Computer Vision and Image Understanding 114 (2010) 155–166

Contents lists available at ScienceDirect

Computer Vision and Image Understanding

journal homepage: www.elsevier.com/locate/cviu

Affinity functions in fuzzy connectedness based image segmentation II: Defining and recognizing truly novel affinities

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ARTICLE INFO

Article history:
Received 9 September 2009
Accepted 9 September 2009
Available online 16 September 2009

Keywords:
Affinity
Fuzzy connectedness
Image segmentation
Equivalence of algorithms

ABSTRACT

Affinity functions — the measure of how strongly pairs of adjacent spels in the image hang together — represent the core aspect (main variability parameter) of the fuzzy connectedness (FC) algorithms, an important class of image segmentation schemas. In this paper, we present the first ever theoretical analysis of the two standard affinities, homogeneity and object-feature, the way they can be combined, and which combined versions are truly distinct from each other. The analysis is based on the notion of equivalent affinities, the theory of which comes from a companion Part I of this paper (Ciesielski and Udupa, in this issue) [11]. We demonstrate that the homogeneity based and object feature based affinities are equivalent, respectively, to the difference quotient of the intensity function and Rosenfeld's degree of connectivity. We also show that many parameters used in the definitions of these two affinities are redundant in the sense that changing their values lead to equivalent affinities. We finish with an analysis of possible ways of combining different component affinities that result in non-equivalent affinities. In particular, we investigate which of these methods, when applied to homogeneity based and object-feature based components lead to truly novel (non-equivalent) affinities, and how this is affected by different choices of parameters. Since the main goal of the paper is to identify, by formal mathematical arguments, the affinity functions that are equivalent, extensive experimental confirmations are not needed — they show completely identical FC segmentations — and as such, only relevant examples of the theoretical results are provided. Instead, we focus mainly on theoretical results within a perspective of the fuzzy connectedness segmentation literature.

1. Introduction and preliminaries

The subject of this article is to study the notion of affinity, the main variability parameter of the image segmentation method known as Fuzzy Connectedness (FC), see [38,36,13]. Since this paper is a second part of the study, we refer the reader to its first part [11] for an overview of the segmentation literature and the role that FC plays within. Similarly, we will present below only the minimum preliminaries needed to follow this paper, referring to [11] for more detailed discussion.

1.1. Fuzzy connectedness framework

Digital space: Let $\boldsymbol{n} \geqslant 2$ and let $\mathbb{Z}^{\boldsymbol{n}}$ stand for the set of all $\boldsymbol{n}$-tuples of integer numbers. By an *$\boldsymbol{n}$-dimensional fuzzy digital space* we will understand a pair $\langle \mathbb{Z}^{\boldsymbol{n}}, \alpha \rangle$, where α is an *adjacency relation*. We will assume in what follows that the adjacency relation α is defined as $\alpha(c,d) = \chi_{[0,1]}(\|c-d\|)$; that is, $\alpha(c,d) = 1$ when $\|c-d\| \leqslant 1$ and $\alpha(c,d) = 0$ for $\|c-d\| > 1$, where $\|c-d\|$ represents the Euclidean distance between c and d. α represents 4-adjacency in two-dimensional space and 6-adjacency in three-dimensional space. The elements of the digital space are called *spels*. (For $\boldsymbol{n} = 2$ also called *pixels*, while for $\boldsymbol{n} = 3$ – *voxels*.)

Digital scene: A *scene* over a fuzzy digital space $\langle \mathbb{Z}^{\boldsymbol{n}}, \alpha \rangle$ is a pair $\mathscr{C} = \langle C, f \rangle$, where $C = \prod_{j=1}^{\boldsymbol{n}} [-b_j, b_j] \subset \mathbb{Z}^{\boldsymbol{n}}$, each $b_j > 0$ being an integer, and $f : C \to \mathbb{R}$ is a *scene intensity function*.[3] The value of f represents either the original acquired image intensity or an estimate of certain image properties (such as gradients and texture measures) obtained from the given image.

Affinity: The notion most important for this paper is that of an affinity function. Let $\preceq$ be a linear order relation [10] on a set L and let C be an arbitrary finite non-empty set, representing a domain of a scene intensity function. We say that a function $\kappa : C \times C \to L$ is an *affinity function* (from C into $\langle L, \preceq \rangle$) provided κ

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[1] Partially supported by NSF Grant DMS-0623906.
[2] Partially supported by NIH Grant EB004395. The authors thank Dr. Ying Zhuge for preparing some images presented in this paper.

[3] For simplicity, we will restrict our attention to scalar valued images. However, all presented results can be generalized to the vectorial images.

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doi:10.1016/j.cviu.2009.09.005

is symmetric (i.e., $\kappa(a,b)=\kappa(b,a)$ for every $a,b\in C$) and $\kappa(a,b)\preceq\kappa(c,c)$ for every $a,b,c\in C$. Since $\kappa(d,d)\preceq\kappa(c,c)$ for every $c,d\in C$, there exists an element in L, which we will denote by a symbol $\mathbf{1}_\kappa$, such that $\kappa(c,c)=\mathbf{1}_\kappa$ for every $c\in C$. Notice that $\mathbf{1}_\kappa$ is the largest element of $L_\kappa=\{\kappa(a,b):a,b\in C\}$, although it does not need to be the largest element of L. In what follows, the strict inequality related to $\preceq$ will be denoted by $\prec$, that is, $a\prec b$ if and only if $a\preceq b$ and $a\neq b$.

We say that κ is a *standard affinity* provided $\langle L,\preceq\rangle=\langle[0,1],\leqslant\rangle$ and $\mathbf{1}_\kappa=1$. We will also use extensively order $\langle L,\preceq\rangle=\langle[0,\infty],\geqslant\rangle$, in which case the order relation $\preceq$ is the reversed standard order relation $\geqslant$. In such a setting, "$\preceq$-stronger" means "less than" in terms of the standard order $\leqslant$. Also, the meanings of the terms min and max are switched: "min in terms of $\preceq$" means "max in terms of $\leqslant$," and "max in terms of $\preceq$" becomes "min in terms of $\leqslant$." For example, the $\preceq$-minimum of a set $S=\{.1,.5,.7\}$ is equal to .7 (since $.7\preceq.5$ and $.7\preceq.1$), while .7 is the maximum of S in the standard order $\leqslant$. We will use symbol $\mathbf{1}$ for denoting the $\preceq$ largest number of these sets, that is, $\mathbf{1}=1$ for $\langle[0,1],\leqslant\rangle$ and $\mathbf{1}=0$ for $\langle[0,\infty],\geqslant\rangle$.

