RESEARCH

Real Analysis Exchange
Vol. 39(1), 2013/2014, pp. 57–72

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SMOOTH PEANO FUNCTIONS FOR PERFECT SUBSETS OF THE REAL LINE

Abstract

In this paper we investigate for which closed subsets P of the real line $\mathbb{R}$ there exists a continuous map from P onto P^2 and, if such a function exists, how smooth can it be. We show that there exists an infinitely many times differentiable function $f:\mathbb{R}\to\mathbb{R}^2$ which maps an unbounded perfect set P onto P^2. At the same time, no continuously differentiable function $f:\mathbb{R}\to\mathbb{R}^2$ can map a compact perfect set onto its square. Finally, we show that a disconnected compact perfect set P admits a continuous function from P onto P^2 if, and only if, P has uncountably many connected components.

1 Introduction and overview

Let P be a nonempty subset of the set $\mathbb{R}$ of real numbers. If P has no isolated points and $n,m\in\{1,2,3,\ldots\}$, then we consider the following classes of smooth functions from P to $\mathbb{R}^m$: $\mathcal{D}^n$ of n-times differentiable functions and $\mathcal{C}^n$ of continuously n-times differentiable functions. In addition, $\mathcal{C}^0$ will stand for the class of all continuous functions and $\mathcal{C}^\infty$ for the class of functions differentiable infinitely many times. For every $n<\omega$ we have $\mathcal{C}^\infty\subset\mathcal{C}^{n+1}\subset\mathcal{D}^{n+1}\subset\mathcal{C}^n$.

A nonempty set $P\subseteq\mathbb{R}$ is called *perfect* if it is closed and has no isolated points. We say that a function $f:P\to\mathbb{R}^2$ is *Peano* if it is onto P^2, that is,

Mathematical Reviews subject classification: Primary: 26A30; Secondary: 26B05
Key words: Peano curve, space filing curve, differentiability
Received by the editors March 4, 2013
Communicated by: Paul D. Humke

when $f[P] = P^2$. For example, the classic result of Peano [7] states that there exists a Peano function $f:[0,1] \to [0,1]^2$ of class $\mathcal{C}^0$. More on this topic can be found in Sagan [9].

It is worth noting that some Peano functions $f: P \to \mathbb{R}^2$ of a given smoothness class can be extended to the entire functions $\widehat{f}: \mathbb{R} \to \mathbb{R}^2$ of the same class.

Proposition 1.1. *Let $P \subset \mathbb{R}$ be a perfect set.*

(a) *Any $\mathcal{C}^0$ Peano function $f: P \to P^2$ may be extended to a $\mathcal{C}^0$ function $\widehat{f}: \mathbb{R} \to \mathbb{R}^2$.*

(b) *Any $\mathcal{D}^1$ Peano function $f: P \to P^2$ may be extended to a $\mathcal{D}^1$ function $\widehat{f}: \mathbb{R} \to \mathbb{R}^2$.*

Proof. (a) follows from the Generalized Tietze extension theorem, see e.g. [5, p. 151]. Part (b) follows from the following extension theorem due to V. Jarník [2]: "Every differentiable function f from a perfect set $P \subset \mathbb{R}$ into $\mathbb{R}$ can be extended to a differentiable function $\widehat{f}: \mathbb{R} \to \mathbb{R}$." More on Jarník's theorem can be found in [4]. The theorem has also been independently proved in [8, theorem 4.5]. □

Proposition 1.1 shows that for the functions from classes $\mathcal{C}^0$ and $\mathcal{D}^1$, the existence of a Peano function for a perfect set $P \subset \mathbb{R}$ is equivalent to the

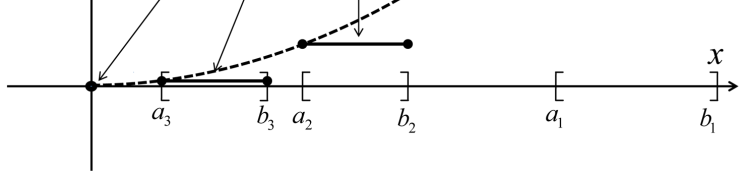

Figure 1: $f(0) = 0$ and $f(x) = (a_n)^2$ for $x \in [a_n, b_n]$.

existence of a function $f: \mathbb{R} \to \mathbb{R}^2$ of the same class with $f \restriction P$ being Peano.

Remark 1.2. For the functions of the higher classes of smoothness such simple equivalence is not achievable. Indeed, in general, a $\mathcal{C}^1$ function f from a perfect set $P \subset [0,1]$ into $\mathbb{R}$ need not be extendable to an entire $\mathcal{C}^1$ function $\widehat{f}: [0,1] \to \mathbb{R}$, even if f is of the $\mathcal{C}^\infty$ class.

Perhaps the simplest example supporting our Remark 1.2 is the function f defined on the set $P = \{0\} \cup \bigcup_{n=1}^\infty [a_n, b_n]$, where $a_n = 2^{-n}$ and $b_n \in (a_n, a_{n-1})$, as $f(0) = 0$ and $f(x) = (a_n)^2$ for $x \in [a_n, b_n]$. See Figure 1. Then, $f'(x) = 0$ for every $x \in P$, so f is $\mathcal{C}^\infty$. However, if we choose b_{n+1}'s such that the quotient $\frac{f(a_n) - f(b_{n+1})}{a_n - b_{n+1}} = \frac{(2^{-n})^2 - (2^{-n-1})^2}{2^{-n} - b_{n+1}}$ equals 1, $b_{n+1} = \frac{2^{n+2}-3}{2^{2n+2}}$ works, then by the mean value theorem any differentiable extension $\widehat{f}: [0,1] \to \mathbb{R}$ of f will have discontinuous derivative at 0.

Remark 1.2 shows that for the functions of at least $\mathcal{C}^1$ smoothness, it makes a difference, if we construct the Peano functions as the restrictions of the entire smooth functions or just on the set P. We pay attention to these details in what follows.

The following theorem summarizes all the results on the Peano functions for the subsets of $\mathbb{R}$ presently known to us.

