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FUNCTIONS CONTINUOUS ON TWICE DIFFERENTIABLE CURVES, DISCONTINUOUS ON LARGE SETS

Abstract

We provide a simple construction of a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ discontinuous on a perfect set P, while having continuous restrictions $F \restriction C$ for all twice differentiable curves C. In particular, F is separately continuous and linearly continuous.

While it has been known that the projection $\pi[P]$ of any such set P onto a straight line must be meager, our construction allows $\pi[P]$ to have arbitrarily large measure. In particular, P can have arbitrarily large 1-Hausdorff measure, which is the best possible result in this direction, since any such P has Hausdorff dimension at most 1.

1 Introduction

In this paper, a *curve* is understood as the range of a continuous injection $h = \langle h_1, h_2 \rangle$ of an interval J into the plane $\mathbb{R}^2$. A curve C is said to be *smooth* (or $\mathcal{C}^1$), if the coordinate functions h_1 and h_2 are continuously differentiable (i.e., are $\mathcal{C}^1$) and $\langle h_1'(t), h_2'(t) \rangle \neq \langle 0, 0 \rangle$ for every $t \in J$; we say that C is *twice differentiable* (or D^2), when it is smooth (so, its derivative nowhere vanishes) and the coordinate functions are twice differentiable. It has been proved by Rosenthal [17] that

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(∗) For any function $G\colon \mathbb{R}^2 \to \mathbb{R}$, if its restriction $G \restriction C$ is continuous for every smooth curve C, then G is continuous. However, there exists a discontinuous function $F\colon \mathbb{R}^2 \to \mathbb{R}$ with $F \restriction C$ continuous for all twice differentiable curves C.[1]

The function F constructed by Rosenthal was discontinuous at a single point. The function constructed in our Theorem 4 seems to be the first example of a function with continuous restrictions to all twice differentiable curves, which has uncountable set of points of discontinuity.

For a family $\mathfrak{C}$ of curves C in the plane $\mathbb{R}^2$, we say that $F\colon \mathbb{R}^2 \to \mathbb{R}$ is $\mathfrak{C}$-*continuous*, provided its restriction $F \restriction C$ is continuous for every $C \in \mathfrak{C}$. The $\mathfrak{C}$-continuous functions for different classes $\mathfrak{C}$ of curves have been studied from the dawn of mathematical analysis. For the class $\mathcal{L}_0$ of straight lines parallel to either of the axis, the $\mathcal{L}_0$-continuity coincides with *separate continuity* (referring to maps F with section functions $F(\cdot, y)$ and $F(x, \cdot)$ continuous for every $x, y \in \mathbb{R}$). Separately continuous functions have been investigated by many prominent mathematicians: Volterra (see Baire [2, p. 95]), Baire (1899, see [2]), Lebesgue (1905, see [13, pp. 201-202]), and Hahn (1919, see [9]). For the class $\mathcal{L}$ of all straight lines, $\mathcal{L}$-continuity is known under the name *linear continuity*. It has been known by J. Thomae (1870, see [20, p. 15] or [11]) that linearly continuous function need not be continuous. A simple example of such a function, which can be traced to a 1884 treatise on calculus by Genocchi and Peano [10], is defined as $F(x, y) = \frac{xy^2}{x^2+y^4}$ for $\langle x, y\rangle \neq \langle 0, 0\rangle$, and $F(0, 0) = 0$. Scheeffer (1890, see [18]) and Lebesgue (1905, see [13, pp. 199-200]) have also noticed that the continuity along all analytic curves does not implies continuity. The question *for what classes* $\mathfrak{C}$ *of curves does* $\mathfrak{C}$*-continuity imply continuity*, apparently addressed in all works cited above, has been elegantly answered in 1955 by Rosenthal, as we stated in (∗).

A next natural question, in this line of research, is about the structure of *the sets* $D(F)$ *of points of discontinuity* of $\mathfrak{C}$-continuous functions F for different classes $\mathfrak{C}$ of curves. Of course, every set $D(F)$ must be F_σ. This follows from a well known result (see [14, thm. 7.1]) that, for arbitrary $F\colon \mathbb{R}^2 \to \mathbb{R}$, $D(F)$ is a union of the closed sets $D_n(F) = \{z \in \mathbb{R}^2\colon \omega_F(z) \geq 2^{-n}\}$, where $\omega_F(z) = \lim_{\delta\to 0^+} \sup\{|F(z) - F(w)|\colon ||z - w|| < \delta\}$ is the *oscillation* of F at z.