Affinity as an operator: The affinity function is usually associated with each scene $\mathscr{C}$ according to some specific rule, such as $\kappa(c,d)=e^{-\|f(c)-f(d)\|^2}$ for all adjacent $c,d\in C$. (See Section 2.) In such a case, we can treat the rule of such association as an operator $\langle\mathscr{C},p\rangle\overset{\mathscr{K}}{\mapsto}\kappa=\mathscr{K}(\mathscr{C},p)$, where p represents all parameters of the affinity function like the expected scene intensity for the object.

Paths and connectivity measure: Fix an affinity $\kappa:C\times C\to\langle L,\preceq\rangle$. To define fuzzy connectedness segmentation of $\mathscr{C}$, we need first to translate the local measure of connectedness given by κ into the global strength of connectedness. For this, we will need the notions of a path and its strength.

A *path in* $A\subseteq C$ is any sequence[4] $p=\langle c_1,\dots,c_l\rangle$, where $l>1$ and $c_i\in A$ for every $i=1,\dots,l$. (Notice that there is no assumption on any adjacency of the consecutive spels in a path.) The family of all paths in A is denoted by $\mathscr{P}^A$. If $c,d\in A$, then the family of all paths $\langle c_1,\dots,c_l\rangle$ in A from c to d (i.e., such that $c_1=c$ and $c_l=d$) is denoted by $\mathscr{P}^A_{cd}$. The *strength* $\mu_\kappa(p)$ *of a path* $p=\langle c_1,\dots,c_l\rangle\in\mathscr{P}^C$ is defined as the strength of its κ-weakest link; that is, $\mu_\kappa(p)\overset{\text{def}}{=}\min\{\kappa(c_{i-1},c_i):1<i\leqslant l\}$. (Note that, if one follows the common practice of defining $\kappa(c,d)$ to be the minimal element of L_κ for any non-adjacent c and d, then only paths with adjacent consecutive spels can have non-minimal strength.)

For $c,d\in A\subseteq C$, the (global) *κ-connectedness strength in A between c and d* is defined as the strength of a strongest path in A between c and d; that is,

$$\mu^A_\kappa(c,d)\overset{\text{def}}{=}\max\{\mu_\kappa(p):p\in\mathscr{P}^A_{cd}\}. \tag{1}$$

Notice that $\mu^A_\kappa(c,c)=\mu_\kappa(\langle c,c\rangle)=\mathbf{1}_\kappa$. We refer to $\mu^A_\kappa:C\times C\to L$ as a *connectivity measure* (on A) induced by κ. For $c\in A\subset C$ and a non-empty $D\subset A$, we also define $\mu^A_\kappa(c,D)\overset{\text{def}}{=}\max_{d\in D}\mu^A_\kappa(c,d)$. We will write μ for μ_κ and μ^A for μ^A_κ when κ is clear from the context. The issue of why μ^A_κ should be defined from κ by the procedure described above is discussed in detail in [30].

FC segmentations: To define fuzzy objects delineated by FC segmentations, we start with a family $\mathbb{S}$ of non-empty pairwise disjoint subsets of C, where each $S\in\mathbb{S}$ represents a set of spels, known as *seeds*, which will belong to the object generated by it. Also, fix a threshold $\theta\in L,\theta\leqslant\mathbf{1}_\kappa$. For every $S\in\mathbb{S}$, put $W=\bigcup(\mathbb{S}\setminus\{S\})$ and, similarly as in [13] (see also [37]), define

- $P^\kappa_{S\theta}=\{c\in C:\theta\preceq\mu^C_\kappa(c,S)\}$;
- $P^\kappa_{S\mathbb{S}}=\{c\in C:\mu^C_\kappa(c,W)\prec\mu^C_\kappa(c,S)\}$;
- $P^{I\kappa}_{S\mathbb{S}}=\bigcup_{i=0}^\infty P^{i,\kappa}_{S\mathbb{S}}$, where sets $P^{i,\kappa}_{S\mathbb{S}}$ are defined inductively by $P^{0,\kappa}_{S\mathbb{S}}=\emptyset$ and $P^{i+1,\kappa}_{S\mathbb{S}}=P^{i,\kappa}_{S\mathbb{S}}\cup\{c\in C\setminus P^{i,\kappa}_{S\mathbb{S}}:\mu^{C\setminus P^{i,\kappa}_{S\mathbb{S}}}_\kappa(c,W)\prec\mu^C_\kappa(c,S)\}$.

Then absolute fuzzy connectedness AFC, relative fuzzy connectedness RFC, and iterative relative fuzzy connectedness IRFC segmentations of C are defined, respectively, as $\mathbb{P}^\theta_\kappa(\mathbb{S})=\{P^\kappa_{S\theta}:S\in\mathbb{S}\}$, $\mathbb{P}_\kappa(\mathbb{S})=\{P^\kappa_{S\mathbb{S}}:S\in\mathbb{S}\}$, and $\mathbb{P}^I_\kappa(\mathbb{S})=\{P^{I\kappa}_{S\mathbb{S}}:S\in\mathbb{S}\}$.

Notice that an AFC object $P^\kappa_{S\theta}$ consists of all spels connected with at least one seed s in S with the κ-connectivity strength at least θ. An RFC object is created via competition of seeds for each spel: a spel c belongs to $P^\kappa_{S\mathbb{S}}$ provided there is a seed s in S for which the κ-connectivity between c and s exceeds the κ-connectivity between c and any other seed indicating another object. Finally, IRFC objects are obtained by refining the RFC competition in an iterative manner.

