Algebra Universalis, **25** (1988) 1–6 0002–5240/88/010001–06$01.50 + 0.20/0

A short ordered commutative domain whose quotient field is not short

KRZYSZTOF CIESIELSKI

Claus Borregaard in his Ph.D. thesis posed the following question (see [3]; compare also [1; problem 47]): "Does there exist a short ordered commutative domain whose quotient field is not short?" where an ordered set is said to be short if it doesn't contain a monotonic sequence of length ω_1. The purpose of this paper is to give an affirmative answer to this question.

Let us recall that a commutative ring $D = \langle D, +, \cdot, 0, 1\rangle$ with a linear ordering $\leq$ is said to be an ordered commutative domain provided for any $a, b, c \in D$.

(P1) $a < b$ implies $a + c < b + c$
(P2) $a < b$ and $0 < c$ imply $a \cdot c < b \cdot c$.

DEFINITION. Let X and Y_x for $x \in X$ be linearly ordered sets such that $0 \in Y_x$ for any $x \in X$. We define a set

$$P(X, \{Y_x\}_{x\in X}) = \left\{f \in \prod_{x\in X} Y_x : \ \{x : f(x) \neq 0\} \text{ is finite}\right\}$$

and an ordering on it by

$$f < g \quad \text{if and only if} \quad f(m) < g(m), \quad \text{where} \quad m = \max\{x : f(x) \neq g(x)\}.$$

We write $P(X, Y)$ instead of $P(X, \{Y_x\}_{x\in X})$ provided $Y_x = Y$ for all $x \in X$.

We will use the following several times

LEMMA 1. *Let X and Y be linearly ordered sets.*

(1) *If $0, 1 \in Y$ and $0 < 1$ then there exists an order-preserving embedding $e : X \hookrightarrow P(X, Y)$.*

Presented by Jan Mycielski. Received June 5, 1987. Accepted for publication in final form June 16, 1987.

(2) *If* $\langle Y, +, 0, \leq \rangle$ *is an ordered commutative semigroup (i.e. the semigroup operation satisfies a condition* (P1)) *with a neutral element* 0 *then so is* $\langle P(X, Y), +, 0_p, \leq \rangle$ *where* $(f+g)(x) = f(x) + g(x)$ *and* $0_p(x) = 0$ *for all* $x \in X$.

(3) *If* $\langle Y, +, 0, \leq \rangle$ *is an ordered commutative group then so is* $\langle P(X, Y), +, 0_p, \leq \rangle$.

(4) *If* $\langle X, +, 0, \leq \rangle$ *is an ordered commutative semigroup with a neutral element* 0 *and* $\langle Y, +, \cdot, 0, 1, \leq \rangle$ *is an ordered commutative domain than* $\langle P(X, Y), +, \cdot, 0_p, 1_p, \leq \rangle$ *is also an ordered commutative domain where*

$$1_p(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

and a product is defined by "polynomial-like" methods

$$(k \cdot l)(x) = \sum \{k(z_1) \cdot l(z_2) : z_1, z_2 \in X, \ z_1 + z_2 = x\}.$$

Proof. (1) It is easy to see that the function

$$e(x)(z) = \begin{cases} 0 & \text{for } x \neq z \\ 1 & \text{for } x = z \end{cases}$$

is one-to-one and preserves order.

The natural and easy proof of (2) and (3) can be found e.g. in [2; Section 1.5] or [5; Section 2].

The proof of (4) is also natural and can be found in [2; Section 8, Example 8.1.10] (see also [4]).

If N is the semigroup of natural numbers then, by Lemma 1 (2), $P(T, N)$ is an ordered commutative semigroup for any linearly ordered set T. In particular, if T is a well ordered set with order type $2^\omega + 1$ ($2^\omega + 1$ being the ordinal successor of 2^ω) and $R = P(P(T, N), Z)$ then, by Lemma 1 (4), R is an ordered commutative domain, where Z is the group of integers. Moreover, by Lemma 1 (1), $T \hookrightarrow P(T, N) \hookrightarrow P(P(T, N), Z)$ and, by construction, R has the cardinality of 2^ω. So we proved

LEMMA 2. *There exists an ordered commutative domain* R *of the power of* 2^ω *which contains an increasing sequence of length* $2^\omega + 1$.

Now we are ready to construct our example. Put

$$\bar{D} = P(P(\mathbb{R}, N), R)$$

where $(\mathbb{R}, \leq)$ is the set of reals. By Lemma 1 (2), (4) and Lemma 2, $\bar{D}$ is an ordered commutative domain but clearly $\bar{D}$ is not short.

Let i be a bijection between $\mathbb{R}$ and R and for a finite set $A \subset \mathbb{R}$ let R_A be the subring of R generated by $\{i(a): a \in A\}$. Note that

(*) R_a is countable for any finite $A \subset \mathbb{R}$.

