RESEARCH

Real Analysis Exchange
Vol. 40(2), 2014/2015, pp. 309–318

Krzysztof Chris Ciesielski, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310 and Department of Radiology, MIPG, University of Pennsylvania, Blockley Hall – 4th Floor, 423 Guardian Drive, Philadelphia, PA 19104-6021. email: `KCies@math.wvu.edu`

Jakub Jasinski, Department of Mathematics, University of Scranton, Scranton, PA 18510-4666. email: `jakub.jasinski@scranton.edu`

ON CLOSED SUBSETS OF $\mathbb{R}$ AND OF $\mathbb{R}^2$ ADMITTING PEANO FUNCTIONS

Abstract

In this note we describe closed subsets of the real line $P \subset \mathbb{R}$ for which there exists a continuous function from P onto P^2, called a *Peano function*. Our characterization of those sets is based on the number of connected components of P. We also include a few remarks on compact subsets of $\mathbb{R}^2$ admitting Peano functions, expressed in terms of connectedness and local connectedness.

1 Introduction

For a topological space X, we say that X *admits a Peano function* provided there exists a continuous map f from X onto X^2, which we will refer to as a *Peano function (for X)*. The classic result of G. Peano [5] states that there exists a Peano function for the interval $[0, 1] \subset \mathbb{R}$.

Throughout this note $\kappa(X)$ denotes the number (cardinality) of connected components of X. Recently K. Ciesielski and J. Jasinski [1] gave the following characterization of compact sets of reals that admit Peano functions.

Theorem 1.1. *If $P \subset \mathbb{R}$ is compact, then P admits a Peano function if and only if either $\kappa(P) = 1$ or $\kappa(P) = \mathfrak{c}$.*

Mathematical Reviews subject classification: Primary: 26A30; Secondary: 26B05
Key words: Peano curve, space filing curve
Received by the editors July 18, 2014
Communicated by: Paul D. Humke

In the next theorem, generalizing Theorem 1.1, we give a complete characterization of non-empty closed subsets of $\mathbb{R}$ that admit Peano functions. Note that if X admits a continuous map $f = \langle f_1, f_2\rangle$ from X onto X^2, then for every $n < \omega$ there exists a continuous map g_n from X onto X^n. For $n = 2^k$ this can be done by induction: $g_{2^{k+1}}\colon X \to X^{2^{k+1}} = X^{2^k} \times X^{2^k}$ can be defined as $\langle g_{2^k} \circ f_1, g_{2^k} \circ f_2\rangle$. Then, for any $n \le 2^k$, g_n can be defined as a composition of g_{2^k} with the projection $\pi\colon X^{2^k} \to X^n$ onto the first n coordinates. Compare also [6].

2 Closed subsets of the real line

Theorem 2.1. *A closed non-empty subset P of $\mathbb{R}$ admits a Peano function if and only if one of the following conditions holds:*

(1) $\kappa(P) = 1$

(2.1) $\kappa(P) = \omega$ *and P is countable, unbounded*

(2.2) *for every $n < \omega$, $\kappa(P \setminus (-n, n)) = \omega$ and $P \setminus (-n, n)$ is uncountable*

(3.1) $\kappa(P) = \mathfrak{c}$ *and P is bounded*

(3.2) $\kappa(P \setminus (-n, n)) = \mathfrak{c}$ *for every $n < \omega$.*

Proof. We first show *the sufficiency of each of the conditions* **(1)**–**(3.2)**.