1.2. Affinity equivalence: definition and results

We say that the affinities $\kappa_1:C\times C\to\langle L_1,\preceq_1\rangle$ and $\kappa_2:C\times C\to\langle L_2,\preceq_2\rangle$ are *equivalent (in the FC sense)* provided, for every $a,b,c,d\in C$

$$\kappa_1(a,b)\preceq_1\kappa_1(c,d)\quad\text{if and only if}\quad\kappa_2(a,b)\preceq_2\kappa_2(c,d).$$

The affinity operators $\mathscr{K}_1$ and $\mathscr{K}_2$ are equivalent provided the associated affinities $\kappa_1=\mathscr{K}_1(\mathscr{C},p)$ and $\kappa_2=\mathscr{K}_2(\mathscr{C},p)$ are equivalent for all scenes $\mathscr{C}$ and appropriate parameters p.

In the following characterization of equivalent affinities $\circ$ stands for the composition of functions, that is, $(g\circ\kappa_1)(a,b)=g(\kappa_1(a,b))$.

Proposition 1 [11, Prop. 1 and Cor. 2]. *Affinity functions $\kappa_1:C\times C\to\langle L_1,\preceq_1\rangle$ and $\kappa_2:C\times C\to\langle L_2,\preceq_2\rangle$ are equivalent if and only if there exists a strictly increasing function g from $\langle L_{\kappa_1},\preceq_1\rangle$ onto $\langle L_{\kappa_2},\preceq_2\rangle$ such that $\kappa_2=g\circ\kappa_1$.*

In particular, if $\kappa:C\times C\to\langle[0,\infty],\geqslant\rangle$ is an affinity, then, for every strictly decreasing function g from $[0,\infty]$ onto $[0,1]$, a map $g\circ\kappa:C\times C\to\langle[0,1],\leqslant\rangle$ is an affinity equivalent to κ.

The following results show that equivalent affinities are indistinguishable in the FC segmentation framework: if any two equivalent affinities are used in the same FC schema to produce two versions of the algorithm, then these algorithms lead to identical segmentations.

Theorem 2 [11, Thm. 5]. *Let $\kappa_1:C\times C\to\langle L_1,\preceq_1\rangle$ and $\kappa_2:C\times C\to\langle L_2,\preceq_2\rangle$ be equivalent affinity functions and let $\mathscr{S}$ be a family of non-empty pairwise disjoint subsets of C. Then for every $\theta_1\preceq_1\mathbf{1}_{\kappa_1}$ in L_1, there exists a $\theta_2\preceq_2\mathbf{1}_{\kappa_2}$ in L_2 such that, for every $S\in\mathbb{S}$ and $i\in\{0,1,2,\dots\}$, we have $P^{\kappa_1}_{S\theta_1}=P^{\kappa_2}_{S\theta_2}$, $P^{\kappa_1}_{S\mathbb{S}}=P^{\kappa_2}_{S\mathbb{S}}$, and $P^{i,\kappa_1}_{S\mathbb{S}}=P^{i,\kappa_2}_{S\mathbb{S}}$. In particular, $\mathbb{P}^{\theta_1}_{\kappa_1}(\mathbb{S})=\mathbb{P}^{\theta_2}_{\kappa_2}(\mathbb{S})$, $\mathbb{P}_{\kappa_1}(\mathbb{S})=\mathbb{P}_{\kappa_2}(\mathbb{S})$, and $\mathbb{P}^I_{\kappa_1}(\mathbb{S})=\mathbb{P}^I_{\kappa_2}(\mathbb{S})$. Moreover, by Proposition 1, we can find a strictly monotone function $g:C\to C$ and then $\theta_2=g(\theta_1)$.*

2. Two commonly used affinities and their natural definitions

In this section, we will study the two main classes of affinities that have been employed in the FC literature, namely, homogeneity based and object-feature based, and examine the connectivity measures they induce. We will consider them with range $\langle L,\preceq\rangle=\langle[0,\infty],\geqslant\rangle$.

We will work here with a fixed digital space $\langle\mathbb{Z}^{\mathbf{n}},\alpha\rangle$ and a scene $C=\langle C,f\rangle$. We will also assume that the scene intensity function has scalar values only, $f:C\to\mathbb{R}$. To make our presentation more transparent, we will assume that f represents not necessarily the original scene intensity function, but rather a result of any filtering that could have been done on such acquired scene. In particular,

[4] Notice that the paths must have length greater than 1. We make this requirement to ease some technical difficulties, while it creates no real restriction as, in whatever we do, a "path" $\langle c\rangle$ can be always replaced by a path $\langle c,c\rangle$.

we will not use any scale based approach to the affinity definitions (see [33]), since any scale-based affinity is essentially equal to a non-scale-based affinity applied to an appropriately filtered version of the intensity function. (This is precisely true for the object feature based affinities used in the literature. In the case of homogeneity based affinities, the affinity obtained by what we suggest above is slightly different from that defined in [33]; however, these two versions are very close to each other.)

2.1. Homogeneity based affinity

Intuitively, this function, denoted $\psi(c,d)$, is defined as the maximum of $|f'(x)|$, with x on the segment joining c and d (where f' is the derivative of f): the higher the magnitude of the slope of f between c and d is, the weaker is the affinity (connectivity) between c and d. Of course, there is more than one way to interpret the symbol $|f'(x)|$. In this section we will interpret this as a magnitude of the directional derivative $D_{\overrightarrow{cd}}f(x)$ in the direction of the vector $\overrightarrow{cd}$. This agrees with the standard FC approach used in the research conducted so far. (See e.g. [38,20,18,25,17].) Alternatively, it is possible to treat $|f'(x)|$ as a gradient magnitude. True gradient induced homogeneity based affinity will be incorporated in our future work. (See e.g. [12].)

The value $|f'(x)| = |D_{\overrightarrow{cd}}f(x)|$ is best approximated by a difference quotient $\psi_0(c,d) = |\frac{f(c)-f(d)}{\|c-d\|}|$. Although this expression has no sense for $c = d$, it should be clear that we should define $\psi_0(c,c)$ as equal to 0, the "highest" possible connectivity in this setting. (Recall that "highest" in terms of $\preceq$ defined as $\geqslant$ translates into "least" in terms of the standard order $\leqslant$. That is, the greater ψ_0 is, the weaker is the affinity between c and d.) Is the definition $\psi_0(c,d) = |\frac{f(c)-f(d)}{\|c-d\|}|$ what we are looking for?