Theorem 1.3. *Let P be a closed subset of $\mathbb{R}$.*

(a) *There exists a $\mathcal{C}^0$ Peano function f from P onto P^2 if, and only if, P is either connected or it has uncountably many components.*

(b) *If P is perfect and has positive Lebesgue measure, then there is no $\mathcal{D}^1$ Peano function f from P onto P^2.*

(c) *If $f: \mathbb{R} \to \mathbb{R}^2$ is a $\mathcal{C}^1$ function and $P \subseteq \mathbb{R}$ is a compact perfect set, then $P^2 \not\subset f[P]$. Hence, $f \restriction P$ is not Peano.*

(d) *There exists a $\mathcal{C}^\infty$ function $f: \mathbb{R} \to \mathbb{R}^2$ and a perfect unbounded subset P of $\mathbb{R}$ such that $f[P] = P^2$, that is, $f \restriction P$ is Peano.*

Proof. (a) is proved in Theorem 4.1.

(b) Let $f = \langle f_1, f_2 \rangle: P \to P^2$ be differentiable. Morayne [6, theorem 3] showed (using the fact that $\mathcal{D}^1$ functions satisfy the Banach condition (T_2)) that $f[P]$ must have the planar Lebesgue measure zero. In particular, if P has positive measure, then $P^2 \not\subset f[P]$.

(c) is proved in Theorem 3.1.

(d) is proved in Theorem 2.2. □

2 A $\mathcal{C}^\infty$ function $f:\mathbb{R}\to\mathbb{R}^2$ with a Peano restriction $f \restriction P$ for some perfect set $P\subset\mathbb{R}$

The idea is to construct a sequence $\langle P_k \subseteq [3k, 3k+2]: k<\omega\rangle$ of perfect sets such that for every $\ell,\ell' < k$ there exists a $\mathcal{C}^\infty$ function $f^k_{\ell,\ell'}$ from $[3k, 3k+2]$ into $\mathbb{R}^2$ which maps P_k onto $P_\ell \times P_{\ell'}$, see Figures 2 and 4. Then, the set $P = \bigcup_{k<\omega} P_k$ will be as required, since for any given sequence $\langle\langle \ell_k, \ell'_k\rangle: 0<k<\omega\rangle$ of all pairs of natural numbers with $\ell_k, \ell'_k < k$, the function $\hat{f} = \bigcup_{0<k<\omega} f^k_{\ell_k,\ell'_k}$ is $\mathcal{C}^\infty$ and it maps $\bigcup_{0<k<\omega} P_k$ onto P^2. Such an $\hat{f}$ can easily be extended to the desired $\mathcal{C}^\infty$ function $f:\mathbb{R}\to\mathbb{R}^2$.

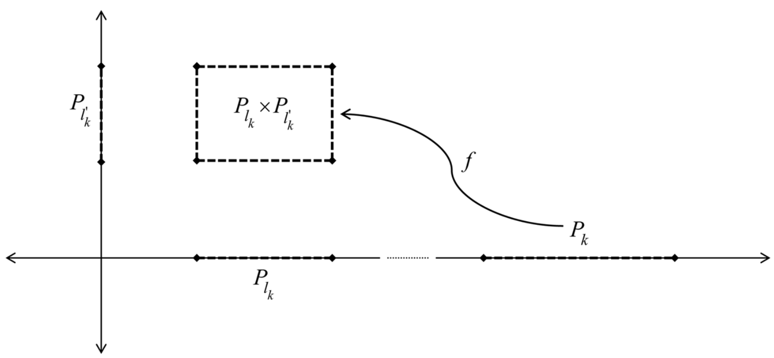

Figure 2: An $f^k_{\ell_k,\ell'_k}$ fragment of the function f.

The construction of the sets P_k will naturally provide continuous mappings $\bar{f}^k_{\ell,\ell'}$ from P_k onto $P_\ell \times P_{\ell'}$. The difficulty will be to ensure that these functions are not only $\mathcal{C}^\infty$, but that they can be also extended to the $\mathcal{C}^\infty$ functions $f^k_{\ell,\ell'}: [3k, 3k+2]\to\mathbb{R}^2$. The tool to insure the extendability is provided by the following Lemma 2.1. Notice, that the lemma can be considered as a version of Whitney extension theorem [10].[1]

Note also, that no analytic function $f:\mathbb{R}\to\mathbb{R}^2$ can have a Peano restriction to any perfect set (since the coordinates, $f_1, f_2:\mathbb{R}\to\mathbb{R}$, of a Peano function need to be constant on some perfect subsets).

Lemma 2.1. *Every real-valued function g_0 from a compact nowhere dense set $K\subset\mathbb{R}$ having the property that for every $k<\omega$ there exists a $\delta_k \in (0,1)$*

[1] *Added in proof.* Actually, Lemma 2.1 follows from [10, thm. 1 p. 65], since "g_0 is of class $\mathcal{C}^\infty$ in K in terms of the functions $f_k \equiv 0$." The authors like to thank Prof. Jan Kolar for pointing this out.

such that

(P_k) $|g_0(x) - g_0(y)| < |x - y|^{k+1}$ *for all* $x, y \in K$ *with* $0 < |x - y| < \delta_k$

can be extended to a $\mathcal{C}^\infty$ *function* $g: \mathbb{R} \to \mathbb{R}$. *Moreover,* $g'(x) = 0$ *for all* $x \in K$.

PROOF. Let $\psi: \mathbb{R} \to \mathbb{R}$ be a monotone C^∞ map such that $\psi[(-\infty, 0)] = \{0\}$ and $\psi[(1, \infty)] = \{1\}$. For $k < \omega$ let

$$M_k = \sup\left\{|\psi^{(i)}(x)|: x \in [0,1] \ \&\ i \le k\right\} \in [1, \infty).$$

Let $\mathcal{K}$ be a family of all connected bounded components (a, b) of $\mathbb{R} \setminus K$. Let $g: \mathbb{R} \to \mathbb{R}$ be an extension of g_0 such that g is constant on the closure of each unbounded component of $\mathbb{R} \setminus K$ and on each $(a, b) \in \mathcal{K}$ function g is defined by a formula

$$g(x) = (g_0(b) - g_0(a))\psi\left(\frac{x-a}{b-a}\right) + g_0(a).$$

In other words, g on (a, b) is a function $\psi \restriction (0, 1)$ shifted and linearly rescaled in such a way that $g \restriction [a, b]$ is continuous. We will show that such defined g is our desired C^∞ function.

