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CONTINUOUS AND SMOOTH IMAGES OF SETS

Abstract

This note shows that if a subset S of $\mathbb{R}$ is such that some continuous function $f\colon \mathbb{R}\to\mathbb{R}$ has the property "$f[S]$ contains a perfect set," then some $\mathcal{C}^\infty$ function $g\colon \mathbb{R}\to\mathbb{R}$ has the same property. Moreover, if $f[S]$ is nowhere dense, then the g can have the stronger property "$g[S]$ is perfect." The last result is used to show that it is consistent with ZFC (the usual axioms of set theory) that for each subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ (the cardinality of the continuum) there exists a $\mathcal{C}^\infty$ function $g\colon \mathbb{R}\to\mathbb{R}$ such that $g[S]$ contains a perfect set.

1 The results

Recall that a proposition

(A) for every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $g\colon \mathbb{R}\to\mathbb{R}$ such that $g[S]=[0,1]$

is independent of the usual ZFC axioms of set theory. More precisely, (A) holds in the iterated perfect set model, as proved by A. Miller [7]. In fact, (A) follows easily from the Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}$, which holds in this

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model, see [2, sec. 1.1]. (A) holds also in two other models of ZFC described in [4] and in [3].

The property (A) is false in any model of ZFC in which there exists either Lusin's or Sierpinski's set [7, 2]. In particular, the continuum hypothesis CH implies that (A) is false.

It is easy to see that (A) is equivalent to a seemingly weaker proposition

(B) for every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $G\colon \mathbb{R} \to \mathbb{R}$ such that $G[S]$ contains a perfect set P.

Indeed, if G satisfies (B) and $h\colon \mathbb{R} \to [0,1]$ is a continuous function such that $h[P] = [0,1]$, then $g = h \circ G$ satisfies (A).

Can (A) hold if g is required to have the stronger condition of being differentiable? The answer is clearly negative. In fact, the requirement of $g[S] = [0,1]$ in (A) fails for any Lebesgue measure zero set S and every differentiable function g, since every differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ satisfies Lusin's condition (N), that is, g maps Lebesgue measure zero sets onto sets of measure zero. (See e.g., [5, p. 355].) This argument does not work any more for the requirement in (B) that $g[S]$ contain a perfect set. In fact, the next theorem shows that in the statement of (B) the function g can also be required to be a $\mathcal{C}^\infty$ function, that is, infinitely many times differentiable.

Theorem 1. *The following conditions are equivalent.*

(A) For every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[S] = [0,1]$.

(B) For every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[S]$ contains a perfect set.

(C) For every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a $\mathcal{C}^\infty$ function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[S]$ contains a perfect set.

The main notion behind our proof of the theorem is that of outer-homeomorphisms: for a continuous function f from a closed subset K of $\mathbb{R}$ into $\mathbb{R}$ we say that f is *outer-homeomorphic* to $g\colon K \to \mathbb{R}$ provided there exists a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that $g = h \circ f$. Obviously, if $\bar{g}$ and g are outer-homeomorphic and $g[S]$ is as in (A) or in (B), then so is $\bar{g}[S]$. In this sense, (A) and (B) are invariant under outer-homeomorphisms. However, differentiability of a function is not invariant under outer-homeomorphisms. (See e.g. [1].) Nonetheless we have the following result, which is of interest by itself.

Proposition 2. *For every continuous function f from a closed subset K of $\mathbb{R}$ into a nowhere dense compact perfect set $P \subset \mathbb{R}$ there exists a $\mathcal{C}^\infty$ function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g \restriction K$ is outer-homeomorphic to f.*

Proposition 2 is proved in the next section. Here we will use it to prove Theorem 1.

Proof of Theorem 1. We already noticed that (A) is equivalent to (B). It is also obvious, that (C) implies (B). Thus, to finish the proof, it is enough to prove that (B) implies (C).