(1) $\kappa(P) = 1$**:** If P is bounded, then P admits a Peano function by the classical Peano result, [5]. So, assume that P is unbounded. Then there exists a sequence $[b_0, c_0] \subset [b_1, c_1] \subset [b_2, c_2] \subset \cdots$ of closed intervals such that $P = \bigcup_{n<\omega}[b_n, c_n]$. We also have $P^2 = \bigcup_{n<\omega}[b_n, c_n]^2$. Moreover, for some $a \in \mathbb{R}$, P contains either $[a, \infty)$ or $(-\infty, a]$. Assume the former case and, for simplicity, that $a = 0$. By the classic Peano result, for each $n < \omega$ there exists a continuous surjection $f_n\colon [2n, 2n + 1] \to [b_n, c_n]^2$. Then, the union $\bigcup_{n<\omega} f_n$ is a continuous surjection from $\bigcup_{n<\omega}[2n, 2n+1] \subset P$ onto $\bigcup_{n<\omega}[b_n, c_n]^2 = P^2$. Since P^2 is convex, by the version of Tietze's theorem from [2], we can extend $\bigcup_{n<\omega} f_n$ to the desired continuous surjection $f\colon P \to P^2$.

(2.1) $\kappa(P) = \omega$ **and** P **is countable:** If P is countable, unbounded, then either $P \cap [n, \infty) \ne \emptyset$ for all $n < \omega$ or $P \cap (-\infty, n] \ne \emptyset$ for all $n < \omega$. Assume the former case. Then, there exists an increasing sequence $\langle d_n \in \mathbb{R}\setminus P\colon n < \omega\rangle$ divergent to ∞ such that $P_n = P \cap (d_n, d_{n+1})$ is non-empty for every $n < \omega$.

Then, the sets P_n, for every $n<\omega$, and $\bigcup_{k<\omega} P_k = P\cap[d_0,\infty)$ are clopen in P. Let $P^2=\{c_n\colon n<\omega\}$ and notice that $f\colon P\to P^2$ defined by

$$f(x)=\begin{cases} c_n & \text{for } x\in P_n,\\ c_0 & \text{for } x\in P\setminus\bigcup_{k<\omega}P_k\end{cases}$$

is continuous and onto P^2.

(2.2) For every $n<\omega$, $\kappa(P\setminus(-n,n))=\omega$ and $P\setminus(-n,n)$ is uncountable: The assumptions imply that for every real $s<t$ the set $P\setminus(s,t)$ contains a component $[a,b]$ with $a<b$. Thus, by an easy induction, we can choose disjoint components $\{[a_n,b_n]\colon n<\omega\}$ of P such that $a_n<b_n$ and $[a_n,b_n]\subset P\setminus(-n,n)$ for every $n<\omega$. Clearly, $\bigcup_{n<\omega}[a_n,b_n]$ is unbounded.

We make the argument more transparent by assuming that $\bigcup_{n<\omega}[a_n,b_n]$ is unbounded towards positive infinity and (by selecting a subsequence, if necessary) that $a_0<b_0<a_1<b_1<\cdots$. It is now easy to see that there exists a sequence of intervals $\langle(c_n,d_n)\colon n<\omega\rangle$ with $c_n,d_n\notin P$, $d_n=c_{n+1}$ such that each set $P_n=P\cap(c_n,d_n)$ contains $[a_n,b_n]$ for $n<\omega$. Then, as before, the sets P_n, for all $n<\omega$, and $\bigcup_{k<\omega}P_k=P\cap[c_0,\infty)$ are clopen in P.

Notice also that $P^2=\bigcup_{n<\omega}R_n$, where each R_n is a bounded closed rectangle $I_n\times J_n$; that is, both I_n and J_n are bounded closed (possibly singleton) intervals. Indeed, each component of P^2 is of the form $C=I\times J$, where both I and J are closed (possibly singleton or unbounded) intervals, while any such set is a countable union of bounded closed rectangles.

For every $n<\omega$, choose a continuous function f_n from P_n onto R_n. It exists, since P_n can be continuously mapped onto $[a_n,b_n]$ and, with the help of the classic Peano result, $[a_n,b_n]$ can always be mapped continuously onto R_n. Let $c\in P^2$ and define

$$f(x)=\begin{cases} f_n(x) & \text{for } x\in P_n,\\ c & \text{for } x\in P\setminus\bigcup_{k<\omega}P_k.\end{cases} \tag{1}$$

Clearly $f\colon P\to P^2$ is a Peano function for P.