Clearly, the restriction $g|_{\mathbb{R}\setminus K}$ of g is infinitely many times differentiable at any $x \in \mathbb{R} \setminus K$. We need to show that the same is true for any $x \in K$. For this, we will show, by induction on $k \ge 1$, that

(I_k) for every $x \in K$, the k-th derivative $g^{(k)}(x)$ exists and is equal 0.

The inductive argument is based on the following estimate, where $k \ge 1$:

(S_k) $\left|\frac{g^{(k-1)}(y) - g^{(k-1)}(z)}{y-z}\right| < M_k(b-a)$ provided $(a, b) \in \mathcal{K}$, $b - a < \delta_k$, and $y, z \in [a, b]$ are distinct.

Let $k \ge 1$. To see (S_k), take y and z as in its assumption. Then,

$$\begin{aligned}
\left|\frac{g^{(k-1)}(y) - g^{(k-1)}(z)}{y-z}\right| &\le \sup_{x\in(a,b)} \left|g^{(k)}(x)\right| && (1)\\
&= \sup_{x\in(a,b)} \frac{|g(b) - g(a)|}{|b-a|^k} \left|\psi^{(k)}\left(\frac{x-a}{b-a}\right)\right| && (2)\\
&\le \frac{|g(b) - g(a)|}{|b-a|^k} M_k && (3)\\
&< \frac{|b-a|^{k+1}}{|b-a|^k} M_k = M_k(b-a), && (4)
\end{aligned}$$

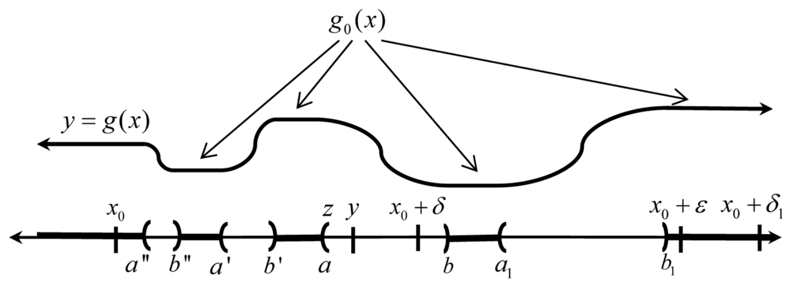

Figure 3: $b - a < \varepsilon/M_1 < b_1 - a_1$.

where (1) follows from the Mean Value Theorem, (2) from the fact that $g^{(k)}(x) = \frac{d^k}{dx^k}\left[(g(b)-g(a))\psi\left(\frac{x-a}{b-a}\right)+g(a)\right] = \frac{g(b)-g(a)}{(b-a)^k}\psi^{(k)}\left(\frac{x-a}{b-a}\right)$ for every $x \in (a,b)$, (3) from the definition of M_k, while (4) is concluded from (P_k) used with $x = b$ and $y = a$.

To show (I_1), fix an $x_0 \in K$ and an $\varepsilon > 0$. We will find a $\delta > 0$ for which

$$\left|\frac{g(y)-g(x_0)}{y-x_0}\right| < \varepsilon \text{ provided } x_0 < y < x_0+\delta. \tag{5}$$

If x_0 is equal to the left endpoint of some component interval of $\mathbb{R} \setminus K$, then the existence δ follows from our definition of the function g on such intervals, specifically because $\psi'(0) = 0$. So, assume that this is not the case, that is, that $(x_0, x_0+\eta) \cap K \neq \varnothing$ for every $\eta > 0$. Let $\delta \in (0, \min\{\varepsilon, \delta_1\})$ be such that $(x_0, x_0+\delta)$ is disjoint with every $(a_1, b_1) \in \mathcal{K}$ for which $b_1 - a_1 \geq \varepsilon/M_1$. See Figure 3. We will show that such δ works.

So, fix a $y \in (x_0, x_0+\delta)$ and let $z = \sup K \cap [x_0, y]$. Since $|z - x_0| < \delta < \delta_1$, by (P_1) we have $\left|\frac{g(z)-g(x_0)}{z-x_0}\right| < \frac{|z-x_0|^{1+1}}{|z-x_0|} = |z-x_0| < \delta < \varepsilon$. If $z = y$, this completes the proof of (5). So, assume that $z < y$. Then, there exists an $(a,b) \in \mathcal{K}$ for which $z = a$ and $y \in (a,b)$. Notice that, by the choice of δ, we have $b - a < \varepsilon/M_1$, see Figure 3. Hence, by (S_1), we have $\left|\frac{g(y)-g(z)}{y-z}\right| < M_1(b-a) < \varepsilon$. Combining this with $\left|\frac{g(z)-g(x_0)}{z-x_0}\right| < \varepsilon$, we obtain $\left|\frac{g(y)-g(x_0)}{y-x_0}\right| \leq \max\left\{\left|\frac{g(y)-g(z)}{y-z}\right|, \left|\frac{g(z)-g(x_0)}{z-x_0}\right|\right\} < \varepsilon$, finishing the proof of the property (5).

Similarly, we prove that there exists a $\delta > 0$ for which $\left|\frac{g(y)-g(x_0)}{y-x_0}\right| < \varepsilon$ provided $x_0 - \delta < y < x_0$. This completes the argument for (I_1).

Next, assume that for some $k \geq 2$ the property (I_{k-1}) holds. We need to show (I_k). So, fix an $x_0 \in K$ and an $\varepsilon > 0$. We will find a $\delta > 0$ for which

$$\left|\frac{g^{(k-1)}(y) - g^{(k-1)}(x_0)}{y - x_0}\right| < \varepsilon \text{ provided } x_0 < y < x_0 + \delta. \tag{6}$$

If x_0 is equal to the left endpoint of some component interval of $\mathbb{R} \setminus K$, then the existence of δ follow from our definition of function g on such intervals. So, assume that this is not the case, that is, that $(x_0, x_0 + \eta) \cap K \neq \emptyset$ for every $\eta > 0$. Let $\delta \in (0, \min\{\varepsilon, \delta_k\})$ be such that $(x_0, x_0 + \delta)$ is disjoint with every $(a_1, b_1) \in \mathcal{K}$ for which $b_1 - a_1 \geq \varepsilon / M_k$. We will show that such δ works.