The structure of sets $D(F)$ for separately continuous functions (i.e., for $\mathfrak{C} = \mathcal{L}_0$) was examined by Young and Young (1910, see [21]) and was fully

[1] Clearly, for any such F, the composition $F \circ h$ is continuous, whenever $h = \langle h_1, h_2\rangle$ is a coordinate system for a D^2 curve. In fact, a little care in constructing such an F (e.g. by using $\mathcal{C}^\infty$ functions h_n in Proposition 1) insures that $F \circ h$ is also D^2. However, it is important here, that the derivative h' never vanishes, as it has been proved by Boman [3] (see also [11]), that if $F \circ \langle h_1, h_2\rangle$ is $\mathcal{C}^1$ for any $\mathcal{C}^\infty$ functions h_1, h_2, then F is continuous.

described in 1943 by Kershner [12] (compare also [4]), who showed that a set $D \subset \mathbb{R}^2$ is equal to $D(F)$ for a separately continuous $F\colon \mathbb{R}^2 \to \mathbb{R}$ if and only if D is F_σ and the projection of D onto each axis is meager. More precisely, the characterization follows from the fact that a bounded set $D \subset \mathbb{R}^2$ is equal to the set $D_n(F) = \{z \in \mathbb{R}^2 \colon \omega_F(z) \geq 2^{-n}\}$ for a separately continuous $F\colon \mathbb{R}^2 \to \mathbb{R}$ if and only if D is closed and its projection onto each axis is nowhere dense. Notice, that this characterization implies, in particular, that a set of points of discontinuity a separately continuous $F\colon \mathbb{R}^2 \to \mathbb{R}$ can have full planar measure.

The structure of sets $D(F)$ for linearly continuous functions $F\colon \mathbb{R}^2 \to \mathbb{R}$ is considerable more restrictive, as can be seen by the following result of Slobodnik [19]. More on separate continuity can be found in [7, 15, 16].

Proposition 1. *If D is the set of points of discontinuity of a linearly continuous function $F\colon \mathbb{R}^2 \to \mathbb{R}$, then*

- ($\bullet$) *D is a union of sets D_n, $n = 1, 2, 3, \ldots$, where each D_n is a rotation of a graph $h_n \restriction P_n$ of a Lipschitz function $h_n\colon \mathbb{R} \to \mathbb{R}$ restricted to a compact nowhere dense set P_n.*

Since the graph of a Lipschitz function has Hausdorff dimension 1 (see e.g. [8, sec. 3.2]), this means that so does any set of points of discontinuity of a linearly continuous function. We have recently shown [5] that the condition ($\bullet$) is actually quite close to the full characterization of sets $D(F)$ for linearly continuous functions F, by proving that: if D is as in ($\bullet$), where each function h_n is either convex or $\mathcal{C}^2$, then D is equal to the set of points of discontinuity of some linearly continuous function. This new result implies, in particular, that any meager F_σ subset of a line is the set of points of discontinuity of some linearly continuous function; so such a set may have positive 1-Hausdorff measure.

The main goal of this paper is to show that a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ with continuous restrictions to all twice differentiable curves can also have a set of points of discontinuity with large 1-Hausdorff measure.

Notice, that any smooth curve C, with associated injection $h = \langle h_1, h_2 \rangle$, is locally (at a neighborhood of an arbitrary point $\langle h_1(t), h_2(t) \rangle$) a function of either variable x (when $h_1'(t) \neq 0$) or of variable y (when $h_2'(t) \neq 0$).

Thus, $\mathfrak{C}(\mathcal{C}^1)$-continuity with respect to the class $\mathfrak{C}(\mathcal{C}^1)$ of all smooth curves is the same as the $\mathcal{C}^1 \cup (\mathcal{C}^1)^{-1}$-continuity, where $\mathcal{C}^1$ is the class of all continuously differentiable functions $g\colon \mathbb{R} \to \mathbb{R}$, and $(\mathcal{C}^1)^{-1} = \{g^{-1} \colon g \in \mathcal{C}^1\}$, with g^{-1} understood as an inverse relation, that is, as $g^{-1} = \{\langle g(y), y \rangle \colon y \in \mathbb{R}\}$. Similarly, $\mathfrak{C}(D^2)$-continuity, where $\mathfrak{C}(D^2)$ is the class of all (smooth) twice differentiable curves, coincides with $D^2 \cup (D^2)^{-1}$-continuity.