(3.1) $\kappa(P)=\mathfrak{c}$ and P is bounded: Such sets P admit Peano functions by Theorem 1.1.

(3.2) $\kappa(P\setminus(-n,n))=\mathfrak{c}$ for every $n<\omega$: We either have $\kappa(P\cap[n,\infty))=\mathfrak{c}$ for all $n<\omega$ or $\kappa(P\cap(-\infty,-n])=\mathfrak{c}$ for all $n<\omega$. Assume the former case. Then, there exists an increasing sequence $\langle d_n\in\mathbb{R}\setminus P\colon n<\omega\rangle$ diverging to ∞

such that for every $P_n = P \cap (d_n, d_{n+1})$ we have $\kappa(P_n) = \mathfrak{c}$. Then $\bigcup_{k<\omega} P_k$ is closed and each P_n is compact and open in P.

Since every closed subset of $\mathbb{R}^2$ is sigma compact, $P^2 = \bigcup_{n<\omega} K_n$, where each $K_n \subseteq \mathbb{R}^2$ is compact. Recall (e.g. [1, p. 70]) that any compact set $Q \subseteq \mathbb{R}$ with $\kappa(Q) = \mathfrak{c}$ can be mapped continuously onto any compact $K \subseteq \mathbb{R}^2$. For every $n < \omega$, choose a continuous function f_n from P_n onto K_n and fix a $c \in P^2$. Notice that the function $f\colon P \to P^2$ defined by the formula (1) above is a Peano function for P.

We now prove *the necessity of the conditions* of Theorem 2.1. For the rest of this section let $P \subseteq \mathbb{R}$ be a non-empty closed set, and let $f\colon P \to P^2$ be a Peano function for P. It is easy to verify that $\kappa(P) \in \{1, \omega, \mathfrak{c}\}$. Indeed, $\kappa(P)$ cannot be finite greater than 1 because it admits a Peano function so $\kappa(f[P]) \leq \kappa(P)$ while $\kappa(P^2) = \kappa(P)^2$. Now if $\kappa(P)$ is infinite, then $\kappa(P) = |P \setminus \mathrm{Int(P)}|$. So, below we discuss the three possible values of $\kappa(P)$.

$\kappa(P) = 1$: The condition (1) from Theorem 2.1 is satisfied.

$\kappa(P) = \omega$: Notice that in this case, by Theorem 1.1, P must be unbounded. If P is countable, then the condition (2.1) is met. So, let us assume that $|P| = \mathfrak{c}$. Let $\langle K_n \colon n < \omega\rangle$ be a sequence of all components of P. Clearly at least one of the K_n's, say K_0, must be uncountable.

Claim 1: *Infinitely many of the K_n's are uncountable.*

Suppose otherwise, that is, that P has a finite number of uncountable components. At the same time, P^2 has infinitely many uncountable components, $K_0 \times K_n$, $n < \omega$. Since a continuous image of a single component must be connected, only finitely many of the uncountable components of P^2 can be covered by a continuous image of P. Thus, P does not admit Peano functions, which proves Claim 1.

It follows that P has infinitely many non-degenerate interval components. At most two of them can be unbounded. Let $\langle I_n\rangle_{n<\omega}$ be a sequence of all non-degenerate bounded interval components of P and set $F = \bigcup_{n<\omega} I_n$.