Fix a $y \in (x_0, x_0 + \delta)$. If $y \in K$, then $\left|\frac{g^{(k-1)}(y) - g^{(k-1)}(x_0)}{y - x_0}\right| = 0 < \varepsilon$ follows from (I_{k-1}). So, we assume that $y \in (a, b)$ for some $(a, b) \in \mathcal{K}$. Then,

$$\begin{aligned}
\left|\frac{g^{(k-1)}(y) - g^{(k-1)}(x_0)}{y - x_0}\right| &= \left|\frac{g^{(k-1)}(y) - g^{(k-1)}(a)}{y - x_0}\right| && (7)\\
&\leq \left|\frac{g^{(k-1)}(y) - g^{(k-1)}(a)}{y - a}\right| \\
&< M_k(b - a) < \varepsilon, && (8)
\end{aligned}$$

where (7) follows from $g^{(k-1)}(x_0) = 0 = g^{(k-1)}(a)$, which is implied by (I_{k-1}), while (8) follows from (S_k), since the choice of $\delta < \delta_k$ implies $b - a < \varepsilon / M_k$. This completes the proof of (6).

Similarly, we prove that there is a $\delta > 0$ for which $\left|\frac{g^{(k-1)}(y) - g^{(k-1)}(x_0)}{y - x_0}\right| < \varepsilon$ provided $x_0 - \delta < y < x_0$. This completes the argument for (I_k) and concludes the proof of the lemma. □

Theorem 2.2. *There exist $\mathcal{C}^\infty$ functions $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ and a perfect set $P \subset \mathbb{R}$ such that $f = \langle f_1, f_2 \rangle$ maps P onto P^2, that is, $f \restriction P$ is a Peano function.*

Proof. The construction will follow the outline indicated at the beginning of the section.

Perhaps the simplest continuous Peano-like function is the following map $h = \langle h^{\text{odd}}, h^{\text{even}} \rangle \colon 2^\omega \to (2^\omega)^2$, whose coordinate functions are the projections defined as $h^{\text{odd}}(s)(i) = s(2i + 1)$ and $h^{\text{even}}(s)(i) = s(2i)$. If we identify 2^ω with the Cantor ternary set $C = \left\{\sum_{i<\omega} \frac{2s(i)}{3^{i+1}} \colon s \in 2^\omega\right\}$, then h becomes a continuous Peano function, from C onto C^2. However, the compression of terms performed by h^{odd} and h^{even} gives us

$$\limsup_{s \to t} \left|\frac{h^{\text{odd}}(s) - h^{\text{odd}}(t)}{s - t}\right| = \infty.$$

Hence, h is not differentiable. In Section 3 we observe that this is a common problem for all compact sets.

To compensate for this compression, we define the sets P_k inductively, creating each P_k by "thickening" P_{k-1} in such a way, that the "condensed" coordinate projections of P_k, via analogs of the maps h^{odd} and h^{even}, may still be mapped onto P_l in a differentiable way as long as $l < k$. Notice that while the "thickening" must be essential enough to obtain the above-mentioned requirement, it cannot be too radical, since the produced sets P_k must be of measure zero. This balancing act will be facilitated by the following functions p_k.

For every $k < \omega$ choose an increasing function $p_k\colon \omega \to [1, \infty)$ such that

$$\lim_{i\to\infty} \frac{p_\ell(i)}{p_k(2i)} = \lim_{i\to\infty} \frac{p_\ell(i)}{p_k(2i+1)} = \infty \text{ for every } \ell < k < \omega. \tag{9}$$

For example, the formula $p_k(i) = (i+1)^{2^{-k}}$ insures (9), as for every $i > 0$ we have

$$\frac{p_\ell(i)}{p_k(2i)} \geq \frac{p_\ell(i)}{p_k(2i+1)} = \frac{(i+1)^{2^{-\ell}}}{(2i+1)^{2^{-k}}} \geq \frac{i^{2^{-\ell}}}{(3i)^{2^{-k}}} = \frac{1}{3^{2^{-k}}} \frac{i^{2^{-\ell}}}{i^{2^{-k}}} = \frac{1}{3^{2^{-k}}} i^{2^{-\ell}-2^{-k}},$$

and $\lim_{i\to\infty} \frac{1}{3^{2^{-k}}} i^{2^{-\ell}-2^{-k}} = \infty$ since $2^{-\ell} - 2^{-k} > 0$.

For $k < \omega$ define $h_k\colon 2^\omega \to [3k, 3k+2]$ as $h_k(s) = 3k + \sum_{n=0}^{\infty} s(n) 3^{-np_k(n)}$. Notice, that h_k is a continuous embedding. Moreover, for every $i < \omega$ we have $\sum_{n=i}^{\infty} 3^{-np_k(n)} \leq \sum_{n=i}^{\infty} 3^{-np_k(i)} \leq 3^{-ip_k(i)} \sum_{n=0}^{\infty} 3^{-n} = \frac{3}{2}\, 3^{-ip_k(i)}$. In particular, for every distinct $s, t \in 2^\omega$, if $i = \min\{n < \omega\colon s(n) \neq t(n)\}$, then

$$\frac{1}{2}\, 3^{-ip_k(i)} \leq |h_k(s) - h_k(t)| \leq \sum_{n=i}^{\infty} 3^{-np_k(n)} \leq \frac{3}{2}\, 3^{-ip_k(i)}, \tag{10}$$

where the first of the inequalities is justified by the following estimation,

$$\begin{aligned} |h_k(s) - h_k(t)| &= \left| \sum_{n=i}^{\infty} (s(n) - t(n)) 3^{-np_k(n)} \right| \geq 3^{-ip_k(i)} - \sum_{n=i+1}^{\infty} 3^{-np_k(n)} \\ &\geq 3^{-ip_k(i)} - \frac{3}{2}\, 3^{-(i+1)p_k(i+1)} \geq 3^{-ip_k(i)} - \frac{3}{2}\, 3^{-(i+1)p_k(i)} \\ &\geq 3^{-ip_k(i)} - \frac{1}{2}\, 3^{-ip_k(i)}. \end{aligned}$$

Let $P_k = h_k[2^\omega]$ and put $P = \bigcup_{k<\omega} P_k$. Clearly P is a perfect subset of $\mathbb{R}$. We will show that it satisfies the theorem.