So, fix a subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$. By (B), there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[S]$ contains a perfect set. Let $P \subset f[S]$ be nowhere dense, compact, perfect and let $K = f^{-1}(P)$. Apply Proposition 2 to $f \restriction K$ to find a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ and a $\mathcal{C}^\infty$ function $g\colon \mathbb{R} \to \mathbb{R}$ such that $h \circ f \restriction K = g \restriction K$. Then $g[S] \supset g[S \cap K] = (h \circ f)[S \cap K] = h[f[S \cap K]] = h[P]$. As h is a homeomorphism, $h[P]$ is a compact perfect subset of $\mathbb{R}$. □

It is also worth noticing that the following strengthened versions of (A) are false.

Proposition 3. *The following properties are false.*

(A') For every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[S]$ is a perfect set.

(C') For every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a real analytic function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[S]$ contains a perfect set.

Proof. Property (A') fails for any dense set $S \subset \mathbb{R}$ of measure zero. Indeed, by way of contradiction, assume that there exists a differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ for which $g[S]$ is perfect. Then g is continuous and $g[S]$ is closed. In particular, $g[S] \subset g[\mathrm{cl}(S)] \subset \mathrm{cl}(g[S]) = g[S]$. Therefore, $g[\mathbb{R}] = g[S]$ is a closed connected set of measure zero. So, $g[S]$ is not perfect.

A set S for which (C') fails can be constructed by transfinite induction. Indeed, let $\{\langle f_\xi, P_\xi\rangle \colon \xi < \mathfrak{c}\}$ be an enumeration all pairs $\langle f, P\rangle$, where f is a non-constant real analytic function and P is a perfect subset of $\mathbb{R}$. By induction choose a sequence $\langle\langle s_\xi, y_\xi\rangle \colon \xi < \mathfrak{c}\rangle$ such that, for every $\xi < \mathfrak{c}$,

(i) $y_\xi \in P_\xi \setminus f_\xi[\{s_\zeta \colon \zeta < \xi\}]$,

(ii) $s_\xi \in \mathbb{R} \setminus \left(\{s_\zeta \colon \zeta < \xi\} \cup \bigcup_{\zeta \le \xi} f_\zeta^{-1}(Y_\xi)\right)$, where $Y_\xi = \{y_\zeta \colon \zeta \le \xi\}$.

The choice in (ii) can be made since every set $f^{-1}(y)$ is at most countable for non-constant real analytic functions f.

Then the set $S = \{s_\xi \colon \xi < \mathfrak{c}\}$ is as required. Indeed, if $f \colon \mathbb{R} \to \mathbb{R}$ is non-constant real analytic and P is a perfect subset of $\mathbb{R}$, then $P \not\subset f[S]$, since there exists a $\xi < \mathfrak{c}$ such that $\langle f_\xi, P_\xi \rangle = \langle f, P \rangle$ and, by construction, $y_\xi \in P_\xi \setminus f_\xi[S]$. □

Proposition 2 implies also the following result.

Corollary 4. *Let $S \subset \mathbb{R}$. If there exists a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f[S]$ contains a perfect set, then there is a $\mathcal{C}^\infty$ function $g \colon \mathbb{R} \to \mathbb{R}$ with the same property. Moreover, if S is nowhere dense, then g can be chosen so that $g[S]$ itself is perfect.*

Proof. Let $P \subset f[S]$ be nowhere dense, compact, and perfect. Let us first show that

(•) $f[S] = P$ for some continuous f whenever S is additionally nowhere dense in $\mathbb{R}$.

Indeed, assume that $K = \mathrm{cl}(S)$ is nowhere dense and let $K_0 = \mathrm{cl}(S \cap f^{-1}(P))$. Then $K_0 \subset K$ and $f \restriction K_0 \colon K_0 \to P$ maps $S \cap K_0$ onto P. By a version of Tietze extension theorem for zero-dimensional spaces, there exists a continuous extension $F_0 \colon K \to P$ of $f \restriction K_0$.[1] Let $F \colon \mathbb{R} \to \mathbb{R}$ be a continuous extension of F_0, which exists by the Tietze extension theorem. Then, $F[S] = P$, since $P = f[S \cap K_0] \subset F[S] \subset F[K] \subset P$. Thus, replacing f with F, if necessary, we can assume that (•) holds.

