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An auto-homeomorphism of a Cantor set with derivative zero everywhere

Krzysztof Chris Ciesielski[a,b], Jakub Jasinski[c,*]

[a] Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, United States
[b] Department of Radiology, MIPG, University of Pennsylvania, Philadelphia, PA 19104-6021, United States
[c] Mathematics Department, University of Scranton, Scranton, PA 18510-4666, United States

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ABSTRACT

We construct a closed bounded subset $\mathfrak{X}$ of $\mathbb{R}$ with no isolated points which admits a differentiable bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}'(x) = 0$ for all $x \in \mathfrak{X}$. We also show that any such function admits a restriction $\mathfrak{f} \restriction P$ to an uncountable closed $P \subseteq \mathfrak{X}$ forming a minimal dynamical system. The existence of such a map $\mathfrak{f}$ *seems* to contradict several well know results. The map $\mathfrak{f}$ marks a limit beyond which Banach Fixed-Point Theorem cannot be generalized.

1. Introduction

Recall, that a subset $X \subseteq \mathbb{R}$ is *perfect*, if it is closed and has no isolated points. A map $f\colon X \to X$ (or, more formally, a pair $\langle X, f\rangle$) is a *minimal dynamical system*, provided X is non-empty, f is surjective, and $f[P] \neq P$ for any non-empty closed proper subset $P \subsetneq X$.

The main contribution of this article is the construction and discussion of a perfect set $\mathfrak{X}$ and a seemingly paradoxical (see Fact 2) map $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$, a bijection with $\mathfrak{f}' \equiv 0$. More importantly, $\mathfrak{f}$ satisfies certain local contraction properties but does not have a fixed point. Hence it indicates the boundaries beyond which local versions of Banach Fixed-Point Theorem cannot be generalized.

Theorem 1. *There exists a non-empty compact perfect set $\mathfrak{X} \subset \mathbb{R}$ and a differentiable bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}'(x) = 0$ for every $x \in \mathfrak{X}$. Moreover,*

(i) *$\mathfrak{f}$ is a minimal dynamical system;*
(ii) *$\mathfrak{f}$ can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.*

* Corresponding author.
E-mail address: jakub.jasinski@scranton.edu (J. Jasinski).

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The identity $\mathfrak{f}' \equiv 0$ readily implies that $\mathfrak{f}$ is *locally radially shrinking* in a sense that

(LRS) for every $x \in \mathfrak{X}$ there exists an $\varepsilon_x > 0$ such that $|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for any $y \in \mathfrak{X}$ with $0 < |x - y| < \varepsilon_x$

and it seems impossible for a function with such property to map an infinite compact set $\mathfrak{X}$ onto itself.

The (incorrect) intuition against the existence of the function $\mathfrak{f}$ from Theorem 1 is also supported by the following three facts.

Fact 2. *Assume that $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$.*

(i) *$X \nsubseteq f[X]$ when X is a bounded closed interval and $|f'| \le \lambda < 1$ on X since then, by the Mean Value Theorem, $|f(y) - f(z)| \le \lambda|y - z|$ for every $y, z \in X$, so that the diameter of $f[X]$ is strictly smaller than the diameter of X. If $\mathfrak{f}' \equiv 0$, then f is constant.*
(ii) *$X \nsubseteq f[X]$ when X has a positive finite Lebesgue measure $m(X)$ and $|f'| \le \lambda < 1$ on X, since then $m(f[X]) \le \lambda m(X)$, see e.g. [9].*
(iii) *$X \nsubseteq f[X]$ when $|f'| < 1$ on a non-empty perfect compact X and f can be extended to a **continuously** differentiable function $F: \mathbb{R} \to \mathbb{R}$. This has been proved by the authors in [5, lemma 3.3].*

The nonexistence of an example such as one from Theorem 1 must have been suspected by Edrei, when in his 1952 paper [8] he made the following conjecture.

If $\langle X, d\rangle$ is a compact metric space and $f: X \to X$ is surjection such that for every $x \in X$ there exists an $\varepsilon_x > 0$ such that $d(f(x), f(y)) \le d(x, y)$ for every $y \in X$ with $d(x, y) < \varepsilon_x$, then every point of X is a point of isometry of f (i.e., for every $x \in X$ there exists an $\delta_x > 0$ such that $d(f(x), f(y)) = d(x, y)$ for every $y \in X$ with $d(x, y) < \delta_x$).

Clearly, Theorem 1 contradicts this conjecture.

In Section 2 we discuss the relation of the dynamical system $\langle \mathfrak{X}, \mathfrak{f}\rangle$ from Theorem 1 to the fixed-point theory of locally contractive functions. Section 3 contains the details of a rather delicate construction of $\langle \mathfrak{X}, \mathfrak{f}\rangle$. In Section 4 we prove that any infinite dynamical system $\langle X, f\rangle$ on a compact space X and with surjective (LRS) map f must contain an uncountable minimal dynamical system. This illuminates the role of property (i) in Theorem 1.

2. The example, minimal dynamics, and Banach Fixed-Point Theorem

Let $\langle X, d\rangle$ be a metric space. A map $f: X \to X$ is *contractive* with a contraction constant $\lambda \in [0, 1)$ if $d(f(y), f(z)) \le \lambda d(y, z)$ for every $y, z \in X$. An $x \in X$ is a *fixed point* of f whenever $f(x) = x$.

A famous 1922 theorem of Banach [1], known as Banach Fixed-Point Theorem or the Contractive Mapping Principle, states that

Theorem 3. *If X is a complete metric space and $f: X \to X$ is contractive, then f has a fixed point.*

Let us recall some notation we need to discuss the dynamics of a continuous function $f: X \to X$. For a number $n \in \omega = \{0, 1, 2, \ldots\}$, the n-th iteration $f^{(n)}$ of f is defined as $f \circ \cdots \circ f$, the composition of n instances of f. In particular, $f^{(1)} = f$ and $f^{(0)}$ is the identity function. The *orbit* of $x \in X$ with respect to f is the set $O(x) = \{f^{(n)}(x): n \in \omega\}$. It is easy to see that f is a minimal dynamical system if, and only if, the orbit $O(x)$ of every $x \in X$ is dense in X (i.e., for every $c \in X$ and $\varepsilon > 0$, the open ball $B(c, \varepsilon) = \{y \in X : d(c, y) < \varepsilon\}$ intersects $O(x)$).

Recall, that a simple application of Zorn's Lemma[1] gives the following 1912 theorem of Birkhoff [2].

Theorem 4. *For every compact X and continuous $f: X \to X$ there exists a non-empty compact $Z \subseteq X$ such that $f \restriction Z$ is a minimal dynamical system.*

Of course, the set Z from Birkhoff's Theorem 4 can be a singleton. Actually, it must be a singleton whenever f is a contraction, since otherwise, the diameter of $f[Z]$ would be smaller than the diameter of Z.