Certainly this is not a local measurement of connectedness when $\|c-d\|$ is large. In this case, the difference quotient is a poor approximation of the definition of the derivative. We also have a better way of estimating the highest slope on the road from c to d: crawl from c to d along a path with steps of length 1, estimating the slope of each step separately. Because of this, it makes sense to consider the number $\psi_0(c,d)$ as a good value for $\psi(c,d)$ only when $\|c-d\| \leqslant 1$, in all other cases we should assign to it the worst possible value; that is, ∞. This leads to the definition $\psi(c,d) = \psi_0(c,d)/\alpha(c,d)$; that is,

$$\psi(c,d) = \begin{cases} |f(c)-f(d)| & \text{for } \|c-d\| \leqslant 1 \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

It is easy to see that ψ satisfies our definition of affinity function. It should be stressed here that such a function approximates only the magnitude of the directional derivative of f in the direction $\overrightarrow{cd}$, and gives no information on the slope of f in a direction perpendicular to $\overrightarrow{cd}$.

If one likes to express this affinity by an equivalent standard affinity, our definition of ψ can be replaced by $g_1(\psi(c,d))$, where g_σ is a Gaussian function $g_\sigma(x) = e^{-x^2/\sigma^2}$. Notice that if $\alpha(c,d) = \chi_{[0,1]}(\|c-d\|)$, as we defined earlier, then $g_1(\psi(c,d)) = \alpha(c,d)\cdot g_1(|f(c)-f(d)|)$, the formula defining purely homogeneity based affinity in [33, pp. 149–150]. (We use the weights $w_1 = 0$ and $w_2 = 1$.) However, if α is an arbitrary fuzzy adjacency relation, then the formula $\alpha(c,d)\cdot g_1(|f(c)-f(d)|)$ disagrees with the derivative intuition. For example, if $\alpha(c,d) = g_1(\|c-d\|)$, then $\alpha(c,d)\cdot g_1(|f(c)-f(d)|) = e^{-(|f(c)-f(d)|^2+\|c-d\|^2)} = g_1(\sqrt{|f(c)-f(d)|^2+\|c-d\|^2})$, rather than the more appropriate $g_1\left(\frac{|f(c)-f(d)|}{\|c-d\|}\right)$ (possibly multiplied by number $\alpha(c,d)$).

In what follows, we will use the homogeneity based affinity $\psi(c,d)$ as defined in (2), rather than $g_1(\psi(c,d))$, as it is more intuitive, and, by Proposition 1, these two affinities are equivalent. We refer the reader to [11, Fig. 1] for an illustration demonstrating the equivalence of $\psi(c,d)$ and $g_\sigma(\psi(c,d))$. Thus, the parameter σ in the homogeneity based affinity $\psi_\sigma = g_\sigma \circ \psi$ is of no consequence for the FC algorithms. However, in all FC literature, this σ has been considered as a parameter of the method in the description of the methods and their evaluation, and different settings have been claimed to give different segmentation accuracies, which has no theoretical basis in view of Theorem 2.

The homogeneity based connectivity measure, $\mu_\psi = \mu^C_\psi$, can be elegantly interpreted if our scene $\mathscr{C} = \langle C, f\rangle$ is considered as a topographical map in which $f(c)$ represents an elevation at the location $c \in C$. Then, $\mu_\psi(c,d)$ is the highest possible step (a slope of f) that one must make in order to get from c to d with each step on a location (spel) from C and of unit length. In particular, the object $P^\psi_{s\theta} = \{c \in C : \theta \geqslant \mu_\psi(s,c)\}$ represents those spels $c \in C$ which can be reached from s without ever making a step higher than θ. Note that all we measure in this setting is the actual change of the altitude while making the step. Thus, this value can be small, even if the step is made on a very steep slope, as long as the path approximately follows the altitude contour lines — this is why on steep hills the roads zigzag, allowing for a small incline of the motion. On the other hand, the measure of the same step would be large, if measured with some form of gradient induced homogeneity based affinity!

2.2. Object feature based affinity

There are two principal differences between the object feature based and the homogeneity based affinities. (1) The definition of the object feature based affinity requires some prior knowledge on the intensities of the objects we like to uncover, while the definition of the homogeneity based affinity is completely independent of such knowledge. (2) The homogeneity based affinity is represented in terms of (the approximation of) the derivative f' of the intensity function f, while the object feature based affinity is defined directly from the intensity function f. In the rest of this subsection, we will consider object feature based affinity for the cases of single and multiple objects separately.

2.2.1. Object feature based affinity: single object case

We will start with the definition of the object feature based affinity, denoted $\phi(c,d)$, in terms of only a single object O. To define ϕ, we need to start with an approximate expected (average) intensity value m for the spels in the object. We will also assume that we have a standard deviation $\sigma > 0$ of the distribution of intensity for this object. Then, the intuition behind ϕ can be expressed with a pseudo-affinity formula $\bar\varphi_0(c) = |f(c)-m|$ — the smaller the value of $\bar\varphi_0(c)$ is, the closer is c's intensity to the object intensity, and the better c is connected to object O. (Since the range of ϕ is $\langle L, \preceq\rangle = \langle[0,\infty], \geqslant\rangle$, the notion of "$\preceq$-stronger" translates into "smaller in the $\leqslant$sense.") It is also convenient, for facilitating a definition of the object feature based affinity for multiple objects, to rescale this formula to $\bar\varphi(c) = |f(c)-m|/\sigma$. (This is related to the Mahalanobis distance [16].)

Now, one may attempt to define the strength of a path $p = \langle c_1, \ldots, c_l\rangle$ as

$$\mu_{\bar\varphi}(p) = \max_{i=1,\ldots,l} \bar\varphi(c_i) \tag{3}$$

and the connectivity measure as $\mu_\varphi(c,d) = \min_{p\in\mathscr{P}^C_{cd}} \mu_\varphi(p)$. (Once again, the use of inverse inequality $\geqslant$ as $\preceq$ makes the $\leqslant$-largest value to be the $\preceq$-smallest value.) However, since in this definition we do not assume that the consecutive spels in a path are adjacent, there is nothing local in this definition. In particular, if $f(c) = f(d) = m$, then $\mu_{\bar\varphi}(\langle c,d\rangle) = 0$ is not a good connectivity measure: the best possible connectivity in $\mu_{\bar\varphi}$-sense, $\mu_{\bar\varphi}(\langle c,d\rangle) = 0$,

means only that the intensities at both spels equal m, and it may still happen that such spels are spatially separated by spels with very different intensities; on the other hand, if distinct c and d are adjacent (next to each other), then the fact that $f(c) = f(d) = m$ is very informative — such spels are indeed perfectly connected.