2 The main result

Our example will be constructed using the following simple, but general result on $\mathfrak{C}$-continuous functions. Recall that the *support* of a function $F\colon \mathbb{R}^2 \to \mathbb{R}$, denoted as $\operatorname{supp}(F)$, is defined as the closure of the set $\{x \in \mathbb{R}^2 \colon f(x) \neq 0\}$. Symbol ω will be used here to denote the first infinite ordinal number, which is identified with the set of all natural numbers, $\omega = \{0, 1, 2, \ldots\}$.

Lemma 2. *Let $\mathfrak{C}$ be a family of curves in $\mathbb{R}^2$ and let $\{D_j \subset \mathbb{R}^2 \colon j < \omega\}$ be a pointwise finite family of open sets such that*

(F) the set $\{j < \omega \colon D_j \cap C \neq \emptyset\}$ is finite for every $C \in \mathfrak{C}$.

Then for every sequence $\langle F_j \colon j < \omega \rangle$ of continuous functions from $\mathbb{R}^2$ into $\mathbb{R}$ such that $\operatorname{supp}(F_i) \subset D_i$ for all $i < \omega$, the function $F \stackrel{\text{def}}{=} \sum_{j<\omega} F_j$ is $\mathfrak{C}$-continuous. Moreover, if

- *the diameters of the sets D_j go to 0, as $j \to \infty$,*
- *$\hat{P}$ is the set of all $z \in \mathbb{R}^2$ for which every open $U \ni z$ intersects infinitely many sets D_j, and*
- *each function F_j is onto $[0, 1]$,*

then $\hat{P} = D(F) = \{z \in \mathbb{R}^2 \colon \omega_F(z) = 1\}$.

Proof. The first part is obvious. The second follows easily from the fact, that, for any $z \in \hat{P}$, every open $U \ni z$ contains infinitely many sets D_j. □

Lemma 2 will be used with $\hat{P} = h \restriction P$, the graph of h restricted to P, where h and P are from the proposition below.

Proposition 3. *For every $M \in [0, 1)$ there exists a $\mathcal{C}^1$ function $h\colon \mathbb{R} \to \mathbb{R}$ and a nowhere dense perfect $P \subset (0, 1)$ of measure M such that for every $\hat{x} \in P$:*

$$h'(\hat{x}) = 0 \text{ and } \lim_{x\to\hat{x}} \frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} = \infty. \tag{1}$$

We will postpone the proof of Proposition 3 till the next section. However, we like to notice here, that the limit $\lim_{x\to\hat{x}} \frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2}$ is a variant of the limit $\lim_{x\to\hat{x}} 2\frac{h(x)-h(\hat{x})}{(x-\hat{x})^2}$, which constitutes a generalized second derivative (related to Peano derivative) of h at $\hat{x}$. Indeed, if $h''(\hat{x})$ exists, finite or infinite, then, by l'Hôpital's Rule, $\lim_{x\to\hat{x}} 2\frac{h(x)-h(\hat{x})}{(x-\hat{x})^2} = \lim_{x\to\hat{x}} 2\frac{h'(x)-0}{2(x-\hat{x})} = \lim_{x\to\hat{x}} \frac{h'(x)-h'(\hat{x})}{x-\hat{x}} = h''(\hat{x})$. We need Proposition 3 in its current form, since there is no $\mathcal{C}^1$ function

h having an infinite second derivative on set of positive measure.[2] But see also remarks at the end of this section.

Theorem 4. *Let h and P be as in Proposition 3. Then $\hat{P} = h \restriction P$ is the set of points of discontinuity of a D^2-continuous function $F\colon \mathbb{R}^2 \to \mathbb{R}$. Moreover, F has oscillation equal 1 at every point from $\hat{P}$.*

Proof. Let $\{J_j\colon j < \omega\}$ be an enumeration, without repetitions, of bounded connected components of $\mathbb{R} \setminus P$. For every $j < \omega$ let the I_j be the open middle third subinterval of J_j and let F_j be a continuous function from $\mathbb{R}^2$ *onto* $[0,1]$ with supp(F_j) contained in $D_j = \{\langle x, y\rangle \in \mathbb{R}^2\colon x \in I_j \ \&\ |y - h(x)| < |I_j|^3\}$, where $|I_j|$ is the length of I_j. We will show that the function $F = \sum_{j<\omega} F_j$ is as required.