> Does it mean, that *the only compact minimal dynamical systems to which Banach Fixed-Point Theorem is applicable are the systems with singleton spaces*?

For the original Banach Fixed-Point Theorem, the answer is affirmative. However, in this note, we discuss its generalizations in which the assumption that f is contractive is relaxed to a "local contracting" condition, see Theorems 6 and 7 below. In particular, under such relaxed assumptions, the interplay between the generalized Banach Fixed-Point Theorems and the minimal dynamical systems is considerably more intricate.

In the rest of this section, we will discuss two notions of locally contractive maps: one defined via standard topological localization technique, the other motivated by a calculus interpretation of contractive maps.

Locally contractive maps via standard localization technique We say that a map $f: X \to X$ is *locally contractive*, (LC), provided for every $x \in X$ there exists an $\varepsilon_x > 0$ such that $f \restriction B(x, \varepsilon_x)$ is contractive with some constant $\lambda_x \in [0, 1)$. For a compact space X, (LC) is equivalent to the following *uniform local contraction* property[2]

Fact 5. *If X is compact, then $f: X \to X$ is locally contractive if, and only if,*

(ULC) *there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that $d(f(y), f(z)) \leq \lambda d(y, z)$ for every $x \in X$ and $y, z \in B(x, \varepsilon)$.*

Recall that an $x \in X$ is a periodic point of a function $f: X \to X$ provided $f^{(n)}(x) = x$ for some $n > 0$. In particular, $x \in X$ is a fixed point of f if, and only if, it is a periodic point of f with period 1, that is, $f^{(1)}(x) = x$. For (LC) functions, using Fact 5, Edelstein's generalizations of Banach Fixed-Point Theorem [7, Remark 5.1], and [6, Theorem 5.2], we obtain the following:

Theorem 6. *Assume that $f: X \to X$ is locally contractive and that X is compact. Then*

(i) *f has a periodic point;*
(ii) *f has a fixed point provided X is connected.*

Notice, that the assumption of connectedness in (ii) is essential, as justified by the function $f: X \to X$, with $X = [-2, -1] \cup [1, 2]$, defined as $f(x) = -\text{sgn}(x) = -\frac{x}{|x|}$ for all $x \in X$. Clearly, it satisfies (LC) with $\lambda = 0$ and it has no fixed point, though points 1 and -1 are periodic.

[1] Applied to the family $\mathcal{Z}$ of all closed non-empty $Z \subseteq X$ such that $f[Z] \subseteq Z$.
[2] Let $\{B(x, \varepsilon_x) : x \in X_0\} \subseteq \{B(x, \varepsilon_x) : x \in X\}$ be a finite subcover of X. Then the number $\lambda = \max_{x \in X_0} \lambda_x \in [0, 1)$ satisfies (LC), though with possibly smaller numbers ε_x.

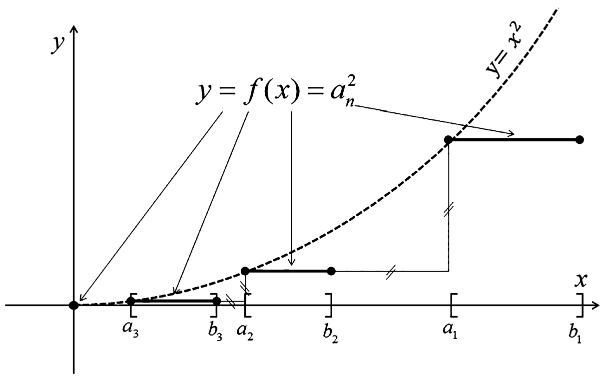

Fig. 1. $f(0) = 0$ and $f(x) = (a_n)^2$ for any $x \in [a_n, b_n]$ and $n = 1, 2, 3, \ldots$.

Locally contractive maps via calculus interpretation Differentiable contractive maps on $\mathbb{R}$ have a very nice characterization. Namely, if $X \subseteq \mathbb{R}$ is a closed interval and $f: X \to X$ is differentiable, then, by the Mean Value Theorem, f is contractive if, and only if,

(D) there exists a $\lambda \in [0, 1)$ such that $|f'(x)| \leq \lambda$ for every $x \in X$.

More generally, notice that if $X \subseteq \mathbb{R}$ has no isolated points, then the standard definition of the derivative makes sense for $f: X \to X$ and, if f is differentiable, then (D) is equivalent to the following property, which uses no notion of the derivative

(LRC) there is a $\lambda \in [0, 1)$ such that for every $x \in X$ there exists an $\varepsilon_x > 0$ with a property that $d(f(x), f(z)) \leq \lambda d(x, z)$ for every $z \in B(x, \varepsilon_x)$.

(LRC) was studied, for arbitrary metric spaces X, by several authors [11,12,14] and was referred to as the *local radial contraction* property of f.

Clearly (ULC) $\Rightarrow$ (LRC). The fact that this implication cannot be reversed is justified by a function $f: X \to X$ depicted in Fig. 1, where $X = \{0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]$, $1 = b_1 > a_1 > b_2 > a_2 > \cdots > \lim_n a_n = 0$, and $f(a_n) - f(b_{n+1}) = a_n - b_{n+1}$ for all $n = 1, 2, 3, \ldots$. This f is (LRC) since $f'(x) = 0$ for every $x \in X$. At the same time (LC) fails for f at $x = 0$, since any open $U \ni 0$ contains distinct a and b with $f(a) - f(b) = a - b$.

Now, returning to Banach Fixed-Point Theorem, the following generalization to (LRC) functions first appeared in a 1978 paper [12] of Hu and Kirk. However, its proof contained a gap, as it relied on a false proposition from [11]. The first complete proof of this theorem appeared in the 1982 paper [14] of Jungck.

Theorem 7. *Assume that X is a complete metric space and that every two points of X can be connected by a path in X of finite length.*[3] *If $f: X \to X$ satisfies (LRC), then f has a fixed point.*

But what happens if, in Theorem 7, we replace all the assumptions on the space X with a simple requirement that X is compact? In other words,

is Theorem 6(i) true for (LRC) maps?

The negative answer is provided by the function $\mathfrak{f}$ from Theorem 1; it shows the limits to the localized generalizations of Banach Fixed-Point Theorem. As $\mathfrak{f}$ forms a minimal dynamical system, it is fair to say

[3] A length of a path $p: [0, 1] \to X$ is defined as a supremum over all numbers $\sum_{i=1}^{n} d(p(t_i), p(t_{i-1}))$, where $0 = t_0 < t_1 < \cdots < t_n = 1$. In particular, every convex subset X of a Banach space is path connected in the sense of Theorem 7.

Table 1
Fixed/periodic point properties implied by various contractive properties of the function $f: X \to X$, where X is compact and either arbitrary, or a convex subspace of a Banach space.