The situation can be rescued if one considers only the paths from the family $\bar{\mathscr{P}}_{cd}$ of all paths from c to d in which the consecutive spels are *distinct* and adjacent. Then, for $c \neq d$, the formula

$$\mu_{\bar{\varphi}}(c,d) = \min_{p \in \bar{\mathscr{P}}_{cd}} \mu_{\bar{\varphi}}(p) \tag{4}$$

agrees with our intuition and with the formula for μ_ϕ defined below. (See (7).) So, why can we not use formula (4) as a definition of μ_ϕ? Although we could, there are two inconveniences connected with this approach: first we would need to replace $\mathscr{P}^C_{cd}$ with $\bar{\mathscr{P}}_{cd}$; second, the value of $\mu_{\bar{\varphi}}(p)$ is not defined by using any affinity function (the pseudo-affinity $\bar{\varphi}(c_i)$ used in (3) cannot be treated as affinity, since it is a function of one variable), so the general results on the FC theory could not be applied to a connectivity measure so defined. Moreover, connectivity formula (4) carries some other dangers, which we will mention below.

Thus, we will define ϕ properly, as a function on the pairs $\langle c,d\rangle$ of spels. We like to define ϕ in such a way that, for every $p \in \bar{\mathscr{P}}_{cd}$, the strength $\mu_\phi(p)$ of p equals $\mu_{\bar{\varphi}}(p)$. To ensure this, for distinct adjacent c and d, $\phi(c,d)$ must be defined as $\max\{\bar{\varphi}(c),\bar{\varphi}(d)\} = \max\{|f(c)-m|,|f(d)-m|\}/\sigma$. Thus, in general, we define $\phi(c,d) = \max\{\bar{\varphi}(c),\bar{\varphi}(d)\}/\alpha(c,d)$; that is,

$$\phi(c,d) = \begin{cases} 0 & \text{for } c = d \\ \max\{|f(c)-m|,|f(d)-m|\}/\sigma & \text{for } \|c-d\| = 1 \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

Clearly function ϕ is an affinity function in the sense of general affinity function defined in Section 1. Moreover,

$$\mu_\phi(p) = \max_{i=1,\ldots,l} \bar{\varphi}(c_i) \text{ for every } p = \langle c_1,\ldots,c_l\rangle \in \bar{\mathscr{P}}_{cd}, \tag{6}$$

since $\mu_\phi(p) = \max_i \max\{\bar{\varphi}(c_i),\bar{\varphi}(c_{i+1})\} = \max_i \bar{\varphi}(c_i)$. In particular, by (3), $\mu_\phi(p) = \mu_{\bar{\varphi}}(p)$ for every $p \in \bar{\mathscr{P}}_{cd}$.

Notice also that for every $c \neq d$ function μ_ϕ agrees with $\mu_{\bar{\varphi}}$:

$$\mu_\phi(c,d) = \mu_{\bar{\varphi}}(c,d), \tag{7}$$

since $\mu_\phi(c,d) = \min_{p\in\mathscr{P}^C_{cd}} \mu_\phi(p) = \min_{p\in\bar{\mathscr{P}}_{cd}} \mu_\phi(p) = \min_{p\in\bar{\mathscr{P}}_{cd}} \mu_{\bar{\varphi}}(p) = \mu_{\bar{\varphi}}(c,d)$. Here the first and the last equations come from (1) and (4), respectively. The third equation follows from the above argument, while the second one is justified by the fact that for every $q \in \mathscr{P}_{cd}$ either $\mu_\phi(q) = \infty$ (when q contains non-adjacent consecutive spels) or $\mu_\phi(q) = \mu_\phi(p)$ for $p \in \bar{\mathscr{P}}_{cd}$ obtained from q by collapsing all constant consecutive subsequences of q to a single occurrence of the repeated value.

Note that, in reference [33], for distinct adjacent spels c and d the authors define $\phi(c,d)$ as $|\frac{f(c)+f(d)}{2} - m|$ in place of $\max\{\bar{\varphi}(c),\bar{\varphi}(d)\}$. Although this carries similar intuitions, the averaging of the values of $f(c)$ and $f(d)$ loses information on how far the intensity of each spel is from m. For example, if $f(c) = m + r$ and $f(d) = m - r$ for some $r > 0$, then $|\frac{f(c)+f(d)}{2} - m| = 0$ and $\mu_\phi(\langle c,d\rangle)$ associated with such affinity equals 0, which does not satisfy (6) and is counterintuitive for large values of r. Another difficulty with affinity defined as $\kappa(c,d) = |\frac{f(c)+f(d)}{2} - m|$ is shown in Fig. 1. Object $P^\kappa_{s,6}$ delineated with κ and $s = c_1$ includes spels c_2, c_3, c_4, c_5, but no other spels adjacent to c_5. (The intensity averages of the consecutive spels in the path $\langle c_1,c_2,c_3,c_4,c_5\rangle$ are, respectively, 37.5, 42.5, 40, 45, that is, closer to $m = 40$ than $\theta = 6$. It does not include any other spel c adjacent to c_5, since for such c the average $\frac{f(c)+f(c_5)}{2} = 60$ is $20 > \theta$ units from m.) Both including the spels c_3, c_4, c_5 in the object as well as, after including c_5, excluding other spels adjacent to c_5 defies intuitions behind the object feature based affinity. Notice also that, the object $P^\phi_{s,6}$ delineated with ϕ does not include c_3, since $\phi(c_2,c_3) = \max\{|35-40|,|50-40|\} = 10 > 6 = \theta$.

Once again, we can replace $\phi(c,d)$ with $g_\sigma(\phi(c,d))$ for some Gaussian-like function to get an equivalent affinity in the standard form. In particular, for $g_\sigma(x) = e^{-x^2/\sigma^2}$ this leads to $\bar{\varphi}(c) = e^{-\frac{(f(c)-m)^2}{\sigma^2}}$, one of the formulas used in [33]. (See also [38,29,32,45].)