Put $G = P(\mathbb{R}, N)$ and

$$D = P(G, \{R_{\mathrm{supp}(g)}\}_{g \in G}) \subset \bar{D}$$

where $\mathrm{supp}\,(g) = \{r : g(r) \neq 0\}$.

We will see that D is a short commutative domain whose quotient field is not short.

LEMMA 3. *D is a commutative domain.*

Proof. Clearly $D \subset \bar{D}$ and $0_{\bar{D}}, 1_{\bar{D}} \in D$. Moreover $-k, k + l \in D$ for any $k, l \in D$, because R_A is an additive group for every finite $A \subset \mathbb{R}$.

So it is enough to prove that $k \cdot l \in D$ provided $k, l \in D$. But let us note that for all $f, g \in G = P(\mathbb{R}, N)$ $\mathrm{supp}\,(g) \cup \mathrm{supp}\,(f) = \mathrm{supp}\,(f + g)$. So for any $h \in G$

$$(k \cdot l)(h) = \sum \{k(f) \cdot l(g) \colon f, g \in G, f + g = h\} \in R_{\mathrm{supp}(h)}$$

because for $f, g \in G$, $f + g = h$ we have $k(f) \in R_{\mathrm{supp}(f)}$, $l(g) \in R_{\mathrm{supp}(g)}$, i.e.

$$k(f) \cdot l(g) \in R_{\mathrm{supp}(f) \cup \mathrm{supp}(g)} = R_{\mathrm{supp}(f+g)} = R_{\mathrm{supp}(h)}.$$

Hence $k \cdot l \in D$ for any $k, l \in D$.

LEMMA 4. *The ordered domain R is embeddable into the quotient field F of D as an ordered structure. In particular F is not short.*

Proof. Let $e : \mathbb{R} \hookrightarrow P(\mathbb{R}, N) = G$ and $e_1 : G \hookrightarrow P(G, R_\varnothing) \subset D$ be embeddings from the proof of Lemma 1 (1). So, for $s \in \mathbb{R}$, $\mathrm{supp}\,(e(s)) = \{s\}$.

For $r \in R$ let $h = e(i^{-1}(r))$, $d(r) = e_1(h)$ and $n(r) = r \cdot d(r)$ (i.e. $n(r)(g) = r \cdot d(r)(g)$ for any $g \in G$). Then, by definition, $d(r) \in D$. Moreover $n(r) \in D$ because $n(r)(g) = 0 \in R_\varnothing$ for $g \neq h$ and $n(r)(h) = r \in R_{\{r\}} = R_{\mathrm{supp}(h)}$.

Therefore we can associate with each $r \in R$ a quotient $n(r)/d(r) = r \cdot d(r)/d(r) \in F$. Clearly, the described function is an order-preserving embedding of the ring R into the field F.

Now to finish the proof it is enough to show that D is short. We use for this the following

LEMMA 5. *Let X be a short linearly ordered set. If Z_x is short linearly ordered set for every $x \in X$ then so is $P(X, \{Z_x\}_{x\in X})$. In particular if the sets Z_x are countable then $P(X, \{Z_x\}_{x\in X})$ is short.*

We will use the lemma only in the case of countable Z_x so we will prove it only in this case. The general case can be found, for example, in [6, p. 167].

Proof. Let us assume by contradiction that there exists a monotonic sequence $\{g_\zeta : \zeta < \omega_1\} \subset P(X, \{Z_x\}_{x\in X})$.

Without lost of generality we can assume that for some natural number $n > 0$ the power $|\{x : g_\zeta(x) \neq 0\}| = n$ for all $\zeta < \omega_1$.

So let $n > 0$ be the least natural number with the property that there exists a monotonic sequence $\{g_\zeta : \zeta < \omega_1\} \subset P(X, \{Z_x\}_{x\in X})$ such that $|\{x : g_\zeta(x) \neq 0\}| = n$.

Let for $\zeta < \omega_1$

$$x_\zeta = \max\{x : g_\zeta(x) \neq 0\}.$$

If a set $\{x_\zeta : \zeta < \omega_1\}$ is countable then we can assume, choosing a subsequence if necessary, that there exists $x \in X$ such that $x_\zeta = x$ for any $\zeta < \omega_1$. But Z_x is countable. So again we can assume that for some $z \in Z_x$, $g_\zeta(x) = z$ for all $\zeta < \omega_1$. Now putting

$$g'_\zeta(y) = \begin{cases} g_\zeta(y) & \text{for} \quad y \neq x \\ 0 & \text{for} \quad y = x \end{cases}$$

we conclude $\{g'_\zeta : \zeta < \omega_1\}$ is monotonic and $|\{x : g'_\zeta(x) \neq 0\}| = n - 1 < n$ for all $\zeta < \omega_1$ which contradicts minimality of n.