Convexity of X assumed?	$f: X \to X$ has periodic/fixed point when f is		
	contractive	locally contractive (LC)	locally radially contractive (LRC)
Yes	fixed point Banach, Theorem 3	fixed point Edelstein, Theorem 6(ii)	fixed point Hu & Kirk, Theorem 7
No	fixed point Banach, Theorem 3	periodic point Edelstein, Theorem 6(i)	neither Ciesielski & Jasinski, Theorem 1

that $\mathfrak{f}$ marks the spot where *the minimal dynamical systems "meet" Banach Fixed-Point Theorem*. See also Theorem 9.

The results discussed in this section are summarized in Table 1.

Remark 8. It is interesting to notice that, according to the property (6) proven below, function $\mathfrak{f}$ from Theorem 1 is (LC) at all points but one. Of course, this single exception is of paramount importance, since, by Theorem 6(i), any everywhere (LC) function has periodic points.

3. Construction of the example from Theorem 1

The adding machine On the set 2^ω of infinite $0-1$ sequences define the following "add one and carry" operation $\sigma: 2^\omega \to 2^\omega$, often referred to as *adding machine* (see e.g. [17] or [4]) and representing odometer-like action: for $s = \langle s_0, s_1, s_2, \ldots\rangle \in 2^\omega$, $\sigma(s) = s + \langle 1, 0, 0, \ldots\rangle$ or, more precisely,

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle & \text{if } s_k = 0 \text{ and } s_i = 1 \text{ for all } i < k. \end{cases}$$

In other words, if for $k < \omega$ we let $w_k \in 2^{k+1}$ to be $w_k = \langle 1, \ldots, 1, 0\rangle$ (a sequence of k-many 1s followed by a single 0) and $z_k \in 2^{k+1}$ to be $z_k = \langle 0, \ldots, 0, 1\rangle$ (a sequence of k-many 0s followed by a single 1), then

$$\sigma(1,1,1,\ldots) = \langle 0,0,0,\ldots\rangle$$
$$\sigma(w_k, s_{k+1}, s_{k+2}, \ldots) = \langle z_k, s_{k+1}, s_{k+2}, \ldots\rangle.$$

It is well known and easy to see that σ is a continuous bijection and that

$$\text{the orbit of every } s \in 2^\omega \text{ is dense in } 2^\omega.\text{[4]} \tag{1}$$

In particular, σ is a minimal dynamical system, see e.g. [16].

For $s \in 2^\omega$ and $\nu < \omega$ let $N_\nu(s) = \sum_{i<\nu} s_i 2^i$, with $N_0(s)$ understood as 0. An important property of σ is that for every $s \in 2^\omega$ and $k < \omega$

$$\text{if } s \restriction (k+1) = w_k, \text{ then } N_\nu(\sigma(s)) = N_\nu(s) + 1 \text{ for every } \nu > k. \tag{2}$$

Let $\bar{1} = \langle 1,1,1,\ldots\rangle$. Then, in particular,

$$N_\nu(s) < N_\nu(\sigma(s)) \text{ for every } s \in 2^\omega \text{ with } s \neq \bar{1} \text{ and any large enough } \nu < \omega.$$

However, the inequality $N_\nu(s) < N_\nu(\sigma(s))$ is false for any $\nu < \omega$, when $s = \bar{1}$.

[4] For $\tau \in 2^n$ let $[\tau] = \{t \in 2^\omega : t \restriction n = \tau\}$. By induction on $n < \omega$, we can easily see that $O(s) \cap [\tau] \neq \emptyset$ for any $s \in 2^\omega$.

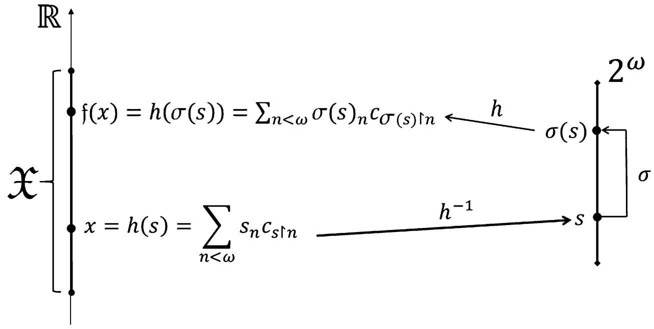

Fig. 2. $\mathfrak{f} = h \circ \sigma \circ h^{-1}$.

Format of the example We will find a continuous injection $h: 2^\omega \to \mathbb{R}$ such that $\mathfrak{X} = h[2^\omega]$ and $\mathfrak{f} = h \circ \sigma \circ h^{-1}$ forms the example from Theorem 1, see Fig. 2. (Note that h^{-1} is a homeomorphism between 2^ω and X.) Since $\mathfrak{f}^{(n)} = h \circ \sigma^{(n)} \circ h^{-1}$ whenever $n < \omega$, (1) implies that for any $x \in \mathfrak{X}$ the orbit $O(x)$ of $\mathfrak{f}$ is dense in $\mathfrak{X}$.

Note that $\mathfrak{f} = h \circ \sigma \circ h^{-1}$ is, what is usually called, a *topological conjugate* of (or *isomorphic* to) the adding machine σ. In particular, the mapping h can be considered as a generator of a metric ρ on 2^ω defined as $\rho(s,t) = |h(s) - h(t)|$.

Format of the function h The map $h: 2^\omega \to \mathbb{R}$ will be defined via formula

$$h(s) = \sum_{n<\omega} s_n c_{s \restriction n} \text{ for every } s \in 2^\omega \tag{3}$$

for appropriately chosen numbers $c_\tau \in \mathbb{R}$ for $\tau \in 2^{<\omega}$. To ensure that $\mathfrak{f}'(x) = 0$ for $x = h(s)$ with $s \in 2^\omega$, it needs to be shown that for every $y = h(t)$ with $t \in 2^\omega$ and $t \neq s$, the numbers

$$\Delta_{st} = \frac{|\mathfrak{f}(x) - \mathfrak{f}(y)|}{|x - y|} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|}$$

converge to 0 when $\ell = \min\{i < \omega: s_i \neq t_i\}$ diverges to infinity.