The difference between ϕ and κ that was illustrated in Fig. 1 above is also demonstrated on the 2D CT slice of a human knee in the following Fig. 2. Fig. 2(b) and (c) show, respectively, the connectivity image and AFC object corresponding to the affinity $g_\sigma \circ \phi$ which, for distinct adjacent c and d, is equal to $g_\sigma(\phi(c,d)) = \min\{e^{-\frac{(f(c)-m)^2}{\sigma^2}}, e^{-\frac{(f(c)-m)^2}{\sigma^2}}\}$. Fig. 2(d) and (e) are similar images obtained for the affinity $\hat{\kappa}$ defined, for distinct adjacent c and d, as $g_\sigma\left(\frac{f(c)+f(d)}{2} - m\right)$. The object shown in (f), generated with affinity $\hat{\kappa}$, is slightly bigger than that for $g_\sigma \circ \phi$, shown in (c). Fig. 2(f) shows the symmetric difference between these two segmentation results.

The object feature based connectivity measure of one object has also a nice topographical map interpretation. For understanding this, consider a modified scene $\bar{\mathscr{C}} = \langle C, |f(\cdot) - m|\rangle$ (called *membership scene* in [38]) as a topographical map. Then the number $\mu_\phi(c,d)$ represents the lowest possible elevation (in $\bar{\mathscr{C}}$) which one must reach (a mountain pass) in order to get from c to d, where each step is on a location from C and is of unit length. Notice that

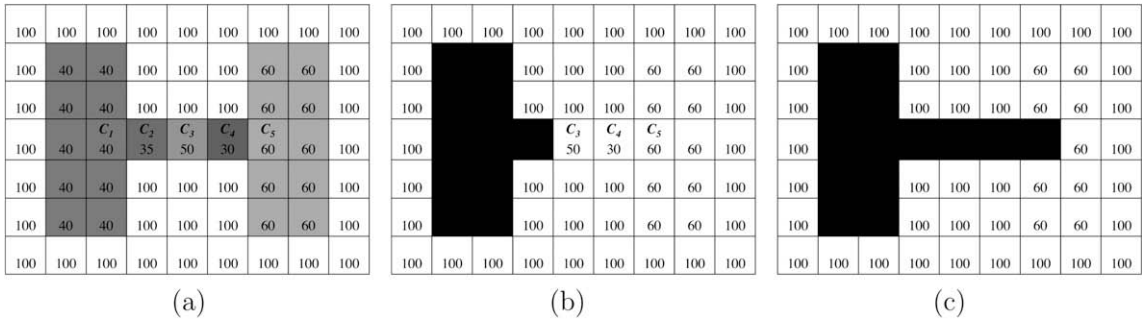

Fig. 1. (a) A schematic scene with each rectangular cell representing a single spel. A number in each spel indicates its intensity. We delineate an object indicated by a seed $s = c_1$, assuming that its average intensity is $m = 40$. We also assume $\sigma = 1$. In (b) the shaded area depicts object $P^\phi_{s,6}$ (i.e., with $\theta = 6$) delineated with the affinity ϕ defined in (5). The region correctly excludes spel c_3, since the difference between its intensity and m exceeds threshold value $\theta = 6$. The shaded region in (c) represents object $P^\kappa_{s,6}$, where $\kappa(c,d) = |\frac{f(c)+f(d)}{2} - m|$. Not only it incorrectly leaks all the way to spel c_5, but it also abruptly stops there, after reaching an area of uniform intensity.

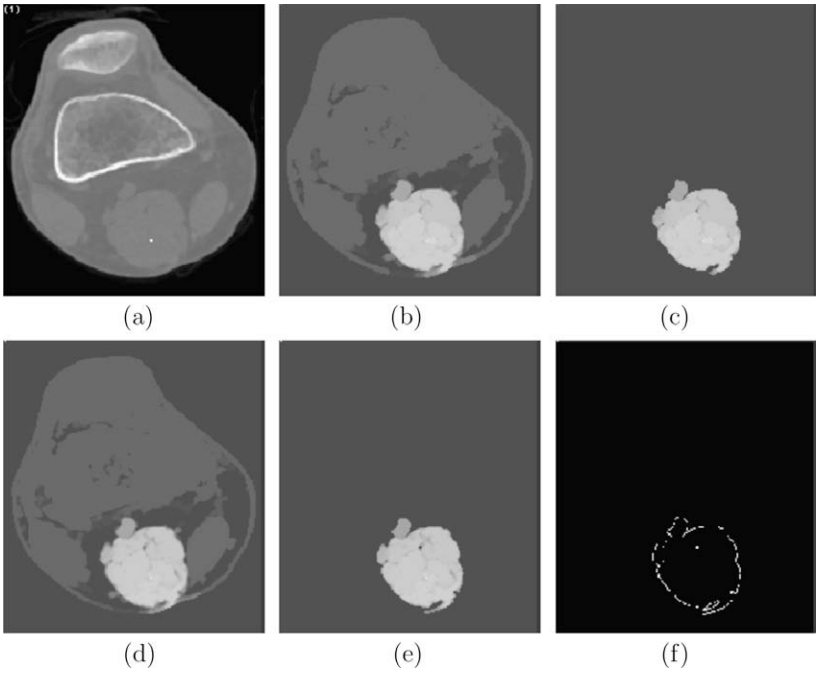

Fig. 2. (a) A 2D scene — a CT slice of a human knee, with an indicated seed. (b) and (c) Connectivity scene and an AFC object corresponding to the indicated seed and affinity $g_\sigma \circ \phi$. (d) and (e) Same as in (b) and (c) but for the affinity defined as $g_\sigma\left(\frac{f(c)+f(d)}{2} - m\right)$. (f) The symmetric difference between images (c) and (e).

$\mu_\phi(c,d)$ is precisely the *degree of connectivity* as defined by Rosenfeld [26–28]. (Compare also with [24], where it is used under the name *pass value*.) By the above analysis, we brought Rosenfeld's connectivity also into the affinity framework introduced by [38], particularly as another object feature component of affinity.

