How Good Is Lebesgue Measure?

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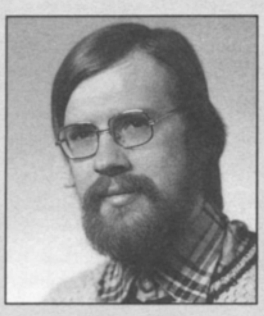
The problem of determining the distance between two points, the area of a region, and the volume of a solid are some of the oldest and most important problems in mathematics. Their roots are in the ancient world. For example, the formula for the area of a circle was already known to the Babylonians of 2000 to 1600 B.C., although they used as π either 3 or $3\frac{1}{8}$. The Egyptians of 1650 B.C. used $\pi = (4/3)^4 = 3.1604\ldots$. The first crisis in mathematics arose from a measurement problem. The discovery of incommensurable magnitudes by the Pythagoreans (before 340 B.C.) created such a great "logical scandal" that efforts were made for a while to keep the matter secret (see [Ev], pages 25, 31, 85, 56).



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For a given subset of $\mathbf{R}^n$, what do we mean by the statement that some number is its area (for $n = 2$), volume (for $n = 3$) or, more generally, its n-dimensional measure? Such a number must describe the size of the set. So the function that associates with some subsets of $\mathbf{R}^n$ their measure must have some "good" properties. How can we construct such a function and what reasonable properties should it have?

This question, in connection with the theory of integration, has been considered since the beginning of the nineteenth century. It was examined by such well-known mathematicians as Augustin Cauchy, Lejeune Dirichlet, Bernhard Riemann, Camille Jordan, Émile Borel, Henri Lebesgue, and Giuseppe Vitali (see [Haw]). Lebesgue's solution of this problem, dating from the turn of the century, is now considered to be the best answer to the question, which is not completely settled even today.

Lebesgue Measure

In his solution Lebesgue constructed a family $\mathcal{S}$ of subsets of Euclidean space $\mathbf{R}^n$ (where $n = 1, 2, \ldots$) and a function $m\colon \mathcal{S} \to [0,\infty]$ that satisfies the following properties:

(a) $\mathcal{S}$ is a σ-algebra, i.e., $\mathcal{S}$ is closed under countable unions (if $A_k \in \mathcal{S}$ for every natural number k then $\bigcup \{A_k\colon k \in N\} \in \mathcal{S}$) and under complementation ($A \in \mathcal{S}$ implies $\mathbf{R}^n \setminus A \in \mathcal{S}$);

(b) $[0,1]^n \in \mathcal{S}$ and $m([0,1]^n) = 1$;

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(c) m is countably additive, i.e., for every family $\{A_k: k \in N\}$ of pairwise disjoint sets from $\mathcal{S}$,

$$m\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} m(A_k);$$

(d) m is isometrically invariant, i.e., for every isometry i of $\mathbf{R}^n$

(1) $A \in \mathcal{S}$ if and only if $i(A) \in \mathcal{S}$;

(2) $m(A) = m(i(A))$ for every A in $\mathcal{S}$.

Recall that a function $i: \mathbf{R}^n \to \mathbf{R}^n$ is an isometry provided it preserves distance, in other words, i is composed of translations and rotations.

The properties (a)–(d) seem to capture perfectly our intuitive notion of area and volume. Lebesgue measure is not sensitive to moving sets around without distorting them. To find the measure of a set it suffices to split it up into a finite or countable family of disjoint sets and then to add up their measures (provided, of course, that each piece is measurable). However, we should ask at least one more question: "With how many sets does Lebesgue measure deal?" or "How big is the family $\mathcal{S}$?" In particular, we should decide whether m is universal, i.e., whether $\mathcal{S}$ is equal to the family $\mathcal{P}(\mathbf{R}^n)$ of all subsets of $\mathbf{R}^n$.

Unfortunately the answer to this question is negative. In 1905 Vitali constructed a subset $\mathbf{V}$ of $\mathbf{R}^n$ such that $\mathbf{V}$ is not in $\mathcal{S}$ (see [Vi], [Haw], p. 123, or [Ru], Thm. 2.22, p. 53). His proof is easy, so we shall repeat it here for $n = 1$.

For x in [0,1] let $E_x = \{y \in [0,1]: y - x \in \mathbf{Q}\}$ and put $\mathcal{E} = \{E_x: x \in [0,1]\}$. Clearly $\mathcal{E}$ is a nonempty family of non-empty pairwise disjoint sets. Therefore, by the Axiom of Choice there exists a set $\mathbf{V} \subset [0,1]$ such that $\mathbf{V} \cap E$ has exactly one element for every $E \in \mathcal{E}$. We prove that $\mathbf{V}$ is not in $\mathcal{S}$.

