_i(p_v^i) = \alpha^*$. Suppose further that we are at the end of an iteration of the main loop and that H no longer contains any point u such that $\sigma[u] \geq \alpha^*$. To prove the lemma, what we need to show is that the following is currently true:

$$\sigma[v] = \psi_i(p_v^i) = \alpha^* \text{ and } \chi^i[v] = \textbf{true}. \tag{6.31}$$

Let $p_v^i = \langle v_0, \ldots, v_l \rangle$, where $v_0 \in S_i$ and $v_l = v$, and for $0 \leq k \leq l$ let p_k denote the initial segment of p_v^i that is of length k (and ends at v_k), so that $p_l = p_v^i$. Then we claim that the following is currently true:

$$\sigma[v_k] \leq \psi_i(p_k) \text{ for } 0 \leq k \leq l. \tag{6.32}$$

Indeed, if $\sigma[v_k] > \psi_i(p_k)$, then (since statement 1 holds when $\alpha = \sigma[v_k] > \psi_i(p_k) \geq \psi_i(p_l) = \alpha^*$) there would exist a recursively $(\Psi, \mathcal{S})$-optimal V-path to v_k whose $(\Psi, \mathcal{S})$-strength is at least $\sigma[v_k] > \psi_i(p_k)$, and this would contradict the recursive $(\Psi, \mathcal{S})$-optimality of $p_l = p_v^i$. So, our claim is correct. We will now use (6.32) to show, by induction on k, that the following is currently true:

$$\sigma[v_k] = \psi_i(p_k) \text{ and } \chi^i[v_k] = \textbf{true} \text{ for } 0 \le k \le l. \quad (6.33)$$

This will prove the lemma, since (6.31) is just the case $k = l$ of (6.33).

Readily, (6.33) is currently true for $k = 0$, since $v_0 \in S_i$ and so $\sigma[v_0] = 1 = \psi_i(p_0)$ and $\chi^i[v_0] = \textbf{true}$ (because of initialization loop 2). Now we suppose as our induction hypothesis that (6.33) is currently true for $k = \kappa - 1$ (where $\kappa \in \{1, \ldots, l\}$) and deduce that (6.33) is currently true for $k = \kappa$.

The induction hypothesis is that currently

$$\sigma[v_{\kappa-1}] = \psi_i(p_{\kappa-1}) \text{ and } \chi^i[v_{\kappa-1}] = \textbf{true} \quad (6.34)$$

which implies $\sigma[v_{\kappa-1}] \ge \psi_i(p_v^i) = \alpha^*$. So, since H no longer contains any point u such that $\sigma[u] \ge \alpha^*$, we must have that $v_{\kappa-1} \notin \mathsf{H}$. Let iteration I be the *most recent* iteration of the main loop at which $v_{\kappa-1}$ was the point w that was removed from H, so that $v_{\kappa-1}$ has never been in H since its removal from H at iteration I. (Iteration I might be the iteration we are currently at the end of.)

By observation (iv), $\sigma[v_{\kappa-1}] = \psi_i(p_{\kappa-1})$ was already true at iteration I; and (6.32) was also true at iteration I, by observation (iii).

We claim that $\chi^i[v_{\kappa-1}] = \textbf{true}$ must have been true at iteration I as well: Otherwise, since $\chi^i[\mathsf{w}]$ does not change at the iteration that removes w from H, there would have to have been a *more recent* iteration than I at which $\chi^i[v_{\kappa-1}]$ was set to **true**; but this is impossible as it could not have been done by line 17 (because, in that case, line 15 would have changed $\sigma[v_{\kappa-1}]$, contrary to observation (iv)) and it also could not have been done by line 19 (because, in that case, $v_{\kappa-1}$ would have been reinserted into H by line 20, contrary to the definition of iteration I). Thus, our claim is valid.

During iteration I we had that $\mathsf{w} = v_{\kappa-1}$, and so, our claim implies $\chi^i[\mathsf{w}]$ was **true** when the inner **foreach** loop on lines 12 – 20 was executed at iteration I. So, at iteration I, the body of that inner loop was executed once with $\mathsf{x} = v_\kappa$ and $\mathsf{i} = i$. (This follows from the fact that $(v_{\kappa-1}, v_\kappa)$ is a Ψ-edge because $\psi_i(v_{\kappa-1}, v_\kappa) \ge \psi_i(p_\kappa) \ge \psi_i(p_l) = \alpha^* > 0$.) At that iteration of the inner **foreach** loop, line 13 would have set σ' to $\min(\sigma[\mathsf{w}], \psi_{\mathsf{i}}(\mathsf{w}, \mathsf{x})) = \min(\sigma[v_{\kappa-1}], \psi_i(v_{\kappa-1}, v_\kappa))$, whence we see from the induction hypothesis (6.34) that the following would have been true immediately afterwards (as both $\sigma[v_{\kappa-1}] = \psi_i(p_{\kappa-1})$ and (6.32) were true at iteration I):

$$\sigma' = \min(\psi_i(p_{\kappa-1}), \psi_i(v_{\kappa-1}, v_\kappa))$$
$$= \psi_i(p_\kappa) \ge \sigma[v_\kappa] = \sigma[\mathsf{x}].$$