2.2.2. Object feature based affinity: case of multiple objects

The single object connectivity measure μ_ϕ can be useful in object definition only if we define it by using absolute connectedness definition, AFC. To find an object via RFC or IRFC methods, we need to have μ_ϕ defined for at least two objects. So, suppose that the scene consists of $n > 1$ objects with expected average intensities $m_1, \ldots, m_n$ and standard deviations $\sigma_1, \ldots, \sigma_n$, respectively. Then we have n different object feature based affinities $\hat{\phi}_i(c,d)$, defined for $c \neq d$ as $\max\{\bar{\varphi}_i(c), \bar{\varphi}_i(d)\}/\alpha(c,d)$, where $\bar{\varphi}_i(c) = \frac{|f(c)-m_i|}{\sigma_i}$, and their respective connectivity measures $\mu_{\hat{\phi}_i}$. We like to combine affinities $\hat{\phi}_i$ to get the cumulative object feature based affinity ϕ. (Obtaining a single affinity at the end becomes essential in order to fulfill the theoretical requirements of fuzzy connectedness. See [29,32].) But how to define such a ϕ? We will build our intuition for such a ϕ by assuming that each object O_i is generated by a single seed s_i with $f(s_i) = m_i$. Although this situation is not general, any discussion of this subject must include this important case. Therefore, we will decide on the form of a definition of ϕ in this situation first, and then argue that the notion we come up with has the desired properties without requiring any extra assumptions.

First note that σ_i's help us to compare different $\hat{\phi}_i$'s. Specifically, each number $\bar{\varphi}_i(c)$ measures the distance $|f(c) - m_i|$ of the image intensity $f(c)$ from the average intensity m_i of the ith object. However, if we like to compare the numbers $\bar{\varphi}_i(c)$ for different i's, we need to fix a reasonable measuring unit. The most natural measuring unit for $\bar{\varphi}_i$ is the associated standard deviation σ_i: with our definition $\bar{\varphi}_i(c) = \frac{|f(c)-m_i|}{\sigma_i}$, the equation $\bar{\varphi}_i(c) = K$ means that the intensity $f(c)$ at c is K standard deviations apart from m_i (like the Mahalanobis distance [16]). Then, equation $\bar{\varphi}_1(c) = \bar{\varphi}_2(c)$ caries the correct intuition: $f(c)$ is the same number of σ_i's apart from m_i for $i = 1$ and $i = 2$.

Now, by Eq. (6), if $p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathscr{P}}_{s_i c}$ and $s_i \neq c$, then the strength of the ith object connectivity between s_i and c on this path p is given by $\mu_{\hat{\phi}_i}(p) = \max_{t=1,\ldots,l} \bar{\varphi}_i(c_t)$. Similarly, the strength of the jth object connectivity between s_j and $c \neq s_j$ on a path $q = \langle d_1, \ldots, d_\ell\rangle \in \bar{\mathscr{P}}_{s_j c}$ is equal to $\mu_{\hat{\phi}_j}(q) = \max_{t=1,\ldots,\ell} \bar{\varphi}_j(d_t)$. Therefore, by the analysis given in the above paragraph, the ith object connectivity strength $\mu_{\hat{\phi}_i}(p)$ of p exceeds (in the $\preceq$ sense) the jth object connectivity strength $\mu_{\hat{\phi}_j}(q)$ of q provided $\mu_{\hat{\phi}_i}(p) = \max_{t=1,\ldots,l} \bar{\varphi}_i(c_t) < \max_{t=1,\ldots,\ell} \bar{\varphi}_j(d_t) = \mu_{\hat{\phi}_j}(q)$. So, by (4), c is better $\hat{\phi}_i$-connected to s_i than it is $\hat{\phi}_j$-connected to s_j precisely when $\mu_{\hat{\phi}_i}(s_i, c) < \mu_{\hat{\phi}_j}(s_j, c)$.

The key results of FC theory (see [29,32,13,30]) insure that the FC objects have the following nice and highly desirable properties.

- *Robustness:* If an FC delineated object P is indicated by a seed s and a spel t belongs to P (or its core, in case of IRFC), then the algorithm returns the same object when seed s is replaced by t.
- *Path Connectedness:* If an FC delineated object P is indicated by a seed s, spel t belongs to P, and a path p from s to t insures that t is in P (has the best strength), then every spel from p belongs to P.

To guarantee these properties, we need to arrive at one affinity defined over the whole scene. We shall examine this issue at the higher level in Section 3. In this section, our goal is to focus on a lower level, that is, to study how to combine the affinities $\hat{\phi}_i$ into a single object feature based affinity ϕ so that it preserves the information given by all affinities $\hat{\phi}_i$ to the fullest possible extent. (The reason why we cannot confidently use two different affinities and define an object via inequality $\mu_{\hat{\phi}_i}(s_i, c) < \mu_{\hat{\phi}_j}(s_j, c)$ is explained below.) In particular, since for every i, the value of $\mu_\phi(s_i, c)$ should approximate, as much as possible, the ith object connectivity strength between s_i and c, it would be most desirable if we could have insured that $\mu_\phi(s_i, c) = \mu_{\hat{\phi}_i}(s_i, c)$. In particular, we would like to insure that $\mu_\phi(s_i, c) < \mu_\phi(s_j, c)$ if and only if $\mu_{\hat{\phi}_i}(s_i, c) < \mu_{\hat{\phi}_j}(s_j, c)$. Unfortunately, we will see below that there is no way to have such a strong property, since in the process of combining $\hat{\phi}_i$'s we always lose some information. Nevertheless, at the very least, we should

insure that inequality $\mu_\phi(s_i,c) < \mu_\phi(s_j,c)$ never happens when $\mu_{\hat\phi_i}(s_i,c) \geqslant \mu_{\hat\phi_j}(s_j,c)$. This can be expressed as

$$\mu_\phi(s_i,c) < \mu_\phi(s_j,c) \quad \text{implies} \quad \mu_{\hat\phi_i}(s_i,c) < \mu_{\hat\phi_j}(s_j,c). \tag{8}$$

This implication represents the most fundamental property that we will impose on the definition of ϕ. In particular, in what follows we will define the object based affinity ϕ which satisfies (8) under some simple assumptions connecting each s_k with m_k. We will also argue (see Example 7 in Appendix) that other seemingly natural definitions of ϕ, like the one used in [29] (compare also [33]), do not satisfy this property.