First notice that if $\mathbf{V} + q = \{x + q: x \in \mathbf{V}\}$ for $q \in \mathbf{Q}$, then $\mathbf{V} + q$ and $\mathbf{V} + p$ are disjoint for distinct $p, q \in \mathbf{Q}$. Otherwise $x + q = y + p$ for some $x, y \in \mathbf{V}$ so x and y both belong to the same $E \in \mathcal{E}$. Then $\mathbf{V} \cap E$ contains more than one element, contradicting the definition of $\mathbf{V}$.

Now if $\mathbf{V}$ is in $\mathcal{S}$, then so is $\mathbf{V} + q$ for every $q \in \mathbf{Q}$, and $m(\mathbf{V}) = m(\mathbf{V} + q)$. But $\bigcup \{\mathbf{V} + q: q \in \mathbf{Q} \cap [0,1]\} \subset [0,2]$ and so

$$\sum_{q \in \mathbf{Q} \cap [0,1]} m(\mathbf{V} + q) = m\left(\bigcup\{\mathbf{V} + q: q \in \mathrm{Q} \cap [0,1]\}\right) \leq m([0,2]) = 2.$$

This implies $m(\mathbf{V}) = m(\mathbf{V} + q) = 0$ for every $q \in \mathbf{Q}$. On the other hand, $[0,1] \subset \mathbf{R} = \cup \{\mathbf{V} + q: q \in \mathbf{Q}\}$ which implies

$$m([0,1]) \leq m\left(\bigcup\{V + q: q \in \mathbf{Q}\}\right) = \sum_{q \in \mathbf{Q}} m(\mathbf{V} + q) = 0,$$

which contradicts property (b) of Lebesgue measure. Thus $\mathbf{V}$ is not in $\mathcal{S}$. (For the general case, modify the above proof using $\mathbf{V} \times [0,1]^{n-1}$.)

Notice that if $\mu: \mathfrak{M} \to [0,\infty]$ satisfies conditions (a)–(d) (we will call such a function an invariant measure) then the above proof shows that $\mathbf{V} \notin \mathfrak{M}$. Thus, $\mathbf{V}$ is not measureable for any invariant measure. In particular, there is no invariant measure $\mu: \mathfrak{M} \to [0,\infty]$ which is universal, i.e., for which $\mathfrak{M}$ equals $\mathcal{P}(\mathbf{R}^n)$.

Another remark we wish to make is that in Vitali's proof we used the Axiom of Choice (hereafter abbreviated AC). At the beginning of the twentieth century the Axiom of Choice was not commonly accepted (see [Mo]) and Lebesgue had his reservations about Vitali's construction (see [Le] or [Haw], p. 123). Today we accept the Axiom of Choice, so we cannot support Lebesgue's complaints. But was Lebesgue completely wrong?

If we do not accept the Axiom of Choice then Vitali's proof will not work. In 1964 Robert Solovay (see [So] or [Wa], Ch. 13) showed something much stronger: we cannot prove that $\mathcal{S} \neq \mathcal{P}(\mathbf{R}^n)$ without AC. More precisely, he proved that there exists a model ("a mathematical world") in which the usual Zermelo-Frankel set theory (ZF) is true and where all subsets of $\mathbf{R}^n$ are Lebesgue measurable, i.e., $\mathcal{S} = \mathcal{P}(\mathbf{R}^n)$. Moreover, although the full power of the Axiom of Choice must fail in this "world," the so-called Axiom of Dependent Choice (DC) is true in the model. In fact, DC tells that we can use inductive definitions and therefore all classical theorems of analysis remain true in this "world."

Solovay's theorem has only one disadvantage. Besides the usual axioms of set theory (ZF), Solovay's proof uses an additional axiom: there exists a weakly inaccessible cardinal (in abbreviation WIC, see [Je], p. 28). The theory ZF + WIC is essentially stronger than ZF. For over 20 years mathematicians wondered whether it is possible to eliminate the hypothesis regarding WIC from Solovay's theorem, but in 1980 Saharon Shelah showed that it cannot be done, proving that the consistency of the theory ZF + DC + "$\mathcal{S} = \mathcal{P}(\mathbf{R}^n)$" implies the consistency of the theory ZF + WIC (see [Ra] or [Wa], p. 209).