For $s \neq \bar{1}$, that is, of the form $\langle w_k, s_{k+1}, s_{k+2}, \ldots \rangle$, the choice of c_τ's will guarantee this convergence by ensuring, for large enough ℓ, and the $u \in \{s,t\}$ with $u_\ell = 1$,

$$\begin{aligned} |h(\sigma(s)) - h(\sigma(t))| &\leq \frac{3}{2} \sum_{n \geq \ell} u_n |c_{\sigma(u)\restriction n}| \\ |h(s) - h(t)| &\geq \frac{1}{2} \sum_{n \geq \ell} u_n |c_{u \restriction n}| > 0 \end{aligned} \tag{4}$$

as well as the existence of a constant $E_k > 0$ depending only on k, and a sequence $\langle \beta_n : n < \omega \rangle$ with $\beta_n^{-1} \searrow 0$ for which

$$\frac{|c_{\sigma(u)\restriction n}|}{|c_{u\restriction n}|} = E_k \beta_n^{-1} \leq E_k \beta_\ell^{-1} \quad \text{for every } n \geq \ell. \tag{5}$$

This guarantees the desired convergence, as then

$$\Delta_{st} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \leq \frac{\frac{3}{2}\sum_{n\geq\ell} u_n |c_{\sigma(u)\restriction n}|}{\frac{1}{2}\sum_{n\geq\ell} u_n |c_{u\restriction n}|} \leq 3E_k \beta_\ell^{-1} \to_{\ell\to\infty} 0. \tag{6}$$

The case $s = \bar{1}$ requires essentially different argument, based on the following two properties, satisfied for $\ell > 0$:

$$|h(\sigma(s)) - h(\sigma(t))| \le \frac{1}{\ell+1}\frac{1}{\ell} \tag{7}$$

and

$$|h(s) - h(t)| \ge \sum_{n\ge\ell} |c_{s\restriction n}| \ge \sum_{n\ge\ell} \frac{1}{(n+2)^{1/2}}\frac{1}{n+2}\frac{1}{n+1}. \tag{8}$$

Since $\sum_{n\ge\ell} \frac{1}{(n+2)^{1/2}}\frac{1}{n+2}\frac{1}{n+1} \ge \sum_{n\ge\ell} \frac{1}{(n+2)^{2.5}} \ge \int_{\ell+2}^{\infty} x^{-2.5}\,dx = \frac{1}{1.5}\frac{1}{(\ell+2)^{1.5}}$, (7) and (8) imply the required convergence:

$$\Delta_{st} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \le \frac{\frac{1}{\ell(\ell+1)}}{\frac{1}{1.5}\frac{1}{(\ell+2)^{1.5}}} = 1.5\frac{(\ell+2)^{1.5}}{\ell(\ell+1)} \to_{\ell\to\infty} 0.$$

Definition of the coefficients $c_{s\restriction n}$ from (3) We can see by now that a lot is expected of the coefficients c_τ. So, their definition is quite delicate and it will not be fully completed until we reach equation (14).

To ensure satisfaction of the properties (4)–(8), for every $s \in 2^\omega$ and $n < \omega$ we let $\beta_n = \ln(n+3) > 1$, and define

$$c_{s\restriction n} = a_{s\restriction n}\beta_n^{-b_{s\restriction n}} d_{s\restriction n}, \tag{9}$$

where $d_{s\restriction n} > 0$ is defined below in (14), $a_{s\restriction 0} = -1$, $b_{s\restriction 0} = 0$, and, for $n > 0$,

$$a_{s\restriction n} = \begin{cases} -1 & \text{when } s \restriction n = \langle 1, 1, \dots, 1\rangle, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad b_{s\restriction n} = N_{\nu_n}(s) = \sum_{i<\nu_n} s_i 2^i,$$

where $\nu_n = \max\left\{m < \omega \colon (\beta_n)^{2^m-1} < \sqrt{n+2}\right\}$. Notice that the definition of ν_n gives $(\beta_n)^{b_{s\restriction n}} \le (\beta_n)^{2^{\nu_n}-1} < \sqrt{n+2}$, that is, that

$$\beta_n^{-b_{s\restriction n}} > \frac{1}{(n+2)^{1/2}} \quad \text{for every } s \in 2^\omega \text{ and } n < \omega. \tag{10}$$

Reduction of property (8) The sole purpose of the coefficients $a_{s\restriction n}$ is to facilitate the following argument for the first inequality from (8), in case $s = \bar{1}$, where the equations hold since $s \restriction n = t \restriction n$ for all $n < \ell$, while $a_{s\restriction n} = -1$ and $a_{t\restriction n} = 1$ for all $n \ge \ell$

$$|h(s) - h(t)| = \left|\sum_{n\ge\ell} s_n c_{s\restriction n} - \sum_{n\ge\ell} t_n c_{t\restriction n}\right| = \left|-\sum_{n\ge\ell} |c_{s\restriction n}| - \sum_{n\ge\ell} t_n |c_{t\restriction n}|\right| \ge \sum_{n\ge\ell} |c_{s\restriction n}|.$$

Also, by (10), for every $n > 0$ we have $|c_{s\restriction n}| = \beta_n^{-b_{s\restriction n}} d_{s\restriction n} \ge \frac{1}{(n+2)^{1/2}} d_{s\restriction n}$. Thus, the second inequality from (8) is ensured by the following requirement:

$$d_{s\restriction n} = \frac{1}{n+2}\frac{1}{n+1} \text{ for every } n < \omega \text{ and } s = \bar{1}. \tag{11}$$

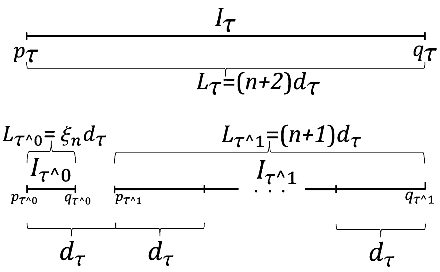

Fig. 3. I_τ, $I_{\tau\hat{}0}$, and $I_{\tau\hat{}1}$ for $\tau \in 2^n$.

Reduction of property (5) For $s = \langle w_k, s_{k+1}, s_{k+2}, \dots\rangle$ and large enough ℓ, the property (5) holds, as long as we ensure that

$$d_{\sigma(s)\restriction n} = E_k d_{s\restriction n} \text{ for every } s = \langle w_k, s_{k+1}, s_{k+2}, \dots\rangle \text{ and } n > k. \tag{12}$$

Indeed, since $\frac{(\beta_n)^{2^{k+1}-1}}{\sqrt{n+2}} = \frac{(\ln(n+3))^{2^{k+1}-1}}{\sqrt{n+2}} \to_{n\to\infty} 0$, there exists an $\ell > k$ such that $(\beta_n)^{2^{k+1}-1} \le \sqrt{n+2}$ for any $n \ge \ell$. This choice of ℓ ensures (5) as then, by the definition of numbers ν_n, for every $n \ge \ell$ we have $k+1 \le \nu_n$. So, by (2), $N_{\nu_n}(\sigma(u)) = N_{\nu_n}(u) + 1$ and

$$\frac{|c_{\sigma(u)\restriction n}|}{|c_{u\restriction n}|} = \frac{\beta_n^{-N_{\nu_n}(\sigma(u))} d_{\sigma(u)\restriction n}}{\beta_n^{-N_{\nu_n}(u)} d_{u\restriction n}} = \beta_n^{-1}\frac{d_{\sigma(u)\restriction n}}{d_{u\restriction n}} = E_k\beta_n^{-1}.$$

To finish the construction, it is enough to define the coefficients $d_{t\restriction n}$ that ensure: the properties (11) and (12), the fact that h is a continuous injection, and the estimates (4) and (7).