Thus, $\sigma' \ge \sigma[\mathsf{x}]$ and $\sigma' = \psi_i(p_k) \ge \alpha^* > 0$ at that time, and so, one of the following must have been true too: Either $\sigma' > \sigma[\mathsf{x}] = \sigma[v_\kappa]$, or $\sigma' = \sigma[\mathsf{x}] = \sigma[v_\kappa]$ and $\chi^i[v_\kappa] = \chi^{\mathsf{i}}[\mathsf{x}] = \textbf{false}$, or $\sigma' = \sigma[\mathsf{x}] = \sigma[v_\kappa]$ and $\chi^i[v_\kappa] = \chi^{\mathsf{i}}[\mathsf{x}] = \textbf{true}$. After executing line 13, the same iteration of the inner **foreach** loop would have executed lines 15–17 in the first case, lines 19–20 in the second case, and none of those lines in the third case. So, it is readily confirmed that, in all three cases, immediately after that iteration of the inner **foreach** loop it must have been true that $\sigma[v_\kappa] = \sigma[\mathsf{x}] = \sigma' = \psi_i(p_\kappa)$ and $\chi^i[v_\kappa] = \chi^{\mathsf{i}}[\mathsf{x}] = \textbf{true}$.

Moreover, $\sigma[v_\kappa] = \psi_i(p_\kappa)$ and $\chi^i[v_\kappa] = \textbf{true}$ would have remained true thereafter, since $\sigma[v_\kappa]$ cannot have increased further (otherwise we would have that $\sigma[v_k] > \psi_i(p_k)$ now, a contradiction of (6.32)) and so $\chi^i[v_\kappa]$ cannot have changed to **false** (by observation (v)). Thus, we have deduced that (6.33) is currently true for $k = \kappa$, and so our inductive proof is complete. □

Completion of the Proof of Proposition 6.12

For any V-path p from $\bigcup_j S_j$ we have that $\Psi_{\mathcal{S}}(p) \in \bigcup_j \psi_j[V \times V]$. Moreover, it is readily confirmed that, every time an element of the array $\sigma[\,]$ is given a new value during execution of Algorithm 5, that value lies in $\bigcup_j \psi_j[V \times V] \cup \{0\}$. Thus, if $\alpha \in (0, 1]$ but $\alpha \notin \bigcup_j \psi_j[V \times V]$, then each of the statements 1, 2, and 3 is vacuously true. So, since the set $\bigcup_j \psi_j[V \times V]$ is finite, if one or more of these statements were false for some $\alpha \in (0, 1]$, $v \in V$, and $i \in \{1, \ldots, M\}$ at the end of some iteration of the main loop, then there would have to be a *greatest* value of α for which this happens. But Lemmas 6.13 and 6.14 imply that such a value of α cannot exist.

Completion of the Proof of Correctness of Algorithm 5

We first verify that the algorithm achieves what is promised by its **Result** lines for every $v \in V$ such that $\sigma[v] = 0$ when the algorithm terminates. For any such v, since $\mathsf{H} = \emptyset$ when the algorithm terminates, we see from statement 3 of Proposition 6.12 that there is no recursively $(\Psi, \mathcal{S})$-optimal V-path of nonzero $(\Psi, \mathcal{S})$-strength to v, and so the $(\Psi, \mathcal{S})$-strength of every recursively $(\Psi, \mathcal{S})$-optimal V-path to v is $\sigma[v] = 0$, which is in accordance with the **Result** lines. Moreover, if $\sigma[v] = 0$ when the algorithm terminates, then it follows from observation (iii) that $\sigma[v] = 0$ has been true at all times during execution of the algorithm. This in turn implies that lines 6–7 have not been executed with $\mathsf{s} = v$, and that neither lines 15–17 nor line 19 have been executed at a time when $\mathsf{x} = v$, so that no statement which assigns **true** to one of the array elements $\{\chi^1[v], \ldots, \chi^M[v]\}$ can ever have been executed and therefore $\chi^i[v]$ must still be **false** for every $i \in \{1, \ldots, M\}$ when the algorithm terminates, as promised by the **Result** lines.