For every $\ell < k < \omega$ let $h_{k,\ell}^{\text{odd}} = h_\ell \circ h^{\text{odd}} \circ h_k^{-1}$. It is easy to see that $h_{k,\ell}^{\text{odd}}$ is a continuous function from P_k onto P_ℓ. The key fact is that $h_{k,\ell}^{\text{odd}}$ satisfies the assumptions of Lemma 2.1, that is, for every $m<\omega$ there exists a $\delta_m\in(0,1)$ such that

$$|h^{\rm odd}_{k,\ell}(x)-h^{\rm odd}_{k,\ell}(y)|<|x-y|^{m+1} \text{ for all } x,y\in P_k \text{ with } 0<|x-y|<\delta_m. \tag{11}$$

Clearly, for any $\delta_m\in(0,1)$, the condition (11) holds for any distinct $x,y\in P_k$ with $h^{\rm odd}_{k,\ell}(x)=h^{\rm odd}_{k,\ell}(y)$. Therefore, we are interested only in the case when $h^{\rm odd}_{k,\ell}(x)\neq h^{\rm odd}_{k,\ell}(y)$. Now, since $P_k=h_k[2^\omega]$, there exist $s,t\in 2^\omega$ with $x=h_k(s)$ and $y=h_k(t)$ and then $h_\ell(h^{\rm odd}(s))=h^{\rm odd}_{k,\ell}(x)\neq h^{\rm odd}_{k,\ell}(y)=h_\ell(h^{\rm odd}((t))$. Since h_ℓ is injective, this implies that $h^{\rm odd}(s)\neq h^{\rm odd}(t)$. In short, we need to study $s,t\in 2^\omega$ for which $h^{\rm odd}(s)\neq h^{\rm odd}(t)$.

So, fix $s,t\in 2^\omega$ for which $h^{\rm odd}(s)\neq h^{\rm odd}(t)$ and define

$$x=h_k(s) \text{ and } y=h_k(t). \tag{12}$$

Let $i=\min\{n<\omega: h^{\rm odd}(s)(n)\neq h^{\rm odd}(t)(n)\}$. By the formula (10) we have the inequality $|h_\ell(h^{\rm odd}(s))-h_\ell(h^{\rm odd}(t))|\le\frac{3}{2}\ 3^{-ip_\ell(i)}$. Moreover, we have $s(2i+1)=h^{\rm odd}(s)(i)\neq h^{\rm odd}(t)(i)=t(2i+1)$. It follows that the number $i_1=\min\{n<\omega: s(n)\neq t(n)\}$ is $\le 2i+1$ and, again by the formula (10), we have $|x-y|=|h_k(s)-h_k(t)|\ge\frac{1}{2}3^{-i_1p_k(i_1)}\ge 3^{-(2i+1)p_k(2i+1)-1}$. In particular

$$\begin{aligned}|h^{\rm odd}_{k,\ell}(x)-h^{\rm odd}_{k,\ell}(y)| &\le \frac{3}{2}\ 3^{-ip_\ell(i)}\\ &= \frac{3}{2}\ \left(3^{-(2i+1)p_k(2i+1)-1}\right)^{\frac{ip_\ell(i)}{(2i+1)p_k(2i+1)+1}}\\ &\le \frac{3}{2}\ |x-y|^{\frac{ip_\ell(i)}{(2i+1)p_k(2i+1)+1}}.\end{aligned}$$

But, by (9), for every $m<\omega$ there is an $i_m<\omega$ with $\frac{ip_\ell(i)}{(2i+1)p_k(2i+1)+1}\ge m+2$ for all $i\ge i_m$. Moreover, since function h_k^{-1} is uniformly continuous, there is a $\delta_m\in(0,1/2)$ such that $|h_k(s)-h_k(t)|<\delta_m$ implies that $s(j)=t(j)$ for all $j\le 2i_m+1$. Notice that this δ_m insures (11).

Indeed, if $|h^{\rm odd}_{k,\ell}(x)-h^{\rm odd}_{k,\ell}(y)|=0$, then the condition certainly holds. Otherwise, with $s=h_k^{-1}(x)$ and $t=h_k^{-1}(y)$, we have $h^{\rm odd}(s)\neq h^{\rm odd}(t)$ and the choice of δ_m insures that $i=\min\{n<\omega: h^{\rm odd}(s)(n)\neq h^{\rm odd}(t)(n)\}$ is greater than i_m. So,

$$|h^{\rm odd}_{k,\ell}(x)-h^{\rm odd}_{k,\ell}(y)|\le\frac{3}{2}\ |x-y|^{\frac{ip_\ell(i)}{(2i+1)p_k(2i+1)}}\le\frac{3}{2}\ |x-y|^{m+2}<|x-y|^{m+1}$$

completing the proof of (11). In a similar manner, whenever $l<k<\omega$ we define $h^{\rm even}_{k,\ell}=h_\ell\circ h^{\rm even}\circ h_k^{-1}$, and obtain that

$$h^{\rm even}_{k,\ell} \text{ satisfies the assumptions of Lemma 2.1.} \tag{13}$$

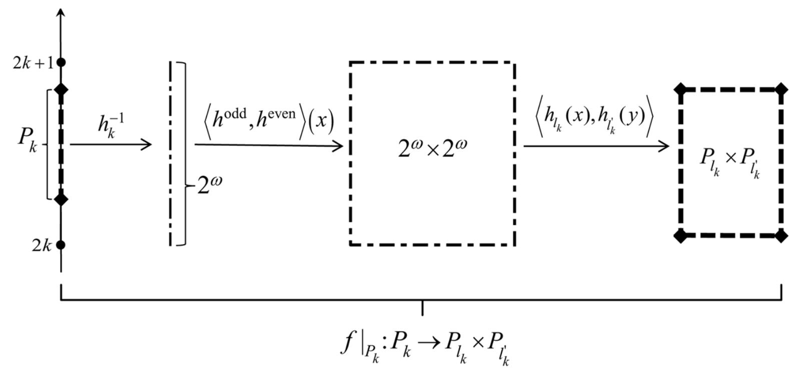

Figure 4: We will define f so that $f \restriction P_k = \langle h_{l_k}, h_{l'_k}\rangle \circ \langle h^{\text{odd}}, h^{\text{even}}\rangle \circ h_k^{-1}$.