Definition of the coefficients $d_{s\restriction n}$ For every $n < \omega$ let

$$\xi_n = \frac{1}{2}\frac{1}{(n+4)^{1/2}}.$$

Then, by (10), for every $s \in 2^\omega$, $\ell < \omega$, and $0 < m < \omega$,

$$\xi_\ell < \frac{1}{2}\beta_\ell^{-b_{s\restriction \ell}} \quad \text{and} \quad \xi_m < \beta_{m-1}^{-b_{s\restriction(m-1)}}. \tag{13}$$

Mimicking the classical construction of Cantor's ternary set, we define, for $\tau \in 2^{<\omega}$, the intervals $I_\tau = [p_\tau, q_\tau]$ in the following way, see Fig. 3. For τ of length 0 (i.e., $\tau = \langle\rangle$), we put $I_\tau = [p_\tau, q_\tau] = [0,1]$. If, for some $\tau \in 2^n$, the interval I_τ is already defined and $\tau\hat{}i \in 2^{n+1}$ is an extension of τ by a term $i \in \{0,1\}$, then $I_{\tau\hat{}1}$ is the terminal $\frac{n+1}{n+2}$-th part of I_τ, while $I_{\tau\hat{}0}$ the initial $\frac{\xi_n}{n+2}$-th part of I_τ. More specifically, if $L_\tau = q_\tau - p_\tau$ is the length of I_τ, then $I_{\tau\hat{}0} = [p_{\tau\hat{}0}, q_{\tau\hat{}0}] = [p_\tau, p_\tau + \frac{\xi_n}{n+2}L_\tau]$, $I_{\tau\hat{}1} = [p_{\tau\hat{}1}, q_{\tau\hat{}1}] = [p_\tau + \frac{1}{n+2}L_\tau, q_\tau]$, $L_{\tau\hat{}0} = \frac{\xi_n}{n+2}L_\tau$, and $L_{\tau\hat{}1} = \frac{n+1}{n+2}L_\tau$. We define

$$d_{s\restriction n} = \frac{1}{n+2}L_{s\restriction n}. \tag{14}$$

Observe that, for any $\tau \in 2^n$ and $i \in \{0,1\}$, we have $L_{\tau\hat{}0} = \frac{\xi_n}{n+2}L_\tau < \frac{n+1}{n+2}L_\tau = L_{\tau\hat{}1}$. So, $L_{\tau\hat{}i} \le L_{\tau\hat{}1} = \frac{n+1}{n+2}L_\tau$ and, by induction on $n < \omega$,

$$L_{s\restriction n} \le L_{\bar{1}\restriction n} = \frac{1}{n+1} \quad \text{for every } s \in 2^\omega \text{ and } n < \omega. \tag{15}$$

Also, an easy inductive argument shows that

$$\sum_{n<\ell} s_n d_{s\restriction n} = p_{s\restriction\ell} \in I_{s\restriction\ell} \quad \text{for every } s \in 2^\omega \text{ and } \ell < \omega.$$

In particular, $\bigcap_{n<\omega} I_{s\restriction n} = \left\{\sum_{n<\omega} s_n d_{s\restriction n}\right\}$ for every $s \in 2^\omega$. Moreover

$$\sum_{n\geq\ell} s_n d_{s\restriction n} \leq L_{s\restriction\ell} \quad \text{for every } s \in 2^\omega \text{ and } \ell < \omega \tag{16}$$

as $p_{s\restriction\ell} + \sum_{n\geq\ell} s_n d_{s\restriction n} = \sum_{n<\omega} s_n d_{s\restriction n} \in I_{s\restriction\ell} = [p_{s\restriction\ell}, p_{s\restriction\ell} + L_{s\restriction\ell}]$. This will be of special importance in the case when $s_\ell = 0$, since then we have $\sum_{n\geq\ell} s_n d_{s\restriction n} = \sum_{n\geq\ell+1} s_n d_{s\restriction n} \leq L_{s\restriction(\ell+1)} = L_{(s\restriction\ell)\hat{}0} = \xi_\ell d_{s\restriction\ell}$, that is,

$$\sum_{n>\ell} s_n d_{s\restriction n} = \sum_{n\geq\ell} s_n d_{s\restriction n} \leq \xi_\ell d_{s\restriction\ell} \text{ for every } s \in 2^\omega \text{ and } \ell < \omega \text{ with } s_\ell = 0. \tag{17}$$

Proof of (11) and (12). The property (11) follows immediately from (14) and (15).

To see (12) notice that for every $\tau, \eta \in 2^m$ and $i \in \{0,1\}$ we have $\frac{L_{\tau\hat{}i}}{L_{\eta\hat{}i}} = \frac{L_\tau}{L_\eta}$. So, an easy induction shows that for every $k < n < \omega$ and $\tau, \eta \in 2^n$ we have

$$\frac{L_{\tau\restriction(k+1)}}{L_{\eta\restriction(k+1)}} = \frac{L_\tau}{L_\eta} \text{ provided } \tau_i = \eta_i \text{ for all } i \text{ with } k < i < n.$$

Since, in (12), $s_i = \sigma(s)_i$ for all i with $k < i < n$, by (14) and the above equation we have $\frac{d_{\sigma(s)\restriction n}}{d_{s\restriction n}} = \frac{L_{\sigma(s)\restriction n}}{L_{s\restriction n}} = \frac{L_{\sigma(s)\restriction(k+1)}}{L_{s\restriction(k+1)}} = \frac{L_{z_k}}{L_{w_k}}$. Thus, (12) holds with $E_k = \frac{L_{z_k}}{L_{w_k}}$.

Proof of the estimate (7). Here $s = \bar{1}$. Then, the use of (17), with $\ell - 1$ in place of ℓ and $\sigma(t)$ in place of s, and (15) gives us the required estimate:

$$\begin{aligned}
|h(\sigma(s)) - h(\sigma(t))| &= \sum_{n\geq\ell} \sigma(t)_n c_{\sigma(t)\restriction n} = \sum_{n\geq\ell-1} \sigma(t)_n \beta_n^{-b_{\sigma(t)\restriction n}} d_{\sigma(t)\restriction n} \\
&\leq \sum_{n\geq\ell-1} \sigma(t)_n d_{\sigma(t)\restriction n} \leq d_{\sigma(t)\restriction(\ell-1)} \xi_{\ell-1} \\
&\leq d_{\sigma(t)\restriction(\ell-1)} = \frac{1}{\ell+1} L_{\sigma(t)\restriction(\ell-1)} \leq \frac{1}{\ell+1}\frac{1}{\ell}.
\end{aligned}$$

Proof of the estimates (4). Here $s = \langle w_k, s_{k+1}, s_{k+2}, \ldots\rangle$ and $\sigma(s) = \langle z_k, s_{k+1}, s_{k+2}, \ldots\rangle$ for some $k < \omega$. Also $t \in 2^\omega$ does not equal s and $\ell = \min\{i < \omega \colon s_i \neq t_i\} > 0$. By symmetry of expressions $|h(s) - h(t)|$ and $|h(\sigma(s)) - h(\sigma(t))|$ we can assume, without loss of generality, that $s_\ell = 1$ and $t_\ell = 0$. So, the estimates will be proved for $u = s$.