It is enough to show that sets D_j satisfy property (F) for $\mathfrak{C} = D^2 \cup (D^2)^{-1}$, since all other assumptions of Lemma 2 are clearly satisfied. To see this, fix a D^2 function $g\colon \mathbb{R} \to \mathbb{R}$. We need to prove that both g and g^{-1} intersect only finitely many sets D_j.

To see that g intersects only finitely many sets D_j, by way of contradiction, assume that there is an infinite set $\{j_n\colon n < \omega\}$ such that $g \cap D_{j_n} \neq \emptyset$. For $n < \omega$ choose $\langle x_n, y_n\rangle \in g \cap D_{j_n}$. Then $g(x_n) = y_n$ for all $n < \omega$. Choosing a subsequence, if necessary, we can assume that $\lim_{n\to\infty} x_n = \hat{x} \in P$. Then, by the definition of sets D_j, we have

$$\lim_{n\to\infty} (y_n - h(x_n)) = \lim_{n\to\infty} \frac{y_n - h(x_n)}{x_n - \hat{x}} = \lim_{n\to\infty} \frac{y_n - h(x_n)}{(x_n - \hat{x})^2} = 0, \qquad (2)$$

as $\lim_{n\to\infty} \left|\frac{y_n - h(x_n)}{(x_n-\hat{x})^2}\right| \le \lim_{n\to\infty} \frac{|y_n - h(x_n)|}{|I_{j_n}|^2} \le \lim_{n\to\infty} |I_{j_n}| = 0$. In particular,

$$g(\hat{x}) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} y_n = \lim_{n\to\infty} (y_n - h(x_n)) + \lim_{n\to\infty} h(x_n) = h(\hat{x})$$

and

$$g'(\hat{x}) = \lim_{n\to\infty} \frac{y_n - h(\hat{x})}{x_n - \hat{x}} = \lim_{n\to\infty} \frac{y_n - h(x_n)}{x_n - \hat{x}} + \lim_{n\to\infty} \frac{h(x_n) - h(\hat{x})}{x_n - \hat{x}} = h'(\hat{x}) = 0.$$

Hence, by l'Hôpital's Rule, $\lim_{x\to\hat{x}} \frac{g(x)-g(\hat{x})}{(x-\hat{x})^2} = \lim_{x\to\hat{x}} \frac{g'(x)-0}{2(x-\hat{x})} = \frac{1}{2} g''(\hat{x})$ and, using (2) once more,

$$\lim_{n\to\infty} \frac{h(x_n) - h(\hat{x})}{(x_n - \hat{x})^2} = \lim_{n\to\infty} \frac{h(x_n) - y_n}{(x_n - \hat{x})^2} + \lim_{n\to\infty} \frac{g(x_n) - g(\hat{x})}{(x_n - \hat{x})^2} = \frac{1}{2} g''(\hat{x}),$$

[2] This follows, for example, from [1, thm. 19] (used with $f = h'$) which says that: for any real-valued continuous function f defined on a set $P \subset \mathbb{R}$ of positive measure there exists a $\mathcal{C}^1$ function $g\colon \mathbb{R} \to \mathbb{R}$ which agrees with f on an uncountable set.

where the first equation is justified by $y_n = g(x_n)$ and $h(\hat{x}) = g(\hat{x})$. But this contradicts the assumption on h that $\lim_{x\to\hat{x}} \frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} = \infty$.

To see that g^{-1} intersects only finitely many sets D_j, by way of contradiction, assume that there is an infinite set $\{j_n : n < \omega\}$ such that $g^{-1} \cap D_{j_n} \neq \emptyset$. For $n < \omega$ choose $\langle x_n, y_n\rangle \in g^{-1} \cap D_{j_n}$. Then $g(y_n) = x_n$ for all $n < \omega$. Choosing a subsequence, if necessary, we can assume that $\lim_{n\to\infty} x_n = \hat{x} \in P$. Then, $\hat{y} \stackrel{\text{def}}{=} \lim_{n\to\infty} y_n = \lim_{n\to\infty}(y_n - h(x_n)) + \lim_{n\to\infty} h(x_n) = h(\hat{x})$ and also $g(\hat{y}) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} x_n = \hat{x}$. Since, by the assumptions from Proposition 3, $h'(\hat{x}) = 0$ we obtain