Extensions of Lebesgue Measure

Let us go back to doing mathematics with the Axiom of Choice. We know that Lebesgue measure is not universal, i.e., $\mathcal{S} \neq \mathcal{P}(\mathbf{R}^n)$. So let us examine the problem of how we can improve Lebesgue measure. The properties (a)–(d) of Lebesgue measure seem to be most desirable. Therefore, we will try to improve $m: \mathcal{S} \to [0,\infty]$ by examining its extensions, the functions $\mu: \mathfrak{M} \to [0,\infty]$ such that $\mathcal{S} \subset \mathfrak{M} \subset \mathcal{P}(\mathbf{R}^n)$ and $\mu(X) = m(X)$ for all $X \in \mathcal{S}$.

By Vitali's theorem, if μ is an invariant measure then $\mathfrak{M} \neq \mathcal{P}(\mathbf{R}^n)$. In this context the following question naturally appears: "How far can we extend Lebesgue measure and what properties can such an extension preserve?" This question has been investigated carefully by members of the Polish Mathematical School and all but one of the results presented here were proved by this group.

Let us first concentrate on the extensions that are invariant measures. The first result in this direction is due to Edward Szpilrajn (who later changed his name to Edward Marczewski). In 1936 he proved that Lebesgue measure is not a maximal invariant measure, i.e., that there exists an invariant measure that is a proper extension of Lebesgue measure. In connection with this result, Wacław Sierpiński (in 1936) posed the following question "Does there exist any maximal invariant measure?" (see [Sz]). Let us notice that by property (b) any such measure should extend Lebesgue measure.

How far can we extend Lebesgue measure and what properties can such an extension preserve?

This question was examined by several mathematicians under different assumptions. The answer was always the same: there is no maximal invariant measure. The first result was noticed by Andrzej Hulanicki in 1962 (see [Hu]) under the additional set-theoretical assumption that the continuum 2^ω is not RVM (this assumption will be discussed later in this article). This result was also obtain by S. S. Pkhakadze (see [Pk]) using similar methods. In 1977 A. B. Harazišvili (from Georgia, USSR) got the same answer in the one-dimensional case without any set-theoretical assumption (see [Har]). Finally, in 1982 Krzysztof Ciesielski and Andrzej Pelc generalized Harazišvili's result to all n-dimensional Euclidean spaces (see [CP] or [Ci]).

The idea of Ciesielski and Pelc's proof is due to Harazišvili. Using the algebraic properties of the group of isometries of $\mathbf{R}^n$ they constructed a family $\{N_j: j = 0,1,2,3,\ldots\}$ of subsets of $\mathbf{R}^n$ with following properties:

(i) $\mathbf{R}^n = \bigcup \{N_j: j = 0,1,2,\ldots\}$;

(ii) if $\mu: \mathfrak{M} \to [0,\infty]$ is an invariant measure then $N_j \in \mathfrak{M}$ implies $\mu(N_j) = 0$;

(iii) for every invariant measure $\mu: \mathfrak{M} \to [0,\infty]$ and for every natural number j there exists an invariant measure $\nu: \mathfrak{N} \to [0,\infty]$ extending μ such that $N_j \in \mathfrak{N}$.

This result easily implies the nonexistence of a maximal invariant measure. If $\mu: \mathfrak{M} \to [0,\infty]$ is an invariant measure then, by (ii), there exists a natural number j such that $N_j \notin \mathfrak{M}$, because otherwise we would have

$$\mu([0,1]^n) \leq \mu(\mathbf{R}^n) = \mu\left(\bigcup_{j=0}^{\infty} N_j\right) \leq \sum_{j=1}^{\infty} \mu(N_j) = 0.$$

Therefore the invariant measure ν as in (ii) is a proper extension of μ and so μ is not maximal.

At this stage of our discussion we are able to give a partial answer to the question of our title. If we restrict our search to invariant measures, then Lebesgue measure is not the richest. However, any other invariant measure has the same defect. This means that if we would like to compare invariant measures only by the size of their domain then the best solution to the measure problem simply does not exist. On the other hand, if we deal only with the subsets of $\mathbf{R}^n$ constructed without the Axiom of Choice but using the induction allowed by DC (this is the case in all constructions in classical analysis) then all sets we are interested in are Lebesgue measurable. Thus, in view of the above arguments and the naturalness of Lebesgue's construction, the Lebesgue measure is the unique reasonable candidate to be a canonical invariant measure.