Now, as $t_\ell = 0$, by (17) and (13), we obtain

$$\sum_{n>\ell} t_n \beta_n^{-b_{t\restriction n}} d_{t\restriction n} \leq \sum_{n>\ell} t_n d_{t\restriction n} \leq \xi_\ell d_{t\restriction\ell} = \xi_\ell d_{s\restriction\ell} \leq \frac{1}{2}\beta_\ell^{-b_{s\restriction\ell}} d_{s\restriction\ell}. \tag{18}$$

Hence, we get the second estimate of (4):

$$\begin{aligned}
h(s)-h(t) &= \sum_{n\geq\ell} s_n\beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>\ell} t_n\beta_n^{-b_{t\restriction n}} d_{t\restriction n} \\
&\geq \sum_{n\geq\ell} s_n\beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \frac{1}{2}\beta_\ell^{-b_{s\restriction \ell}} d_{s\restriction \ell} \\
&\geq \sum_{n\geq\ell} s_n\beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \frac{1}{2}\sum_{n\geq\ell} s_n\beta_n^{-b_{s\restriction n}} d_{s\restriction n} \\
&= \frac{1}{2}\sum_{n\geq\ell} s_n\beta_n^{-b_{s\restriction n}} d_{s\restriction n} = \frac{1}{2}\sum_{n\geq\ell} s_n|c_{s\restriction n}| > 0.
\end{aligned}$$

The first estimate of (4) is obtained as follows:

$$\begin{aligned}
|h(\sigma(s))-h(\sigma(t))| &= \left|\sum_{n\geq\ell} s_n c_{\sigma(s)\restriction n} - \sum_{n>\ell} t_n c_{\sigma(t)\restriction n}\right| \qquad (19)\\
&\leq \sum_{n\geq\ell} s_n|c_{\sigma(s)\restriction n}| + \sum_{n>\ell} t_n|c_{\sigma(t)\restriction n}| \\
&= \sum_{n\geq\ell} s_n\beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} + \sum_{n>\ell} t_n\beta_n^{-b_{\sigma(t)\restriction n}} d_{\sigma(t)\restriction n} \\
&\leq \sum_{n\geq\ell} s_n\beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} + \frac{1}{2}\beta_\ell^{-b_{\sigma(s)\restriction \ell}} d_{\sigma(s)\restriction \ell} \qquad (20)\\
&\leq \sum_{n\geq\ell} s_n\beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} + \frac{1}{2}\sum_{n\geq\ell} s_n\beta_n^{-b_{\sigma(s)\restriction n}} d_{\sigma(s)\restriction n} \\
&= \frac{3}{2}\sum_{n\geq\ell} s_n|c_{\sigma(s)\restriction n}|,
\end{aligned}$$

where (19) is ensured by the fact that $\sigma(s)_n = s_n$ and $\sigma(t)_n = t_n$ for every $n \geq \ell$ and by the equation $\sigma(s) \restriction \ell = \sigma(t) \restriction \ell$, while (20) follows from (18) applied to the pair $\sigma(s)_\ell$ and $\sigma(t)_\ell$.

Proof of continuity of h. By (9), (16), and (15), for any $s \in 2^\omega$ and $\ell < \omega$ we have $\left|\sum_{n\geq\ell} s_n c_{s\restriction n}\right| \leq \sum_{n\geq\ell} s_n|c_{s\restriction n}| \leq \sum_{n\geq\ell} s_n d_{s\restriction n} \leq L_{s\restriction\ell} \leq \frac{1}{\ell+1}$. Therefore, for distinct $s,t \in 2^\omega$ and $\ell = \min\{i<\omega\colon s_i \neq t_i\}$, $|h(s)-h(t)| = \left|\sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n\geq\ell} t_n c_{t\restriction n}\right| \leq \left|\sum_{n\geq\ell} s_n c_{s\restriction n}\right| + \left|\sum_{n\geq\ell} t_n c_{t\restriction n}\right| \leq \frac{2}{\ell+1}$, that is, h is continuous.

Proof of injectivity of h. To see that the function h is one-to-one, fix distinct $s,t \in 2^\omega$ and let $\ell = \min\{i < \omega\colon s_i \neq t_i\}$. By symmetry, we can assume that $s_\ell = 1$ and $t_\ell = 0$. Then, we have

$$h(s)-h(t) = \sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n\geq\ell} t_n c_{t\restriction n} = \sum_{n\geq\ell} s_n c_{s\restriction n} - \sum_{n>\ell} t_n c_{t\restriction n}.$$

We need to show that $h(s)-h(t) \neq 0$. For this we will consider the following cases.

Case 1: *s equals* $\bar{1} = \langle 1,1,1,\ldots\rangle$. Then $a_{s\restriction n} = -1$ for all $n<\omega$ and $a_{t\restriction n} = 1$ for all $n > \ell$. Hence

$$h(s)-h(t) = -\sum_{n\geq\ell} s_n\beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>\ell} t_n\beta_n^{-b_{t\restriction n}} d_{t\restriction n} < 0.$$

Case 2: *there exists an $i < \ell$ such that $t_i = s_i = 0$.* Then, $a_{s\restriction n} = a_{t\restriction n} = 1$ for all $n \geq \ell$. So, using the fact that $\beta_n^{-b_{t\restriction n}} \leq 1$ for all $n < \omega$ and the equations $s_\ell = 1$ and $s \restriction \ell = t \restriction \ell$, and, afterwards, applying (17) to t, followed by (13), we get

$$\begin{aligned} h(s) - h(t) &= \sum_{n\geq\ell} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>\ell} t_n \beta_n^{-b_{t\restriction n}} d_{t\restriction n} \\ &\geq s_\ell \beta_\ell^{-b_{s\restriction \ell}} d_{s\restriction \ell} - \sum_{n>\ell} t_n d_{t\restriction n} \\ &\geq \beta^{-b_{t\restriction \ell}} d_{t\restriction \ell} - \xi_\ell d_{t\restriction \ell} = d_{t\restriction \ell}(\beta_\ell^{-b_{t\restriction \ell}} - \xi_\ell) > 0. \end{aligned}$$