Returning to the proof of the corollary, we apply Proposition 2 to the function $f \restriction f^{-1}(P)$ to find a homeomorphism h and a $\mathcal{C}^\infty$ function g such that $g \restriction f^{-1}(P) = h \circ f \restriction f^{-1}(P)$. Then $h[P] = h[f[S \cap f^{-1}(P)]] = g[S \cap f^{-1}(P)] \subset g[S]$, so $g[S]$ contains the perfect set $h[P]$. Moreover, if S is nowhere dense, then, by (•), there is an f such that $S \cap f^{-1}(P) = S$ and thereby, for this f, $g[S] = h[P]$. □

The fact that property (A') is false shows that in Corollary 4 the additional property that $g[S]$ is a perfect set cannot be ensured in absence of an additional assumption on S. Nevertheless, if we allow replacement of S with its topological copies $T \subset \mathbb{R}$, then we can always find a T and a $\mathcal{C}^\infty$ function g with $g[T]$ being perfect, as proved below. Here we require that the topological spaces S and T be homeomorphic, not necessarily through homeomorphisms of $\mathbb{R}$.

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[1] Explicitly, if (a,b) is a bounded component of $\mathbb{R} \setminus K_0$, then F can be defined on $K \cap [a,b]$ by choosing a component (a',b') of $(a,b) \setminus K$ and defining F on $[a,a'] \cap K$ to be $F_0(a)$ and on $[b',b] \cap K$ to be $F_0(b)$. For an unbounded component (a,b) of $\mathbb{R} \setminus K_0$, define F on $(a,b) \cap K$ to be the appropriate $F_0(a)$ or $F_0(b)$.

Fact 5. *Let $S \subset \mathbb{R}$. If there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[S]$ contains a perfect set, then there exists a topological copy $T \subset \mathbb{R}$ of S and a $\mathcal{C}^\infty$ function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[T]$ is a perfect set.*

PROOF. As we are considering topological copies, we may assume $S \subset (0,1)$. If S contains an open interval, then clearly there is a $\mathcal{C}^\infty$ function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g[S] = [0,1]$. So, assume that $[0,1] \setminus S$ contains a countable dense subset D_1. We assume also that 0 and 1 are in D_1. Let D_2 to be the set of dyadic numbers in $[0,1]$, that is, rational numbers the form $\frac{k}{2^n}$. By a well known theorem, there is an order-preserving bijection $\varphi\colon [0,1] \to [0,1]$ such that $\varphi[D_2] = D_1$.

Let $\varphi_0\colon [0,1] \to [0,1]$ be the classical Cantor function (i.e., nondecreasing continuous function that maps the Cantor ternary set $\mathfrak{C}$ onto $[0,1]$) and denote the end-point set of $\mathfrak{C}$ by E. Observe that $\varphi_0 \restriction (\mathfrak{C} \setminus E)$ is an order-preserving bijection of $\mathfrak{C} \setminus E$ onto $[0,1] \setminus D_2$. So $T = (\varphi \circ \varphi_0 \restriction (\mathfrak{C} \setminus E))^{-1}(S)$ is a topological copy of S, and T is nowhere dense in $\mathbb{R}$. Let $f_0 = f \circ \varphi \circ \varphi_0$. Clearly $f_0[T] = f[S]$ and f_0 is continuous. Let f_1 be a continuous extension of f_0 and apply Corollary 4 to complete the proof. □

Is the conclusion of Corollary 4 true if we only assume the existence of a continuous function $f\colon S \to \mathbb{R}$ such that $f[S]$ contains a perfect set? Certainly, the answer is positive in any model of ZFC in which the property (A) holds. However, in general, this is false for some models of ZFC, as shown by the following result.

Proposition 6. *Under the Continuum Hypothesis there exists a set $S \subset \mathbb{R}$ for which there is a continuous function from S onto a perfect set, but such that $g[S]$ contains a perfect set for no continuous function $g\colon \mathbb{R} \to \mathbb{R}$.*

PROOF. Recall that a set $S \subset \mathbb{R}$ is concentrated on $\mathbb{Q}$ (the set of rational numbers) provided $S \cap K$ is countable for every closed set $K \subset \mathbb{R} \setminus \mathbb{Q}$. Rothberger constructed, under the Continuum Hypothesis, a set $S \subset \mathbb{R}$ concentrated on $\mathbb{Q}$ which can be mapped onto a perfect set by a continuous function from $\mathbb{R} \setminus \mathbb{Q}$ into $\mathbb{R}$. (See e.g. [8].)