For all other v in V we have that $\sigma[v] \in (0, 1]$ when the algorithm terminates. For all such v, since $\mathsf{H} = \emptyset$ when the algorithm terminates, we see from observation (vi) and statements 2 and 3 of Proposition 6.12 that the values $\sigma[v]$ and $\chi^1[v], \ldots, \chi^M[v]$ will then have the properties stated by the **Result** lines.

Correctness of Algorithm 4

It is readily confirmed that the array $\sigma[\,]$ which results from executing Algorithm 4 is the same as the array $\sigma[\,]$ which results from executing Algorithm 5 in the case $M = 1$, $\psi_1 = \psi$, and $S_1 = S$.[16] Thus, the correctness of Algorithm 4 follows from the correctness of Algorithm 5.

7 Concluding Remarks

Fuzzy connectedness (FC) image segmentation, which finds objects based on user-specified seed sets and fuzzy affinity functions, is a computationally efficient segmentation methodology that is commonly used in practical image segmentation tasks (especially in biomedical imaging). An example of a fast FC segmentation algorithm is our Algorithm 5, which is reminiscent of Dijkstra's shortest path algorithm for weighted digraphs [24] but is necessarily less simple because it allows the use of a different fuzzy affinity function for each of the objects to be delineated. For any fixed positive integer M, under mild assumptions (that are usually satisfied in practical applications) which allow its priority queue to be efficiently implemented as an array of doubly linked lists, the algorithm will segment an image into M objects in linear time with respect to the number of points in the image.

[16] Indeed, suppose $M = 1$, $\psi_1 = \psi$, and $S_1 = S$ in Algorithm 5. As $M = 1$, the effect of line 16 of Algorithm 5 is undone by line 17, so we can omit line 16. Also, each i on lines 13–19 can be replaced by 1. Moreover, for each point $v \in V$ the value of $\sigma[v]$ can become nonzero only if line 6 or line 15 is executed when s or x is the point v, and when that happens $\chi^1[v]$ will be set to **true** by line 7 or line 17. So line 12 can be omitted; this allows execution of lines 13 – 20 even if $\chi^1[\mathsf{w}] =$ **false**, but in that case execution of those lines would have no significant effect—for if $\chi^1[\mathsf{w}] =$ **false** we see from the previous sentence that $\sigma[\mathsf{w}]$ is zero, so execution of line 13 would set σ' to zero and then the conditions on lines 14 and 18 would not be satisfied. In fact the condition on line 18 is never satisfied since $\chi^1[\mathsf{x}] =$ **true** if $\sigma[\mathsf{x}]$ is nonzero, and so lines 18 – 20 can be omitted too. After these simplifications there are no statements whose execution is conditional on the contents of the array $\chi^1[\,]$, and if we ignore the lines that only involve $\chi^1[\,]$ then the algorithm is equivalent to Algorithm 4.

Previous work on FC segmentation has developed along two tracks: the MOFS and (I)RFC tracks. This paper presents a unified mathematical theory of FC segmentations which shows how MOFS and (I)RFC-track segmentations relate to each other. We generalize (I)RFC segmentations to allow the use of affinity functions that are not necessarily symmetric, and provide new path-based mathematical characterizations of IRFC and MOFS segmentations. One fact which emerges quickly from our theory is that, when the same single affinity function is used for MOFS as well as IRFC segmentation, each IRFC object consists of those points of the corresponding MOFS object which do not lie in any of the other MOFS objects. It follows from this fact that any MOFS segmentation algorithm can also be used to compute IRFC segmentations. When $M > 2$, a fast MOFS algorithm such as Algorithm 5 is likely to compute an M-object IRFC segmentation more quickly than commonly used IRFC segmentation algorithms that compute IRFC objects one at a time (except possibly when the tie-zone of the segmentation is very large, in which case we show that the IRFC segmentation must be unstable with respect to tiny changes in affinity values).

Our analysis of MOFS segmentation (which, unlike (I)RFC segmentation, allows each object to have its own affinity function) is based on two new theoretical concepts: *recursively optimal* paths and the *core* of an MOFS object. Using these new concepts, we prove results that show how MOFS segmentations are robust with respect to small changes in seed sets (in the sense that the objects of these segmentations usually stay the same when the seed sets are slightly changed) even when different affinities are used for different objects and the affinities are not necessarily symmetric. Our results include MOFS analogs of (I)RFC-track robustness results that previously had no counterpart in the MOFS-track literature.

Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Research involving Human Participants and/or Animals The authors declare that research leading to this paper involved neither human nor animal participation.

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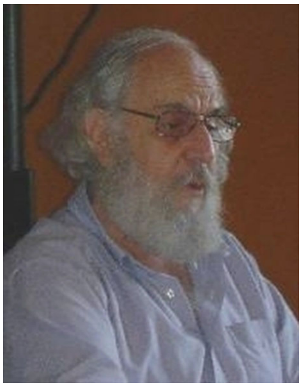

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