Claim 2: *F is unbounded.*

By way of contradiction assume that F is bounded. Then, there exists an $n < \omega$ such that $F \subseteq [-n, n]$. Observe that P cannot have unbounded components. Indeed, if P did have an unbounded component, then P^2 would have infinitely many unbounded components. However, since $F \subseteq [-n, n]$, a continuous image of P can have at most two unbounded components, since

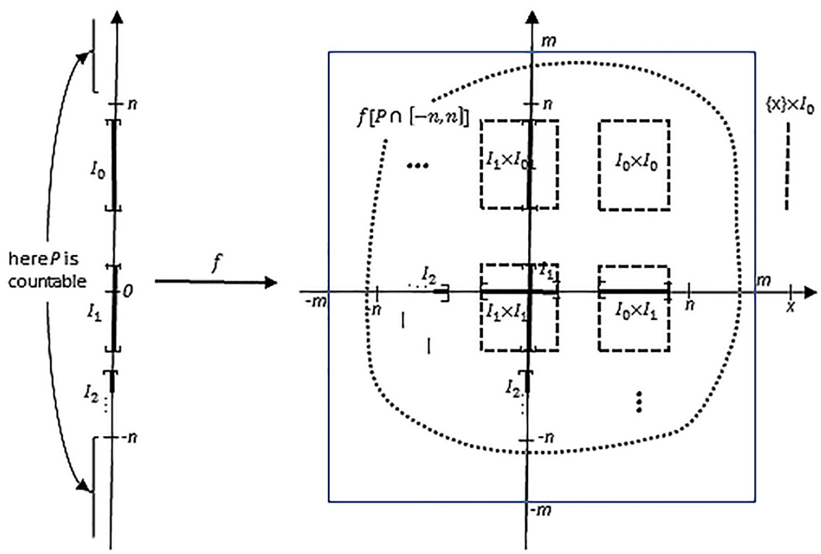

Figure 1: Illustration why the set F in Claim 2 must be unbounded.

the image of $P \cap [-n, n]$ is compact. So, such P would not admit a Peano function. Thus, our P does not have any unbounded components so $P \setminus [-n, n]$ is countable. Therefore, once again, no continuous function f from P can be onto P^2, since $f[P \cap [-n, n]]$, being compact, is contained in $[-m, m]^2$ for some $m < \omega$, and for any $x \in P \setminus [-m, m]$ the uncountable set $\{x\} \times I_0 \subset P^2$ cannot be covered by the countable set $f[P \setminus [-n, n]]$, see Figure 1. So, F must be unbounded, which proves Claim 2.

Clearly, the two claims imply (2.2).

$\kappa(P) = \mathfrak{c}$: If P is bounded, then we have condition (3.1). So assume that the set P is unbounded. We must show that for all $n < \omega$, $\kappa(P \setminus (-n, n)) = \mathfrak{c}$. Suppose otherwise, that there is an $n_0 < \omega$ with $\kappa(P \setminus (-n_0, n_0)) \leq \omega$. Since $f[P \cap [-n_0, n_0]]$ is compact, it is contained in $[-m, m]^2$ for some $m < \omega$. For any $x \in P \setminus [-m, m]$, the set $\{x\} \times P \subset P^2 \setminus f[P \cap [-n_0, n_0]]$ intersects uncountably many components of P^2, so it cannot be covered by a continuous image of $P \setminus (-n_0, n_0)$ since it has only countably many components. $\square$

3 Compact subsets of the plane

The case of connected and locally connected sets is made clear by the following theorem.

Theorem 3.1. *If P is a compact, connected and locally connected metric space, then P admits a Peano function.*

Proof. We may assume that P has two different points, $a, b \in P$. Let d be the metric on P and define a function $g\colon P \to [0,\infty)$, $g(x) = d(x,b)$. The space P is compact and connected so, by the intermediate value theorem (see e.g. [4, Theorem 24.3]), $g[P] = [0,\alpha]$ for some $\alpha \in (0,\infty)$. Also, since P^2 is still a compact, connected and locally connected metric space [4, Theorem 23.6], by Hahn-Mazurkiewicz Theorem [3, p. 129], there exists a continuous function h from $[0,\alpha]$ onto P^2. Then, $f = h \circ g$ is a Peano function for P. $\square$

For compact connected subsets P of $\mathbb{R}^2$ that are not locally connected, the situation is not that simple, since such a space need not be path connected.