However, if $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $g[S]$ cannot contain a perfect set. Indeed, by way of contradiction assume that $g[S]$ contains a perfect set P_0. Let $\mathcal{P}$ be a family of cardinality $\mathfrak{c}$ of pairwise disjoint perfect subsets of P_0. Then, there is a $P \in \mathcal{P}$ such that $K = f^{-1}(P)$ is disjoint with $\mathbb{Q}$. Therefore, $K \cap S$ is countable and so is $f[K \cap S]$. As $P \subset P_0 \subset f[S]$ implies $f[K \cap S] = P$, we have the contradiction that some perfect set is countable. □

2 Proof of Proposition 2

We say that $g\colon K\to\mathbb{R}$ is *locally Lipschitz* provided for every $x\in K$ there is an open set $U\ni x$ in $\mathbb{R}$ and a constant L such that $|g(a)-g(b)|\le L|a-b|$ for all $a,b\in K\cap U$. Proposition 2 is proved in two steps. First, in Lemma 7, we show that every function f as in the assumption of Proposition 2 is outer-homeomorphic to a locally Lipschitz function. Then, we show that Proposition 2 holds if we additionally assume that f locally Lipschitz function.

Lemma 7. *Every continuous function f from a closed subset K of $\mathbb{R}$ into a nowhere dense compact subset P of $\mathbb{R}$ is outer-homeomorphic to a locally Lipschitz function $g\colon K\to\mathbb{R}$.*

Proof. Let $h_0\colon\mathbb{R}\to\mathbb{R}$ be a homeomorphism such that $h_0[P]$ is a subset of the classical ternary Cantor set $\mathfrak{C}$. Replacing f with $h_0\circ f$, if necessary, we may assume that $P=\mathfrak{C}$, that is, that $f\colon K\to\mathfrak{C}$.

The proof is a straightforward inductive construction. As usual, 2^n denotes the set of all maps from $\{0,\dots,n-1\}$ into $\{0,1\}$ and $2^{<\omega}=\bigcup_{n<\omega}2^n$.

We say that a family $\mathcal{J}=\{J_s\colon s\in 2^{<\omega}\}$ of closed intervals in $[0,1]$ is an *interval tree* provided for every $s\in 2^n$ the intervals $J_{s\hat{}0}$ and $J_{s\hat{}1}$ are disjoint subsets of J_s and $J_{s\hat{}0}<J_{s\hat{}1}$. Recall that for every interval tree $\mathcal{J}=\{J_s\colon s\in 2^{<\omega}\}$ the set $R_{\mathcal{J}}=\bigcap_{n<\omega}\bigcup_{s\in 2^n}J_s$ is perfect and nowhere dense. Also, if $\mathcal{I}=\{I_s\colon s\in 2^{<\omega}\}$ is an interval tree such that $I_\emptyset=[0,1]$ and, for every $s\in 2^n$, $I_s\setminus(I_{s\hat{}0}\cup I_{s\hat{}1})$ is the middle third subinterval of I_s, then $R_{\mathcal{I}}$ is the classical ternary Cantor set $\mathfrak{C}$.

For every $n<\omega$ choose a $\delta_n\in(0,1)$ such that $|f(x)-f(y)|<3^{-n}$ for every $x,y\in[-n,n]\cap K$ with $|x-y|<\delta_n$. Such a δ_n exists, since $f\restriction[-n,n]\cap K$ is uniformly continuous. We will also assume that $\delta_n\searrow 0$. Construct, by induction on $n<\omega$, the families $\mathcal{J}_n=\{J_s\colon s\in 2^n\}$ such that $\mathcal{J}=\{J_s\colon s\in 2^{<\omega}\}$ is an interval tree. In the inductive construction we will require that the length of each interval in $\mathcal{J}_n$ is less than δ_{n+1}. Let h_1 be a strictly increasing function from $P=\mathfrak{C}$ onto $R_{\mathcal{J}}$ that maps each set $\mathfrak{C}\cap I_s$ into J_s and let $h\colon\mathbb{R}\to\mathbb{R}$ be an increasing homeomorphism extending h_1. This is our desired homeomorphism.