Therefore the set $\{x_\zeta : \zeta < \omega_1\}$ is uncountable. So we can assume that

$$x_\zeta \neq x_\eta \quad \text{for any} \quad \zeta < \eta < \omega_1$$

and either $g_\zeta(x_\zeta) > 0$ for every $\zeta < \omega_1$ or $g_\zeta(x_\zeta) < 0$ for every $\zeta < \omega_1$.

Let us now assume that our sequence $\{g_\zeta : \zeta < \omega_1\}$ is increasing and $g_\zeta(x_\zeta) > 0$ for every $\zeta < \omega_1$. For the other three cases the proof is similar.

Then, for $\zeta < \eta < \omega_1$, $g_\zeta < g_\eta$ i.e. $g_\zeta(m) < g_\eta(m)$ where $m = \max\{x : g_\zeta(x) \neq g_\eta(x)\} \in \{x_\zeta, x_\eta\}$. But if $m = x$ then $g_\zeta(x_\zeta) = g_\zeta(m) < g_\eta(m) = g_\eta(x_\zeta) = 0$ which contradicts our assumption $g_\zeta(x_\zeta) > 0$. Hence $m = x_\eta$ i.e. $x_\zeta < x_\eta$. Therefore $x_\zeta < x_\eta$ for any $\zeta < \eta < \omega_1$, so X is not short.

Now we are ready to prove

THEOREM. *There exists a short ordered commutative domain D whose quotient field F contains an increasing sequence of length* $2^{\omega}+1$. *In particular F is not short.*

Proof. Let $D = P(G, \{R_{\text{supp}(g)}\}_{g\in G})$ where $G = P(\mathbb{R}, N)$. By Lemma 3 D is an ordered commutative domain and by Lemma 2 and 4 its quotient field F contains an increasing sequence of length $2^{\omega}+1$.

Moreover, by Lemma 5, G is short and so, again by Lemma 5 and condition $(*)$, D is also short.

In fact it is easy to prove by transfinite induction that an ordered field containing an increasing sequence of length $2^{\omega}+1$ contains an increasing sequence of length α for any ordinal $\alpha < (2^{\omega})^{+}$. So we obtained the best possible result in this direction because of

PROPOSITION. *If F is a quotient field of an ordered commutative domain* $(D, \leq)$ *and the cardinality* $|F| > 2^{\omega}$ *then D is not short.*

Proof. $|F| > 2^{\omega}$ implies $|D| \geq (2^{\omega})^{+}$. Let $<$ be some well ordering of D and let for $a, b \in D$, $a \leq b$

$$f(\{a, b\}) = \begin{cases} 1 & \text{for} \quad a < b \\ 0 & \text{otherwise.} \end{cases}$$

Then by the Erdös-Rado partition relation theorem $(2^{\omega})^{+} \to (\omega_1)^2_2$ (see e.g. [8; Thm. 6.4, p. 392]) there exist an uncountable $U \subset D$ and $i \in \{0, 1\}$ such that $f\{a, b\} = i$ for any $a, b \in U$, $a \neq b$. So U is a monotonic sequence of length at least ω_1. (The above proof can be also found in [7]. Compare also [9].)

REFERENCES

[1] ——, *The Black Swamp Problem Book*, an ongoing book of problems in ordered structures kept by Charles Holland at Bowling Green State University, Bowling Green, Ohio.

[2] A. Bigard, K. Keimel, S. Wolfenstein, *Groupes et Anneaux Réticulés*, Lecture Notes in Math. 608, Springer-Verlag 1977 (French).

[3] C. Borregaard, Cut completion and homogeneity properties of totally ordered sets, Ph.D. Thesis, University of Copenhagen, 1977, hand-written (Danish).

[4] P. Conrad, J. Dauns, *An embedding theorem for lattice-ordered fields*, Pacific J. of Math. 30, No. 2, 1969, 385–398.

[5] P. Conrad, J. Harvey, Ch. Holland, *The Hahn embedding theorem for abelian lattice-ordered groups*, Trans. of AMS 108, No. 1, 1963, pp. 143–169.

[6] F. HAUSDORFF, *Grundzüge der Mengenlehre*, Leipzig, Verlag von Veit & Comp., 1914 (German).
[7] I. JUHÁSZ, *Cardinal function in topology – ten years later*, Math. Centrum, Amsterdam, 1980.
[8] K. KUNEN, *Combinatorics*, Handbook of Mathematical Logic, North Holland, 1977.
[9] P. URYSON, *Remarque sur ma note "un théorème sur la puissance des ensembles ordonnés"*, Fund. Math. *6* (1924), p. 278 (French).

Warsaw University
Warsaw
Poland

Department of Mathematics
University of Louisville
Louisville, Kentucky 40292
U.S.A.