Let $\langle\langle \ell_k, \ell'_k\rangle : k = 1,2,3,\ldots\rangle$ be a list of pairs from $\omega \times \omega$ such that for all $k \geq 1$, $\ell_k < k$ and $\ell'_k < k$. For each $k \geq 1$ define $\bar{f}_1$ on P_k as $h^{\text{odd}}_{k,\ell_k}$ and $\bar{f}_2$ on P_k as $h^{\text{even}}_{k,\ell'_k}$. In addition, we define $\bar{f}_1$ and $\bar{f}_2$ on P_0 as constant equal 0. Since sets P_k are separated, (11) and (13) ensure that $\bar{f}_1$ and $\bar{f}_2$ satisfy the assumptions of Lemma 2.1. Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^\infty$ extensions of $\bar{f}_1$ and $\bar{f}_2$, respectively. The proof will be complete as soon as we show that $f = \langle f_1, f_2\rangle$ maps P onto P^2. We have $f \restriction P_k = \langle h_{l_k}, h_{l'_k}\rangle \circ \langle h^{\text{odd}}, h^{\text{even}}\rangle \circ h_k^{-1}$, see Figure 4. Since h_k^{-1} maps P_k onto 2^ω, $\langle h^{\text{odd}}, h^{\text{even}}\rangle$ maps 2^ω onto $2^\omega \times 2^\omega$, $h_{l_k}[2^\omega] = P_{l_k}$, and $h_{l'_k}[2^\omega] = P_{l'_k}$, we have $f[P_k] = P_{\ell_k} \times P_{\ell'_k}$. Therefore, $f[P] = \bigcup_{k<\omega} f[P_k] = \{0\} \cup \bigcup_{k=1}^\infty P_{\ell_k} \times P_{\ell'_k} = P^2$, completing the proof. □

3 There is no $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{R}^2$ with Peano restriction to a compact perfect set

Theorem 3.1. *For any compact perfect $P \subset \mathbb{R}$ and any $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{R}^2$ we have $P^2 \not\subset f[P]$.*

The proof is based on the following two lemmas.

Lemma 3.2. *Let P be a perfect subset of $\mathbb{R}$ and $f = \langle f_1, f_2\rangle$ be a continuous function from P into $\mathbb{R}^2$ such that the coordinate function f_1 is differentiable. If $E = \{x \in P : f'_1(x) \neq 0\}$, then $f[E] \cap P^2$ is meager in P^2.*

Proof. Since the derivative of a coordinate function $f_1 : P \to \mathbb{R}$ is Baire class one (see e.g. [8]), the set E is σ-compact and so is $f[E]$. Also, for every compact $K \subset E$, every level set $(f_1 \restriction K)^{-1}(y) = \{x \in K: f_1(x) = y\}$ of $f_1 \restriction K$ is finite. In particular, each vertical section of $f[K] = \{\langle f_1(x), f_2(x)\rangle: x \in K\}$ is finite, so $f[K] \cap P^2$ is nowhere dense in P^2. □

Lemma 3.3. *Let $g: \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. If P is a compact perfect subset of $\mathbb{R}$ such that $P \subset g[P]$, then there exists an $x \in P$ such that $|g'(x)| \geq 1$.*

PROOF. By way of contradiction, assume that $|g'(x)| < 1$ for every $x \in P$. Since P is compact and g' continuous, there exists an $M < 1$ such that $|g'(x)| < M$ for all $x \in P$. Notice that there exists a $\delta > 0$ such that

$$\left|\frac{g(x)-g(y)}{x-y}\right| < M \text{ for every } x, y \in P \text{ with } 0 < |x - y| \leq \delta. \tag{14}$$

Indeed, otherwise for every $n < \omega$ there exist $x_n, y_n \in P$ for which we have $0 < y_n - x_n \leq 2^{-n}$ and $\left|\frac{g(x_n)-g(y_n)}{x_n-y_n}\right| \geq M$. By the mean value theorem, there exist points $\xi_n \in (x_n, y_n)$ for which $|g'(\xi_n)| \geq M$. Choosing a subsequence, if necessary, we can assume that $\langle x_n\rangle_n$ converges to an $x \in P$. Then also $\langle \xi_n\rangle_n$ converges to x, which contradicts continuity of g', since $\langle |g'(\xi_n)|\rangle_n$ does not converge to $|g'(x)| < M$.

For every $k < \omega$ let $\mathcal{U}_k$ be a collection of the families $\{I_j: j < k\}$ of intervals such that each interval I_j has length $|I_j| \leq \delta$ and $P \subset \bigcup_{j<k} I_j$. Fix a $k < \omega$ for which the $\mathcal{U}_k$ is not empty and let $L = \inf\left\{\sum_{j<k} |I_j|: \{I_j: j < k\} \in \mathcal{U}_k\right\}$. Notice, that $L > 0$, even if P has measure 0. In fact, if P_0 is any subset of P containing $k + 1$ points, then L is greater than or equal to the minimal distance between distinct points in P_0.

Choose $\{I_j: j < k\} \in \mathcal{U}_k$ with $\sum_{j<k} |I_j| < L/M$. For every $j < k$ let J_j be the shortest interval containing $g[P \cap I_j]$. Then, by (14), $|J_j| \leq M|I_j|$. In particular, $\sum_{j<k} |J_j| \leq \sum_{j<k} M|I_j| < L$, so $\bigcup_{j<k} J_j \supset \bigcup_{j<k} g[P \cap I_j] = g[P]$ does not cover P. □

PROOF OF THEOREM 3.1. Let $P \subseteq \mathbb{R}$ be compact and $f = \langle f_1, f_2\rangle: \mathbb{R} \to \mathbb{R}^2$ be of class $\mathcal{C}^1$. By way of contradiction assume that $P^2 \subset f[P]$, and let $P_0 = \{x \in P: f_1'(x) = 0\}$. Then P_0 is closed, since f_1' is continuous. Let $E = P \setminus P_0$. Then, by Lemma 3.2, $f[E]$ is meager in P^2, so $f[P_0] \supset P^2 \setminus f[E]$ is dense in P^2. Therefore, $P^2 \subset f[P_0]$, as $f[P_0]$ is compact.

Next, let E_0 be the set of all isolated points of P_0 and let $P_1 = P_0 \setminus E_0$. Then, P_1 is compact perfect and E_0 is countable. Therefore, as above, we conclude that $P^2 \subset f[P_1] \subset f_1[P_1] \times f_2[P_1]$. Hence, $P_1 \subset P \subset f_1[P_1]$.