Another idea to improve Lebesgue measure was to extend it to a universal measure $\mu: \mathcal{P}(\mathbf{R}^n) \to [0,\infty]$ by weakening some part of properties (a)–(d), thus avoiding Vitali's argument. The most popular approach is to drop isometric invariantness and to consider as measures the functions $\mu: \mathfrak{M} \to [0,\infty]$ that satisfy conditions (a), (b), and (c). This definition of measure is common today mostly because we can easily generalize it from $\mathbf{R}^n$ to an arbitrary set. To answer the question of when there exists a universal measure extending Lebesgue measure, let us consider the following sentence: "There exists a measure $\mu: \mathcal{P}(\mathbf{R}) \to [0,\infty]$ satisfying (a)–(c) such that $m(\{x\}) = 0$ for all $x \in \mathbf{R}$." If this sentence holds, we say that 2^ω is RVM, i.e., on the continuum there is a real-valued measure.

It is well known that if 2^ω is RVM, then there exists a universal extension of Lebesgue measure (the converse implication is obvious). We shall sketch the proof of this theorem for the one-dimensional case by finding the measure $\mu: \mathcal{P}([0,1]) \to [0,\infty]$ extending Lebesgue measure. So let $\nu: \mathcal{P}(\mathbf{R}) \to [0,\infty]$ be a measure such that $\nu(\{x\}) = 0$ for $x \in \mathbf{R}$. For every $X \subset \mathbf{R}$ there exists $X_0 \subset X$ such that $\nu(X_0) = ½\, \nu(X)$ (for the proof see [Je], Ex. 27.3 and Thm. 66, p. 297). So by an easy induction we can define the family of sets $\{X_s: s$ is a finite sequence of 0s and 1s$\}$ such that: $\nu(X_\emptyset) = 1$ ($\emptyset$ is considered to be a sequence of length 0), $X_{s\wedge 0}, X_{s\wedge 1} \subset X_s$, and $\nu(X_{s\wedge i}) = ½\, \nu(X_s)$ where $s\wedge i$ means an extension of sequence s by digit i. For $A \subset [0,1]$ we defined $\mu(A)$ by

$$\mu(A) = \nu\left(\bigcup_{a\in A} \bigcap_{n=1}^{\infty} X_{a(n)}\right),$$

where $a(n)$ is the sequence of the first n digits of the binary representation of a. It is not difficult to see that this μ really extends Lebesgue measure on $[0,1]$.

The systematic investigation of the statement "2^ω is RVM" was started in 1930 by Stanislaw Ulam, when he proved that it implies the existence of a weakly inaccessible cardinal (i.e., axiom WIC), so it cannot be proved under the usual axioms of set theory (see [Ul] or [Je], Thm. 66, p. 297). Although this result was fundamental for one of the most interesting parts of modern set theory, namely the theory of large cardinals, for our discussion it is only a disadvantage. It means that in order to have a universal countably additive measure extending Lebesgue measure we not only lose isometric invariantness, we must also assume a very strong additional axiom that is usually not accepted by mathematicians.

Finitely Additive Extensions of Lebesgue Measure

Let us consider another approach to the universal extension of Lebesgue measure by asking: "Is it really necessary to assume that measure is countably additive? Isn't it enough to deal with finitely additive measures?" More precisely, let us consider the properties:

(a') $\mathcal{S}$ is an algebra, i.e., $\mathcal{S}$ is closed under unions (if $A,B \in \mathcal{S}$ then $A \cup B \in \mathcal{S}$) and under complementation ($A \in \mathcal{S}$ implies $\mathbf{R}^n \setminus A \in \mathcal{S}$);

(c') m is finitely additive, i.e., for every disjoint pair A and B from $\mathcal{S}$,

$$m(A \cup B) = m(A) + m(B).$$

A function $m\colon \mathfrak{M} \to [0,\infty]$ satisfying the properties (a'), (b'), (c') and (d) will be called a finitely additive isometrically invariant measure. Does there exist a universal finitely additive isometrically invariant measure extending Lebesgue measure?

Stefan Banach (see [Ba] or [Wa]) proved in 1923 that such a measure exists on the plane and on the line (i.e., for $n = 2$ and $n = 1$). This beautiful result seems to be the only reasonable improvement of Lebesgue measure. But what is going on with n-dimensional Euclidean spaces for $n \geqq 3$? The answer, due to Stefan Banach and Alfred Tarski (1924; see [BT] or [Wa]), is surprising: there is no universal finitely additive isometrically invariant extension of Lebesgue measure for $n \geqslant 3$. But undoubtedly more surprising is the result that leads to this conclusion: the Banach-Tarski Paradox.