Let $g=h\circ f$. To see that h is locally Lipschitz it is enough to prove that for every $k<\omega$

(i) $|g(x)-g(y)|<|x-y|$ for all $x,y\in K\cap[-k,k]$ with $0<|x-y|<\delta_k$.

To see (i), fix $x,y\in K\cap[-k,k]$ with $0<|x-y|<\delta_k$. Then, there exists an $n\ge k$ such that $|x-y|\in[\delta_{n+1},\delta_n)$. So, $|f(x)-f(y)|<3^{-n}$, that is, $f(x)$ and $f(y)$ must belong to the same I_s for some $s\in 2^n$. Then, by the construction of h, both $g(x) = h(f(x))$ and $g(y) = h(f(y))$ belong to J_s, so $|g(x) - g(y)| < \delta_{n+1} \le |x - y|$, finishing the proof. □

Lemma 8. *Let $U \subset \mathbb{R}$ be open, $x \in U$, and let $\bar\varphi$, $\bar g$, and $\bar b$ be functions from $\mathbb{R}$ to $\mathbb{R}$ such that: $\bar g$ is Lipschitz on U, $\bar b$ is bounded on U, and $\bar\varphi'(\bar g(x)) = 0$. If either $\bar b$ is a constant function or $\bar\varphi(\bar g(x)) = 0$, then the function $G(\bar x) = \bar b(\bar x) \cdot \bar\varphi(\bar g(\bar x))$ is differentiable at x and $G'(x) = 0$.*

Proof. Let $\varepsilon > 0$. We need to find a $\delta > 0$ such that

$$\left|\tfrac{G(y)-G(x)}{y-x}\right| < \varepsilon \text{ whenever } 0 < |x - y| < \delta. \tag{1}$$

This is certainly true when $\bar g(y) = \bar g(x)$, since then, under each assumption, we have $G(y) = G(x)$. So, assume that $\bar g(y) \ne \bar g(x)$. Then

$$\left|\tfrac{G(y)-G(x)}{y-x}\right| = |\bar b(y)| \left|\tfrac{\bar\varphi(\bar g(y))-\bar\varphi(\bar g(x))}{\bar g(y)-\bar g(x)}\right| \left|\tfrac{\bar g(y)-\bar g(x)}{y-x}\right|.$$

In particular, if $L > 0$ a Lipschitz constant for $\bar g$ on U and $M > 0$ is a bound for $|\bar b|$ on U, then for every $y \in U$ with $\bar g(y) \ne \bar g(x)$

$$\left|\tfrac{G(y)-G(x)}{y-x}\right| \le ML \left|\tfrac{\bar\varphi(\bar g(y))-\bar\varphi(\bar g(x))}{\bar g(y)-\bar g(x)}\right|. \tag{2}$$

Since $\bar\varphi'(\bar g(x)) = 0$, we can find a $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$ and $\left|\tfrac{\bar\varphi(z)-\bar\varphi(\bar g(x))}{z-\bar g(x)}\right| < \tfrac{\varepsilon}{ML}$ provided $0 < |z - \bar g(x)| \le \delta L$. Then, for every y for which $\bar g(y) \ne \bar g(x)$ and $|y - x| < \delta$ we have $0 < |\bar g(y) - \bar g(x)| \le L|y - x| \le \delta L$, so $\left|\tfrac{\bar\varphi(\bar g(y))-\bar\varphi(\bar g(x))}{\bar g(y)-\bar g(x)}\right| < \tfrac{\varepsilon}{ML}$. Therefore, (1) follows from (2). □

Proof of Proposition 2. Let $f\colon K \to P$ be as in the assumptions. By Lemma 7, there is a homeomorphism $h_0\colon \mathbb{R} \to \mathbb{R}$ such that $g_0 = h_0 \circ f$ is locally Lipschitz. Let $\bar g\colon \mathbb{R} \to \mathbb{R}$ be the natural continuous linear extension of g_0, that is, such that $\bar g$ is linear on each component interval of $\mathbb{R} \setminus K$, constant on the unbounded components. It is easy to see that $\bar g$ is still locally Lipschitz. (However, for the endpoints of component intervals of $\mathbb{R} \setminus K$ the local Lipschitz constant may change.)