Case 3: *neither Case 1 nor Case 2 hold.* Let $m = \min\{i < \omega : s_i = 0\}$. Then $m > \ell$, $s_{m-1} = 1$, and $s_m = 0$. Hence, as $a_{s\restriction n} = -1$ for $n \leq m$ and $a_{s\restriction n} = 1$ for $n > m$, using (17) we get

$$\begin{aligned} -\sum_{n\geq\ell} s_n c_{s\restriction n} &= \sum_{\ell\leq n\leq m} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} - \sum_{n>m} s_n \beta_n^{-b_{s\restriction n}} d_{s\restriction n} \\ &\geq s_{m-1} \beta_{m-1}^{-b_{s\restriction (m-1)}} d_{s\restriction (m-1)} - \sum_{n>m} s_n d_{s\restriction n} \\ &= \beta_{m-1}^{-b_{s\restriction (m-1)}} d_{s\restriction (m-1)} - \sum_{n\geq m} s_n d_{s\restriction n} \\ &\geq \beta_{m-1}^{-b_{s\restriction (m-1)}} d_{s\restriction (m-1)} - \xi_m d_{s\restriction m}. \end{aligned}$$

Now, $d_{s\restriction m} = \frac{1}{m+2} L_{s\restriction m} = \frac{1}{m+2} L_{(s\restriction(m-1))^\frown 1} = \frac{1}{m+2}\frac{m}{m+1} L_{s\restriction(m-1)} = \frac{m}{m+2} d_{s\restriction(m-1)}$ so that $d_{s\restriction(m-1)} = \frac{m+2}{m} d_{s\restriction m} \geq d_{s\restriction m}$. Thus, by (13),

$$\begin{aligned} -\sum_{n\geq\ell} s_n c_{s\restriction n} &\geq \beta_{m-1}^{-b_{s\restriction (m-1)}} d_{s\restriction (m-1)} - \xi_m d_{s\restriction m} \\ &\geq \beta_{m-1}^{-b_{s\restriction (m-1)}} d_{s\restriction m} - \xi_m d_{s\restriction m} = d_{s\restriction m}\left(\beta_{m-1}^{-b_{s\restriction (m-1)}} - \xi_m\right) > 0. \end{aligned}$$

So, $h(t) - h(s) = \sum_{n>\ell} t_n c_{t\restriction n} - \sum_{n\geq\ell} s_n c_{s\restriction n} \geq -\sum_{n\geq\ell} s_n c_{s\restriction n} > 0$.

Proof of (i) and (ii) of Theorem 1. Item (i) was addressed earlier, see (1) and the discussion in Section 4 below.

Item (ii) follows from a theorem of Jarník [13] that *every differentiable function f from a compact perfect subset of $\mathbb{R}$ into $\mathbb{R}$ can be extended to a differentiable function $F: \mathbb{R} \to \mathbb{R}$.* (More on Jarník's theorem can be found in [15]. The theorem has also been independently proved in [18, theorem 4.5].)

This concludes the proof of Theorem 1.

4. Must the example be based on a minimal dynamics?

Recall that for a metric space X, a function $f: X \to X$ is *locally radially shrinking* if

(LRS) for every $x \in X$ there exists an $\varepsilon_x > 0$ such that $d(f(x), f(y)) < d(x, y)$ for any $y \in B(x, \varepsilon_x)$, $y \neq x$.

The function $\mathfrak{f}$ from Theorem 1(i), constructed in Section 3, is (LRS) and forms a minimal dynamical system. Our goal here is to prove, that this is not a coincidence, since any surjective (LRS) self map of an infinite compact space X contains a minimal dynamics of an uncountable $Y \subset X$:

Theorem 9. *Let X be an infinite compact metric space and assume that a map $f: X \to X$ is an (LRS) surjection. Then there exists a perfect subset $Y \subseteq X$ such that $f \restriction Y$ is a minimal dynamical system.*

The proof of this theorem is based on several lemmas. We will also use the following standard notation: for $\delta > 0$ and non-empty $A \subseteq X$ we define $B(A, \delta) = \bigcup_{a \in A} B(a, \delta)$.

Lemma 10. *If $X_0 \subseteq X$, $f: X_0 \to X$ satisfies* (LRS)*, and finite $A \subseteq X_0$ is such that $f[A] \subseteq A$, then exists a $\delta > 0$ such that $f[X_0 \cap B(A, \varepsilon)] \subseteq B(A, \varepsilon)$ for every $\varepsilon \in (0, \delta]$.*

Proof. For every $a \in A$ let $\delta_a > 0$ be such that $d(f(x), f(a)) \le d(x, a)$ whenever $x \in X_0 \cap B(a, \delta_a)$. Then $\delta = \min_{a \in A} \delta_a > 0$ is as needed.

Indeed, fix an $\varepsilon \in (0, \delta]$ and choose an $x \in X_0 \cap B(A, \varepsilon)$. To see that $f(x) \in B(A, \varepsilon)$ pick an $a \in A$ with $x \in B(a, \varepsilon)$. Then, since $f(a) \in A$, we have $d(f(x), A) \le d(f(x), f(a)) \le d(x, a) < \varepsilon$ so that $f(x) \in B(A, \varepsilon)$, as needed. □

Our next lemma states that the existence of a surjection with (LRS) property implies that the space X must be uncountable. In the proof, we use the notion of Cantor–Bendixon rank, defined as follows. For a metric space X we let $(X)'$ to be the set of all accumulation points of X. For the ordinal numbers $\alpha, \lambda < \omega_1$, where λ is a limit ordinal, we define

$$X^{(0)} = X,\ X^{(\alpha+1)} = (X^{(\alpha)})',\ \text{and}\ X^{(\lambda)} = \bigcap\nolimits_{\alpha<\lambda}\ X^{(\alpha)}.$$

An easy inductive argument shows that for every $\alpha < \omega_1$, if $A \subseteq B \subseteq X$, then $A^{(\alpha)} \subseteq B^{(\alpha)}$.

We define the *Cantor–Bendixon rank* of X, denoted $|X|_{CB}$, to be the least ordinal number $\alpha < \omega_1$ such that $X^{(\alpha+1)} = X^{(\alpha)}$. Recall, that if X is compact, then $\alpha = |X|_{CB}$ is either zero or a successor ordinal, that is, of the form $\alpha = \beta + 1$. Moreover, if X is also countable, then $\alpha > 0$ and $X^{(\alpha)} = \emptyset$.

Lemma 11. *If $X_0 \subseteq X$ is infinite compact and $f: X_0 \to X$ is a surjection with* (LRS) *property, then X_0 is uncountable.*

Proof. Assume, towards a contradiction, that there exists a function f as in the lemma with a countable infinite X_0. Let X_0 be such an example with the smallest possible Cantor–Bendixon rank $\alpha = |X_0|_{CB}$. Then, since X_0 is compact and infinite, $\alpha = \beta + 1$ for some ordinal $\beta \ge 1$. Clearly $X \subseteq f[X_0]$ implies that $X^{(\beta)} \subseteq f[X_0]^{(\beta)}$. Also, an easy inductive argument shows that $f[X_0]^{(\beta)} \subseteq f[(X_0)^{(\beta)}]$. (See e.g. (I_β) in [5, lemma 4.3].) It follows that $X^{(\beta)} \subseteq f[(X_0)^{(\beta)}]$. Since, $(X_0)^{(\beta)} \subseteq X^{(\beta)}$ and, by compactness of X_0, $(X_0)^{(\beta)}$ is finite, the inclusions $(X_0)^{(\beta)} \subseteq X^{(\beta)} \subseteq f[(X_0)^{(\beta)}]$ imply the equality $(X_0)^{(\beta)} = f[(X_0)^{(\beta)}]$. The set $A = (X_0)^{(\beta)}$ satisfies the assumptions of Lemma 10. So, let $\delta > 0$ be as in this lemma.