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{g(y_n)-g(\hat{y})}{y_n-\hat{y}} \cdot \frac{y_n-\hat{y}}{g(y_n)-g(\hat{y})} \\ &= \lim_{n\to\infty} \frac{g(y_n)-g(\hat{y})}{y_n-\hat{y}} \cdot \lim_{n\to\infty} \frac{y_n-h(\hat{x})}{x_n-\hat{x}} \\ &= g'(\hat{y}) \cdot h'(\hat{x}) = g'(\hat{y}) \cdot 0 = 0, \end{aligned}$$

a contradiction. □

It is also worth to notice here, that if $h\colon \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ homeomorphism and P is a perfect set such that $h''(\hat{x}) = \lim_{x\to\hat{x}} \frac{h'(x)-h'(\hat{x})}{x-\hat{x}} = \infty$ for every $\hat{x} \in P$, then a small modification of the above proof gives a D^2 continuous function $F\colon \mathbb{R}^2 \to \mathbb{R}$ with $D(F) = h \restriction P$. This remark is of interest here, since such an h is easily constructed with standard calculus tools, see e.g. [6, Example 4.5.1]. However, as mentioned above, for such an h, neither can P have positive measure, nor can we have $h'(x) = 0$ for more than finitely many points x from P. So, in the modified argument for g, the fraction $\frac{h(x_n)-h(\hat{x})}{(x_n-\hat{x})^2}$ would need to be replaced with $\frac{h(x_n)-[h'(\hat{x})(x_n-\hat{x})+h(\hat{x})]}{(x_n-\hat{x})^2}$. Moreover, the same argument that we used to show that $g \notin D^2$ would need to be repeated for g^{-1}, however, this would require more restrictions in the definition of the sets D_j to allow for the reversed role of the variables x and y.

3 Proof of Proposition 3

Function h described below is a minor modification of a map f from [1, thm. 18].

Let $\varepsilon \in (0,1)$ be such that $M < 1-\varepsilon$ and let K be a symmetrically defined Cantor-like subset of $[0,1]$ of measure $1-\varepsilon$. More precisely, the set K is defined as $K = \bigcap_{n<\omega} \bigcup_{s\in 2^n} I_s = [0,1] \setminus \bigcup_{s\in 2^{<\omega}} J_s$, where: 2^n denotes the set of all sequences from $n = \{0,1,\ldots,n-1\}$ into $2 = \{0,1\}$; $2^{<\omega} = \bigcup_{n<\omega} 2^n$

is the set of all finite 0-1 sequences; $I_\emptyset = [0,1]$, and, for any $s \in 2^n$, J_s is an open interval of length $\frac{\varepsilon}{3^{n+1}}$ sharing the center with I_s, while $I_{s\hat{}0}$ and $I_{s\hat{}1}$ are the left and right component intervals of $I_s \setminus J_s$, respectively. Note that $|J_s| = \frac{\varepsilon}{3^{n+1}} < \frac{1}{3^{n+1}} < |I_s| \leq \frac{1}{2^n}$ for every $s \in 2^n$, so the choice of J_s is always possible. Clearly the set K has the desired measure of $1 - \sum_{s \in 2^{<\omega}} |J_s| = 1 - \sum_{n<\omega} 2^n \frac{\varepsilon}{3^{n+1}} = 1 - \varepsilon$.

For every $s \in 2^n$ let f_s be a function from $\mathbb{R}$ onto $[0, 1/(n+1)]$ defined as $f_s(x) = \frac{2}{(n+1)|J_s|} \operatorname{dist}(x, \mathbb{R} \setminus J_s)$, where $\operatorname{dist}(x,T) = \inf\{|x-t| \colon t \in T\}$ denotes the distance from x to T. Then, the function $h_0 = \sum_{s \in 2^{<\omega}} f_s \colon \mathbb{R} \to [0,1]$ is continuous and our $\mathcal{C}^1$ function $h \colon \mathbb{R} \to \mathbb{R}$ is defined as $h(x) = \int_0^x h_0(t)\, dt$. Note that h is strictly increasing on $[0,1]$.

Let P be an arbitrary perfect subset of K of measure M, which is disjoint with the set of all endpoints of the intervals J_s, $s \in 2^{<\omega}$. We will show that h and P are as required.