Let $T = h_0[P]$. Then T is compact, perfect, nowhere dense and $\bar g[K] \subset T$. Let $\varphi_0\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $\varphi_0(x) = 0$ for all $x \in T$ and $\varphi_0(x) > 0$ for all $x \in \mathbb{R} \setminus T$. Then $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined as $\varphi(x) = \int_0^x \varphi_0(t)dt$ is a strictly increasing $\mathcal{C}^\infty$ function such that $\varphi^{(k)}(x) = 0$ for all $k = 1, 2, 3, \ldots$ and $x \in T$. In fact, it is a homeomorphism between $\mathbb{R}$ and $\varphi[\mathbb{R}]$ and it is easy to ensure also that $\varphi[\mathbb{R}] = \mathbb{R}$. Let $g = \varphi \circ \bar g$. We will show that g is the required function.

Notice that $h = \varphi \circ h_0$ is a homeomorphism and the restriction condition holds, as $h \circ f = \varphi \circ h_0 \circ f = \varphi \circ g_0 = \varphi \circ \bar{g} \restriction K = g \restriction K$. To finish the proof, it is enough to show that g is infinitely many times differentiable.

For $k \geq 1$ let $b_k \colon \mathbb{R} \to \mathbb{R}$ be defined as $b_k(x) = \left(\frac{\bar{g}(a)-\bar{g}(b)}{a-b}\right)^k$ for every x in a bounded component (a,b) of $\mathbb{R} \setminus K$, and $b_k(x) = 0$ for all other points x. Notice that each b_k is locally bounded, since $\bar{g}$ is locally Lipschitz. Define also b_0 as a constant 1 function. We will show, by induction on $k < \omega$, that

(I_k) the kth derivative of g exists and is equal $g^{(k)}(x) = b_k(x)\varphi^{(k)}(\bar{g}(x))$.

Clearly it is true for $k = 0$. So, assume (I_k) for some $k < \omega$. We need to prove that $g^{(k+1)}(x) = b_{k+1}(x)\varphi^{(k+1)}(\bar{g}(x))$ for every $x \in \mathbb{R}$.

For $x \in K$ this follows immediately from Lemma 8 applied to $\bar{\varphi} = \varphi^{(k)}$, $\bar{g}$, and $\bar{b} = b_k$, since then $g^{(k+1)}(x) = G'(x) = 0 = b_{k+1}(x)\varphi^{(k+1)}(\bar{g}(x))$. For x from a bounded component (a,b) of $\mathbb{R} \setminus K$ the result holds, since on such interval $g^{(k)}(x) = \left(\frac{\bar{g}(a)-\bar{g}(b)}{a-b}\right)^k \varphi^{(k)}(\bar{g}(x))$ and $\bar{g}$ is a linear function with the slope $\frac{\bar{g}(a)-\bar{g}(b)}{a-b}$. Finally, the formula holds for an unbounded component of $\mathbb{R} \setminus K$ (if it exists), since on such interval $\bar{g}$ is constant and b_{k+1} is equal 0. $\square$

Remark 9. For the compact set $K = [0,1]$, connections between outer-homeomorphisms and differentiation of real valued functions are discussed in A. M. Bruckner's book [1]. Of course, closed subsets of $\mathbb{R}$ need not be bounded. Consequently, the need for the proof of Lemma 7 for unbounded closed sets K presented itself. Also, for $[0,1]$, there is the corresponding notion of inner-homeomorphism. Connections between inner-homeomorphisms and differentiation are discussed in the books by Bruckner [1] and by C. Goffman, T. Nishiura and D. Waterman [6].

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