If $\beta = 1$, put $B = B(A, \delta)$. Then $f[X_0 \cap B] \subseteq B$. We need to show that the inclusion is proper. Indeed, $X_0 \cap B$ is closed, since it contains $A = (B)'$. So, there exists an $x \in X_0 \cap B$ of maximal distance $\eta = d(x, A)$ to A. Notice, that $\eta > 0$, as $X_0 \cap B \nsubseteq A$. We claim that $x \ne f(z)$ for every $z \in B$. Indeed, it is obvious when $z \in A$, since then $d(f(z), A) = 0 < \eta = d(x, A)$; otherwise, there is an $a \in A$ with $0 < d(z, a) = d(z, A) \le \eta < \delta$ and so, by (LRS), $d(f(z), A) \le d(f(z), f(a)) < d(z, a) \le \eta = d(x, A)$, once again giving $x \ne f(z)$. So, we proved that $f[B] \subsetneq B$.

The contradiction is obtained by noticing that, f being surjective, $f[X_0 \setminus B]$ must contain $X \setminus f[X_0 \cap B]$, which is impossible, since finite a set $X_0 \setminus B$ cannot be mapped onto its proper superset $X \setminus f[X_0 \cap B]$.

If $\beta > 1$, then, for some $\varepsilon \in (0, \delta]$, the set $X_1 = X_0 \setminus B(A, \varepsilon)$ is infinite and contains a limit point. Moreover, $X_1 \subseteq f[X_1]$ and X_1 has the Cantor–Bendixon rank less than α, contradicting the choice of α. □

Lemma 12. *If X is compact and $f: X \to X$ satisfies* (LRS)*, then, for every positive $m < \omega$, the set $F_m = \{x \in P: f^{(m)}(x) = x\}$ is finite.*

Proof. An easy induction shows that $f^{(m)}$ also satisfies (LRS). If F_m was infinite, then, being compact, it would contain an accumulation point, say $x \in F_m$. But then, $f^{(m)}$ would not be shrinking in any neighborhood of x, a contradiction. $\square$

In what follows, we will use notation F_m to the sets from Lemma 12.

Lemma 13. *If X is compact uncountable and $f: X \to X$ is a surjective map satisfying* (LRS)*, then there exists an open $U \subseteq X$ containing $\bigcup_{m=1}^{\infty} F_m$ such that $X \setminus U$ is uncountable and $X \setminus U \subseteq f[X \setminus U]$.*

Proof. Let μ be a Borel probability measure on X vanishing on points (e.g., defined as an appropriate product measure on a copy of a Cantor set in X). Clearly the orbit $O(x)$ of each $x \in F_m$ is finite. So, by Lemma 12, the set $A_m = \bigcup_{x \in F_m} O(x)$ is finite and, clearly, $f[A_m] \subseteq A_m$.

By Lemma 10 applied to $A = A_m$, for every positive $m < \omega$ there is a $\delta_m > 0$ such that $f[B(A_m, \varepsilon)] \subseteq B(A_m, \varepsilon)$ for every $\varepsilon \in (0, \delta_m]$. Choose $\varepsilon_m \in (0, \delta_m]$ small enough so that $\mu(B(A_m, \varepsilon_m)) \leq 2^{-(m+2)}$. Then $U = \bigcup_{m=1}^{\infty} B(A_m, \varepsilon_m)$ is as desired, since $\mu(U) \leq 1/2 < \mu(X)$ and $f[U] \subseteq U$. $\square$

Proof of Theorem 9. Let f and X be as in Theorem 9. Then, by Lemma 11, X is uncountable. Hence, we can use Lemma 13. Let U be as in Lemma 13 and put $T = X \setminus U$. Then,

$(*)$ $T \subseteq f[T]$.

A simple application of Zorn's Lemma, following an idea from Birkhoff [2], implies that there exists a minimal non-empty compact $Y \subseteq T$ satisfying $(*)$. Notice, that this minimality of Y implies $Y = f[Y]$, as otherwise $Y \cap f^{-1}(Y)$ would be a proper closed subset of Y satisfying $(*)$.

To finish the argument, notice that Y is infinite, since otherwise it would be contained in $\bigcup_{m=1}^{\infty} F_m \subseteq U$, which is disjoint with Y. Thus, by Lemma 11, Y is uncountable and, being minimal, it must be perfect. (It cannot have isolated points, since the orbit of any point of Y must be dense in Y.) $\square$

Finally, notice that the careful choice of a metric on a copy $\mathfrak{X}$ of the Cantor set 2^ω is essential to the example from Theorem 1.

Remark 14. If d is the standard metric on 2^ω defined, for distinct $s, t \in 2^\omega$ as $d(s,t) = 2^{-\min\{n<\omega : s_n \neq t_n\}}$, then $2^\omega \not\subseteq f[2^\omega]$ for every (LRS) map f on $\langle 2^\omega, d \rangle$. Indeed, $\langle 2^\omega, d \rangle$ is ultrametric (i.e., satisfies $d(s,u) \leq \max\{d(s,t), d(t,u)\}$ for every $s, t, u \in 2^\omega$) while F. George has recently proved [10] that $X \not\subseteq f[X]$ for any (LC) map f on a compact ultrametric space $\langle X, d \rangle$. However, George's proof works for the (LRS) functions as well, because for any $Y \subseteq X$, and $a \in Y$ the diameter of Y equals $\sup\{d(a,y) : y \in Y\}$, see [19, p. 49].

Notice that the perfect subsets X of $\mathbb{R}$ that admit a function f as in Theorem 1 are rare, in a sense that they are of first category in the space $\mathcal{K}$ of non-empty compact subsets of $\mathbb{R}$ furnished with the Hausdorff metric. This has been proved by Bruckner and Steele in [3].

The fact that the set $\mathfrak{X}$ is compact is crucial. The examples of this kind for non-compact complete metric spaces are considerably easier to come by. In particular, Hu and Kirk [12] give an example of a complete metric ρ on $\mathbb{R}$, inducing the standard topology, such that the map $f(x) = x + 1$ has derivative zero everywhere in a sense that $\lim_{y \to x} \frac{\rho(f(y), f(x))}{\rho(y,x)} = 0$ for all $x \in \mathbb{R}$.

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