Example 3.2. If $P = \{\langle x, \sin(1/x)\rangle \colon x \in (0,1]\} \cup (\{0\} \times [-1,1])$ is the topologist's sine curve [4, p.157], then there is no continuous function from P onto P^2.

Proof. Since a continuous image of a path connected set is path connected, the two path components of P cannot be mapped onto four path components of P^2. $\square$

What about compact path connected subsets P of $\mathbb{R}^2$ that are not locally connected? Here is another counterexample. In its statement, for every $r \in \mathbb{R}$, the set I_r represents the closed line segment in $\mathbb{R}^2$ connecting $\langle 0,1\rangle$ with $\langle r,0\rangle$. Moreover, $S = \{0\} \cup \{1/n \colon n = 1,2,3,\ldots\}$.

Example 3.3. Let $P = \bigcup_{r\in S} I_r$ be the closed infinite broom, see Figure 2. (Compare [4, p. 162].) Then, there is no continuous function from P onto P^2.

Proof. An argument bares some similarity to the fact that S cannot be continuously mapped onto S^2, a special case of [1, lemma 4.3]. Recall that P is not locally connected at the point $\langle 0,0\rangle$. Actually, for every open V containing $\langle 0,0\rangle$ and of diameter less than 1, the sets $V \cap I_{1/n}$ are pairwise disjoint, closed in V, and non-empty for all but finitely many n.

By way of contradiction, assume that P admits a Peano function f. We will show that this assumption implies that

- $\{\langle 0,0\rangle\} \times P \subset f[I_0]$.

However, • is impossible, since a continuous image of an interval must be locally connected [3, p. 129]. At the same time, no locally connected subspace T of P^2 can contain $\{\langle 0,0\rangle\} \times P$, since for such T every connected $C \subset P^2$ of diameter less than 1 and with $\langle\langle 0,0\rangle, \langle 0,0\rangle\rangle \in C$ must be contained in I_0^2, so it has empty interior in T.

To see •, fix a $p \in P$ and let $t = \langle\langle 0,0\rangle, p\rangle$. We need to show that $t \in f[I_0]$. For this, consider the sequence $\langle t_n\rangle_n = \langle\langle\langle 1/n, 0\rangle, p\rangle\rangle_n$, in $P \times \{p\}$, converging to t and notice, that

($*$) for every $s \in S$, the set $f[I_s]$ can contain only finitely many t_n's.

Indeed, otherwise, we would have $t \in f[I_s]$, and every open U in $f[I_s]$ containing t would intersect infinitely many sets $K_n = I_{1/n} \times P$, see Figure 2. Since, for every U of diameter less than 1, the sets $U \cap K_n$ are pairwise disjoint and clopen in U, this would mean that $f[I_s]$ is not locally connected at t.

Using ($*$) and the assumption that f is onto P^2, we can choose a subsequence $\langle t_{n_i}\rangle_i$ and distinct points $s_i \in S$ such that $t_{n_i} \in f[I_{s_i}]$ for every i. Let