Clearly, for every $\hat{x} \in P \subset K$ we have $h'(\hat{x}) = h_0(\hat{x}) = 0$. To see the other condition, first notice that for $n > 1/\ln(4/3)$

$$\text{if } \hat{x}, x_0 \in K \cap I_s \text{ for } s \in 2^n \text{ and } \hat{x} \neq x_0, \text{ then } \frac{|h(x_0)-h(\hat{x})|}{(x_0-\hat{x})^2} \geq \frac{\varepsilon}{6} \frac{(4/3)^n}{(n+1)}. \tag{3}$$

To argue for (3), choose the largest $m < \omega$ such that $\hat{x}, x_0 \in I_t$ for some $t \in 2^m$. Then $m \geq n$, $\hat{x}$ and x_0 are separated by the interval J_t, and

$$\frac{|h(x_0)-h(\hat{x})|}{(x_0-\hat{x})^2} = \frac{|\int_{\hat{x}}^{x_0} h_0(t)\, dt|}{(x_0-\hat{x})^2} \geq \frac{|\int_{J_t} h_0(t)\, dt|}{|I_t|^2} = \frac{\frac{1}{2}|J_t| \frac{1}{(m+1)}}{|I_t|^2} \geq \frac{\frac{1}{2} \frac{\varepsilon}{3^{m+1}} \frac{1}{(m+1)}}{(1/2^m)^2}.$$

Hence, $\frac{|h(x_0)-h(\hat{x})|}{(x_0-\hat{x})^2} \geq \frac{\frac{1}{2} \frac{\varepsilon}{3^{m+1}} \frac{1}{(m+1)}}{(1/2^m)^2} = \frac{\varepsilon}{6} \frac{(4/3)^m}{(m+1)} \geq \frac{\varepsilon}{6} \frac{(4/3)^n}{(n+1)}$, as required, where the last inequality holds, since the function $f(x) = \frac{(4/3)^x}{x+1}$ is increasing for $x > 1/\ln(4/3)$, having derivative $f'(x) = \frac{(4/3)^x[\ln(4/3)(x+1)-1]}{(x+1)^2}$.

Next, notice that

$$\text{if } s \in 2^n,\ x \in J_s, \text{ and } x_0 \text{ is an endpoint of } J_s, \text{ then } \frac{|h(x)-h(x_0)|}{(x-x_0)^2} \geq \frac{3^{n+1}}{4(n+1)\varepsilon}. \tag{4}$$

To argue for (4), let x_1 be the midpoint between x_0 and x. Then h_0 is linear on the interval between x_0 and x_1 with the slope $\pm \frac{2}{(n+1)|J_s|}$. Hence, indeed,

$$\frac{|h(x)-h(x_0)|}{(x-x_0)^2} > \frac{|h(x_1)-h(x_0)|}{4(x_1-x_0)^2} = \frac{\frac{1}{2}(x_1-x_0)^2 \frac{2}{(n+1)|J_s|}}{4(x_1-x_0)^2} = \frac{3^{n+1}}{4(n+1)\varepsilon}.$$

Finally, fix an $\hat{x} \in P$. We need to show that $\lim_{x \to \hat{x}} \frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} = \infty$. For this, we fix an arbitrarily large N and show that $\frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} \geq N$ for the points x close enough to $\hat{x}$.

Let n_0 be such that $\min\left\{\frac{\varepsilon}{6}\frac{(4/3)^n}{(n+1)}, \frac{3^{n+1}}{4(n+1)\varepsilon}\right\} \geq 4N$ for all $n \geq n_0$ and let $s \in 2^{n_0}$ be such that $\hat{x} \in I_s$. Notice that $\hat{x}$ belongs to the interior U of I_s, as $\hat{x} \in P$. Hence, it is enough to show that $\frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} \geq N$ for every $x \neq \hat{x}$ from U. So, fix such an x.

If $x \in K$, then $\frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} \geq N$ follows immediately from (3). So, assume that $x \notin K$. Then $x \in J_t$ for some $t \supset s$. Let x_0 be the end point of J_t between x and $\hat{x}$. Notice, that $x_0 \neq \hat{x}$, since $\hat{x} \in P$. Then, since h is increasing on $[0,1]$, properties (3) and (4) imply

$$\begin{aligned}\frac{|h(x)-h(\hat{x})|}{(x-\hat{x})^2} &= \frac{|h(x)-h(x_0)|}{(x-x_0)^2}\frac{(x-x_0)^2}{(x-\hat{x})^2} + \frac{|h(x_0)-h(\hat{x})|}{(x_0-\hat{x})^2}\frac{(x_0-\hat{x})^2}{(x-\hat{x})^2} \\ &\geq 4N\frac{(x-x_0)^2}{(x-\hat{x})^2} + 4N\frac{(x_0-\hat{x})^2}{(x-\hat{x})^2} \geq N,\end{aligned}$$

finishing the proof. □

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