Applying Lemma 3.3 to $g = f_1$ and P_1, we conclude that there is an $x \in P_1$ such that $f_1'(x) \geq 1$. But this contradicts the definition of $P_0 \supset P_1$. □

4 Compact sets $P \subset \mathbb{R}$ with $\mathcal{C}^0$ Peano functions $f: P \to P^2$

The goal of this section is to give a full characterization of compact subsets P of $\mathbb{R}$ for which there exists a $\mathcal{C}^0$ Peano function $f: P \to P^2$. This is provided by the following theorem.

Theorem 4.1. *Let $P \subset \mathbb{R}$ be compact and let κ be the number of connected components in P. Then there exists a $\mathcal{C}^0$ Peano function $f: P \to P^2$ if, and only if, either $\kappa = 1$ or $\kappa = \mathfrak{c}$.*

Actually, since the classical Peano curve covers the case when P is connected ($\kappa = 1$) only disconnected sets P are of true interest in this result. For such sets the theorem can be reformulated as follows.

Corollary 4.2. *A disconnected compact set $P \subset \mathbb{R}$ admits a $\mathcal{C}^0$ Peano function $f: P \to P^2$ if, any only if, P has uncountably many components.*

The proof of the theorem will be based on the following two lemmas. To formulate them, we need to recall the following classical definitions. See Kechris [3, pp. 33-34].

For an $X \subseteq \mathbb{R}$ let $(X)'$ be the set of all accumulation points of X. For the ordinal numbers $\alpha, \lambda < \omega_1$, where λ is a limit ordinal, we define

$$X^{(0)} = X,\ X^{(\alpha+1)} = (X^{(\alpha)})',\ \text{and}\ X^{(\lambda)} = \textstyle\bigcap_{\alpha<\lambda} X^{(\alpha)}. \tag{15}$$

For a closed countable set $X \subset \mathbb{R}$, we define its *Cantor-Bendixon rank*, denoted $|X|_{CB}$, to be the least ordinal number $\alpha < \omega_1$ such that $X^{(\alpha)} = \varnothing$.

Lemma 4.3. *If $P \subset \mathbb{R}$ is a countable compact set and a function $f: P \to \mathbb{R}$ is countable, then $|f[P]|_{CB} \le |P|_{CB}$.*

Proof. We will show, by induction on β, that the condition

(I_β) $f[P]^{(\beta)} \subseteq f[P^{(\beta)}]$

holds for every $\beta < \omega_1$. This clearly implies the result.

So, assume that, for some $\beta < \omega_1$, the inclusion $f[P]^{(\alpha)} \subseteq f[P^{(\alpha)}]$ holds for all $\alpha < \beta$. We need to show (I_β). We will consider three cases.

$\beta = 0$: Then $f[P]^{(\beta)} = f[P] = f[P^{(\beta)}]$, so (I_β) holds.

$\beta>0$ is a limit ordinal number: First notice that

($\bullet$) $\bigcap_{\alpha<\beta} f[P^{(\alpha)}] \subseteq f[\bigcap_{\alpha<\beta} P^{(\alpha)}]$.

To see this, fix a point $y \in \bigcap_{\alpha<\beta} f[P^{(\alpha)}]$ and choose an increasing sequence $\langle \alpha_n < \beta : n < \omega\rangle$ cofinal with β, that is, such that $\lim_n \alpha_n = \beta$. Then, for every $n<\omega$, there exists an $x_n \in P^{(\alpha_n)} \subseteq P$ such that $y = f(x_n)$. By compactness of P, choosing a subsequence if necessary, we can assume that $\langle x_n\rangle_n$ converges to some $x \in P$. Since the sequence $\langle P^{(\alpha_n)}\rangle_n$ is decreasing, we have $x \in \bigcap_{n<\omega} P^{(\alpha_n)} = \bigcap_{\alpha<\beta} P^{(\alpha)}$. Therefore, $y = f(x) \in f[\bigcap_{\alpha<\beta} P^{(\alpha)}]$, as required for proving ($\bullet$).

Now, by ($\bullet$),

$$f[P]^{(\beta)} = \bigcap_{\alpha<\beta} f[P]^{(\alpha)} \subseteq \bigcap_{\alpha<\beta} f[P^{(\alpha)}] \subseteq f[\bigcap_{\alpha<\beta} P^{(\alpha)}] = f[P^{(\beta)}],$$

where the first inclusion is justified by (I_α). So, once again, (I_β) holds.

β is a successor ordinal: Suppose $\beta = \alpha+1$ and fix a $y \in f[P]^{(\beta)} = (f[P]^{(\alpha)})'$. Then, there exists a one-to-one sequence $\langle y_n \in f[P]^{(\alpha)} : n<\omega\rangle$ converging to y. By the inductive assumption $y_n \in f[P]^{(\alpha)} \subseteq f[P^{(\alpha)}]$, so, for every $n<\omega$, there exists an $x_n \in P^{(\alpha)}$ with $y_n = f(x_n)$. Since the sequence $\langle y_n : n<\omega\rangle$ is one-to-one, so is $\langle x_n \in P^{(\alpha)} : n<\omega\rangle$. By compactness of $P^{(\alpha)}$, choosing a subsequence if necessary, we can assume that $\langle x_n\rangle_n$ converges to some $x \in P^{(\alpha)}$. Since $\langle x_n\rangle_n$ is one-to-one, $x \in (P^{(\alpha)})' = P^{(\beta)}$. Finally, $f(x) = f(\lim_n x_n) = \lim_n f(x_n) = \lim_n y_n = y$, so $y = f(x) \in f[P^{(\beta)}]$, as needed for the proof of (I_β). □

Lemma 4.4. *Let P be a countable compact subset of $\mathbb{R}$. If P is infinite, then $|P|_{CB} < |P\times P|_{CB}$.*

Proof. Let $|P|_{CB} = \beta$. The compactness of P implies that β is a successor ordinal, say $\beta = \alpha+1$. We need to show that $((P\times P)^{(\alpha)})' = (P\times P)^{(\alpha+1)} \neq \varnothing$.

Notice, that $X'\times Y \subseteq (X\times Y)'$ for every $X, Y \subset \mathbb{R}$. From this, an obvious inductive argument shows that $X^{(\alpha)}\times Y \subseteq (X\times Y)^{(\alpha)}$. In particular, we have $P^{(\alpha)}\times P \subseteq (P\times P)^{(\alpha)}$. Thus, it is enough to show that $(P^{(\alpha)}\times P)' \neq \varnothing$. But this is obvious, since $P^{(\alpha)} \neq \varnothing$ and P is infinite. □

Proof of Theorem 4.1. The argument naturally leads to the following four cases.