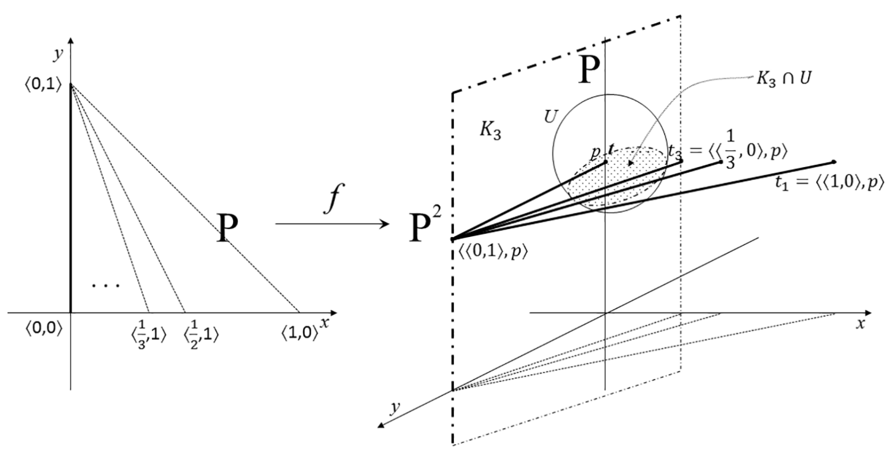

Figure 2: There is no Peano function for the infinite broom P. The outlined parallelogram represents the set $K_3 = I_{1/3} \times P$.

$x_i \in I_{s_i}$ be such that $f(x_i) = t_{n_i}$. Since P is compact, choosing a subsequence if necessary, we can assume that $\langle x_i \rangle_i$ converges to an $x \in P$. Since the points s_i are distinct, $x \in I_0$. Thus, $t = \lim_i t_{n_i} = \lim_i f(x_i) = f(\lim_i x_i) = f(x) \in f[I_0]$, as we were to prove. □

Can the argument from the above example be generalized to any compact path connected subset P of $\mathbb{R}^2$ that is not locally connected? A negative answer is given by the following example. In its statement, $C \subset [0,1]$ stands for the Cantor ternary set.

Example 3.4. Let $P = \bigcup_{r \in C} I_r$ be the closure of Knaster-Kuratowski exploding set. Then P is compact, path connected and not locally connected. Moreover, there exists a continuous function from P onto P^2.

Proof. Suppose $p = \langle p_1, p_2 \rangle \colon C \times [0,1] \to [0,1]^2$ is defined as $p(x,y) = \langle x(1-y), y \rangle$ and notice that $P = p[C \times [0,1]]$, since $p[\{r\} \times [0,1]] = I_r$. Clearly p is a continuous closed map. As such, it is a quotient map, see for example [4, p. 137].

Let $k = \langle k_1, k_2 \rangle$ and $h = \langle h_1, h_2 \rangle$ be the Peano functions for C and $[0,1]$, respectively. Moreover, assume that $h(1) = \langle 1, 1 \rangle$, which is satisfied for the standard Peano curve. Let $g = \langle p \circ \langle k_1, h_1 \rangle, p \circ \langle k_2, h_2 \rangle \rangle \colon C \times [0,1] \to P^2$. Clearly, g is continuous and onto P^2. Moreover, g is constant on any set $p^{-1}(\{z\})$ with $z \in P$. (Indeed, if $p(c_0, y_0) = p(c_1, y_1)$ for distinct $\langle c_0, y_0 \rangle$ and $\langle c_1, y_1 \rangle$ from $C \times [0,1]$, then $y_0 = y_1 = 1$. So, by $h(1) = \langle 1, 1 \rangle$, $g(c_i, y_i) = p(k_i(c_i), 1) = \langle 0, 1 \rangle$ for $i < 2$.) Therefore, by [4, Theorem 22.2], there exists a continuous function f from P onto P^2, the desired Peano map. □

Problem 1. Characterize the compact connected (or just path connected) subsets of the plane that admit a Peano function.

References

[1] K. C. Ciesielski and J. Jasinski, *Smooth Peano functions for perfect subsets of the real line*, Real Anal. Exchange, **39(1)** (2014), 57–72.

[2] J. Dugunji, *An extension of Tietze's theorem*, Pacific J. Math., **1(3)** (1951), 353–366.

[3] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, 1961.

[4] J. R. Munkres, *Topology*, Prentice Hall, 2nd ed., 2000.

[5] G. Peano, *Sur une courbe, qui remplit toute une aire plane*, Math. Ann., **36** (1890), 157–160.

[6] H. Sagan, *Space-Filling Curves*, Springer-Verlag, 1994.