To state this paradox let us introduce the following terminology. We say that a set $A \subset \mathbf{R}^n$ is congruent to $B \subset \mathbf{R}^n$ if we can cut A into finitely many pieces and rearrange them (i.e., transform each of these pieces using some isometry of $\mathbf{R}^n$) to form the set B. The Banach-Tarski Paradox is the theorem that a ball B with volume 1 is congruent to the union of two similar disjoint balls B_1 and B_2 each having the same volume 1! There is even more: we can do this by cutting the ball B into only five pieces. This result is so paradoxical that our discomfort with it can probably be compared only with the Pythagorean "logical scandal" connected with the discovery of incommensurable line segments. This is also one of the strongest arguments against the use of the Axiom of Choice (see [Mo], p. 188).

But what about our measures on $\mathbf{R}^3$? It is easy to see for every universal finitely additive isometrically invariant measure μ that if $X \subset \mathbf{R}^n$ is congruent to $Y \subset \mathbf{R}^n$, then $\mu(X) = \mu(Y)$. In particular $\mu(B) = \mu(B_1 \cup B_2)$. But for 3-dimensional Lebesgue measure $m(B) = 1 \neq 2 = m(B_1 \cup B_2)$, so μ doesn't extend Lebesgue measure m.

Connected with this paradox is a nice open problem, the Tarski Circle-Squaring Problem from 1925, about which Paul Erdös wrote (see [Ma], p. 39):

> It is a very beautiful problem and rather well known. If it were my problem I would offer $1000 for it—a very, very nice question, possibly very difficult.

Tarski noticed that although in 3-dimensional Euclidean space all bounded sets with nonempty interior are congruent, this is not the case on the plane where sets with different Lebesgue measure are not congruent. But what about subsets of the plane with the same measure? In particular he formulated his Circle-Squaring Problem: "Is a square with unit measure congruent to a circle with the same measure?" This problem seems to be so difficult that Stan Wagon wrote about it (see [Wa], p. 101): "The situation seems not so different from that of the Greek geometers who considered the classical straight-edge-and-compass form of the circle-squaring problem."

Conclusion

So, what is the answer to the question "How good is Lebesgue measure?" In the class of invariant measures, Lebesgue measure seems to be the best candidate to be a canonical measure. In the class of countably additive not necessarily invariant measures, to find a universal measure we have to use a strong additional set-theoretical assumption and this seems to be too high a price. Thus the best improvement of Lebesgue measure seems to be the Banach construction of a finitely additive isometrically invariant extension of Lebesgue measure on the plane and line. However, such a measure does not exist on $\mathbf{R}^n$ for $n \geqslant 3$, and to keep the theory of measures uniform for all dimensions we cannot accept the Banach measure on the plane as the best solution to the measure problem.

From this discussion it seems clear that there is no reason to depose Lebesgue measure from the place it has in modern mathematics. Lebesgue measure also has a nice topological property called regularity: for every $E \in \mathcal{S}$ and every $\epsilon > 0$, there exists an open set $V \supset E$ and closed set $F \subset E$ such that $m(V \setminus F) < \epsilon$. It is not difficult to prove that Lebesgue measure is the richest countably additive measure having this property (see [Ru], Thm. 2.20, p. 50).

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В кругу друзей за праздничным столом
Мы выход книги нашей замечали.
Держа в руках хрустящий новый том,
Все от души нас с Изей поздравляли.
Да, наступил наш долгожданный час,
И даже скептик мрачный улыбнулся.
Он подошёл, чтобы поздравить нас,
А я от радости… проснулся.

Из сна М. Г. Крейна в Новогоднюю ночь 1963 г.
По свидетельству И. С. Иохвидова.

Around the festive table all our friends
Have come to mark our new book's publication.
The new and shiny volume in their hands,
They offer Is and me congratulation.
Our long-awaited hour has come at last.
The sourest skeptic sees he was mistaken
And smiling comes to cheer us like the rest;
And I am so delighted . . . I awaken.

From M. G. Kreĭn's dream, New Year's 1963, as reported by I. S. Iokhvidov.

English version by Chandler Davis. (The book dreamt of, *Introduction to the theory of linear nonselfadjoint operators* by I. C. Gohberg and M. G. Kreĭn, was published only in 1965.)

. . . Einstein was always rather hostile to quantum mechanics. How can one understand this? I think it is very easy to understand, because Einstein had been proceeding on different lines, lines of pure geometry. He had been developing geometrical theories and had achieved enormous success. It is only natural that he should think that further problems of physics should be solved by further development of geometrical ideas. Now, to have $a \times b$ not equal to $b \times a$ is something that does not fit in very well with geometrical ideas; hence his hostility to it.

P. A. M. Dirac