$\kappa = 1$: In this case the classical Peano curve works.

$\kappa > 1$ **is finite:** Let $f: P \to \mathbb{R}^2$ be continuous. Then $f[P]$ can have at most κ-many components. Since P^2 has κ^2 components and $\kappa^2 > \kappa$, $f[P]$ cannot be equal P^2.

κ **is countable infinite:** This means that $\kappa = \omega$. We need to show that there is no $\mathcal{C}^0$ Peano function $f: P \to P^2$.

First we note that this is true when P is totally disconnected (i.e., it has only one-point components):

($*$) if an infinite compact totally disconnected set P has countably many components, then there is no continuous function from P onto $P^2 = P \times P$.

Indeed, if $f: P \to \mathbb{R}^2$ is continuous then, by Lemma 4.3, $|f[P]|_{CB} \leq |P|_{CB}$. So, $f[P]$ cannot be equal P^2 since, by Lemma 4.4, $|P|_{CB} < |P^2|_{CB}$. The general case will be reduced to ($*$).

By way of contradiction, suppose that there exists a continuous function $f = \langle f_1, f_2 \rangle$ from P onto P^2. Let $\sim$ be an equivalence relation defined as: $x \sim y$ if, and only if, x and y belong to the same component of P. The equivalence class of $x \in P$ with respect to $\sim$ will be denoted $[x]$. Let $P/{\sim} = \{[x]: x \in P\}$ be the quotient space, that is, $U \subseteq P/{\sim}$ is declared open if, and only if, the set $\hat{U} = \bigcup\{[x]: [x] \in U\}$ is open in P. Notice that $P/{\sim}$ is homeomorphic to a subset of $\mathbb{R}$, since

$$P/{\sim} \text{ is compact, Hausdorff, totally disconnected.}$$

Indeed, if $\{U_j: j \in J\}$ is an open cover of $P/{\sim}$, then $\{\hat{U}_j: j \in J\}$ is an open cover of P. So, there is a finite $J_0 \subseteq J$ such that $\{\hat{U}_j: j \in J_0\}$ covers P. Therefore, $\{U_j: j \in J_0\}$ is a cover of $P/{\sim}$, implying compactness of $P/{\sim}$. To see the other two properties, take $x, y \in P$ with $[x] \neq [y]$. We can assume that $x < y$. Then, there exists an $r \in \mathbb{R} \setminus P$ such that $[x] \subset (-\infty, r)$ and $[y] \subset (r, \infty)$. In particular, if $U = P \cap (r, \infty)$, then $\hat{U}$ is a clopen subset of $P/{\sim}$ containing $[x]$ but not $[y]$. It is worth noting that our space $P/{\sim}$ falls into a broader class of quotient spaces which are metrizable, see e.g. [1, theorem 4.2.13.].

Let $i \in \{1, 2\}$. Since f_i is a continuous function from P into itself, we have $f_i([x]) = [f_i(x)]$ for every $x \in P$. In particular, the function $g_i: (P/{\sim}) \to (P/{\sim})$ given by $g_i([x]) = [f_i(x)]$ is well defined and it is continuous, since for every U open in $P/{\sim}$, the set $W = g_i^{-1}(U)$ is open in $P/{\sim}$, as $\hat{W} = f_i^{-1}(\hat{U})$.

The above shows that function $g = \langle g_1, g_2 \rangle: (P/{\sim}) \to (P/{\sim})^2$ is well defined and continuous. Moreover, it is onto $(P/{\sim})^2$, since $f[P] = P^2$. The space $P/{\sim}$ is countable so this contradicts ($*$), completing the proof of this case.

κ **is uncountable:** In this case $\kappa = \mathfrak{c}$. Recall, that 2^ω can be mapped onto any compact metric space, see e.g. [3, theorem 4.18]. In particular, there exists a continuous function 2^ω onto P^2.

Also, there exists a continuous function g from P onto 2^ω. Indeed, we can define a Cantor-like tree $\{P_s : s \in 2^{<\omega}\}$ of compact subsets of P such that $P_\varnothing = P$ and every P_s is split into two clopen subsets, P_{s0} and P_{s1}, each containing uncountably many components of P. For $t \in 2^\omega$ put $g(x) = t$ if, any only if, $x \in \bigcap_{n<\omega} P_{t\upharpoonright n}$. Then g is as required.

Finally notice that $f = h \circ g$ is continuous and maps P onto P^2. □

5 Final remarks and open problems

Although we proved that for a compact perfect $P \subset \mathbb{R}$ there is no Peano function f from P onto P^2 which can be extended to a $\mathcal{C}^1$ function $\hat{f}: \mathbb{R} \to \mathbb{R}^2$, the argument used in the proof of Theorem 3.1 does not work without the extendability assumption of f. Of course, by Proposition 1.1(b), the extendability would play no role if we could prove a version of Theorem 3.1 with the class $\mathcal{C}^1$ replaced by $\mathcal{D}^1$. But, once again, our argument does not seem to generalize to this case.

In light of this discussion, the following question seems to be of interest.

Problem 1. Does there exist a compact perfect set $P \subset \mathbb{R}$ and a $\mathcal{D}^1$ function f from P onto P^2? If so, can such an f be $\mathcal{C}^1$? (See Remark 1.2.)

Also, Theorem 4.1 gives a full characterization of compact sets P admitting $\mathcal{C}^0$ Peano functions. It would be interesting to find analogous characterization that includes also the unbounded closed sets. However, if there exists such a characterization (in terms of a structure of connected components), it seems it would be quite complicated in nature.

Finally, in the example given in Theorem 2.2, the $\mathcal{C}^\infty$ Peano function f from P onto P^2 is extendable to a $\mathcal{C}^\infty$ function $\hat{f}: \mathbb{R} \to \mathbb{R}^2$. Is this always the case? More precisely it seems to us that the following question should have a negative answer.

Problem 2. Let $P \subset \mathbb{R}$ be a perfect subset of $\mathbb{R}$ for which there is a $\mathcal{C}^\infty$ function from P onto P^2. Does this imply that there exists a $\mathcal{C}^\infty$ function $f: \mathbb{R} \to \mathbb{R}^2$ such that $f[P] = P^2$?

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