Another way to look at property (8) is that, when $n = 2$, it insures that the RFC object $P^\phi_{s_i\{s_j\}}$ is contained in a set $O_{ij} = \{c \in C : \mu_{\hat\phi_i}(s_i,c) < \mu_{\hat\phi_j}(s_j,c)\}$. One may wonder whether we should consider sets O_{ij} (or their intersections $O_i = \bigcap_{j\neq i} O_{ij}$, if $n > 2$) as our objects. The argument against this consideration can be given at two levels. The simple one is that there is a very nice theory for the objects defined with a single connectivity measure and this theory does not extend, in general, to sets defined as in O_{ij}. (Of course, IRFC sets are also defined in this form, but the different connectivity measures used there have a very specific form.)

A slightly deeper argument is that the sets O_{ij} do not have nice properties. For example, it was proved in [29] that, unlike $P^\phi_{s_i\{s_j\}}$, the object O_{ij} has neither robustness nor path connectedness property. In fact, the failure of path property for $O_{1,2}$ can also be seen in a scene in which the spels s_1, s_2, c are on a consecutive path (with s_1 and c not adjacent), have respective intensities $0, 10, 18$, they are surrounded by spels with intensities equal to 300, and we have $m_1 = 0$, $m_2 = 10$, $\sigma_1 = 3$, $\sigma_2 = 1$ — we have $c \in O_{1,2}$ and $\mu_{\hat\phi_1}(s_1,c) = 6 < 8 = \mu_{\hat\phi_2}(s_2,c)$, while the unique $\hat\phi_1$-strongest path from s_1 to c goes through $s_2 \notin O_{1,2}$. Note also that the segmentation of this scene becomes the undesirable pair $\{O_{1,2}, O_{2,1}\}$ if the algorithm from paper [9] is applied to it.

The idea behind the formula for ϕ is to define $\phi(c,d)$ as the best among all numbers $\hat\phi_i(c,d)$. One possible choice for $\phi(c,d)$ is $\min_{i=1,\dots,n}\hat\phi_i(c,d)$. The problem with this choice is that we never know which value of $\hat\phi_i(c,d)$ was used to determine $\phi(c,d)$. Since the values of $\hat\phi_i(c,d) = \max\{\bar\varphi_i(c), \bar\varphi_i(d)\}/\sigma_i$ are the most valuable when this number is small and because difficulties occur when $\hat\phi_i(c,d) = \hat\phi_j(c,d)$ for $i\neq j$, we will eliminate the information in $\bar\varphi_i(c)$ when this value exceeds $\bar\varphi_j(c)$ for some j. This is made formal below.

For distinct $i,j \in \{1,\dots,n\}$, let $\delta^j_i \geqslant 0$ be the largest number with the property that $\frac{|x-m_i|}{\sigma_i} < \frac{|x-m_j|}{\sigma_j}$ for every $x \in (m_i - \delta^j_i, m_i + \delta^j_i)$. (If $\sigma_i = \sigma_j$, then δ^j_i is just half of the distance between m_i and m_j.) Thus, if $x^j_i \in \{m_i - \delta^j_i, m_i + \delta^j_i\}$ is between m_i and m_j, then for each $c \in C$

$$\bar\varphi_i(c) < \frac{|x^j_i - m_i|}{\sigma_i} = \frac{\delta^j_i}{\sigma_i} = \frac{|x^j_i - m_j|}{\sigma_j} < \bar\varphi_j(c) \tag{9}$$

provided $|f(c) - m_i| < \delta^j_i$. Let $\epsilon_i = \min_{j\neq i}\delta^j_i$ and $I_i = (m_i - \epsilon_i, m_i + \epsilon_i)$. Then intervals $I_i, i \in \{1,\dots,n\}$, are pairwise disjoint. Function φ_i is defined as a truncation of $\bar\varphi_i$ to the interval I_i, that is, by a formula

$$\varphi_i(c) = \varphi^{I_i}_i(c) = \begin{cases} \bar\varphi_i(c) & \text{for } f(c) \in I_i \\ \infty & \text{otherwise.} \end{cases}$$

Then $\varphi_i(c) < \infty$ implies $f(c) \in I_i = (m_i - \epsilon_i, m_i + \epsilon_i)$. Fig. 3 gives an example of the graphical representation for numbers δ^j_i and intervals I_i. For $c\neq d$ put $\phi_i(c,d) = \max\{\varphi_i(c), \varphi_i(d)\}/\alpha(c,d)$; that is, $\phi_i(c,d) = 0$ when $c = d$, $\phi_i(c,d) = \max\{\varphi_i(c), \varphi_i(d)\}$ for $||c - d|| = 1$, and $\phi_i(c,d) = \infty$ otherwise, and let

$$\phi(c,d) = \min_{i=1,\dots,n} \phi_i(c,d). \tag{10}$$

We define $\mu_\phi(p)$ for a path p and a connectivity measure μ^A_ϕ according to our general method. The following theorem shows that ϕ so

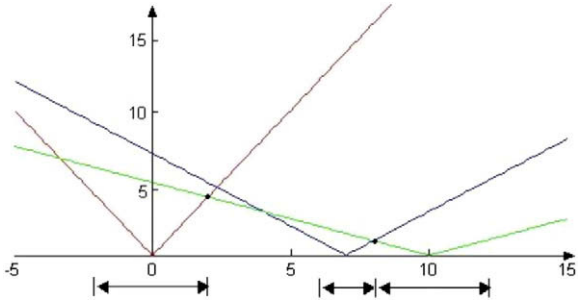

Fig. 3. The graphs of three functions $\bar\varphi_i$ with $m_1 = 0, m_2 = 7, m_3 = 10, \sigma_1 = 0.5, \sigma_2 = 1$, and $\sigma_3 = 2$. We have $\delta^3_1 = 2 < \delta^2_1$, leading to $I_1 = (-2, 2)$. Also, $\epsilon_2 = \delta^3_2 = 1$, so $I_2 = (6, 8)$ and $\epsilon_3 = \delta^2_3 = 2$ leading to $I_3 = (8, 12)$.

defined satisfies (8) we promised. The proof of this theorem and the associated machinery are provided in Section A of Appendix.

Theorem 3. *Fix $i \in \{1,\dots,n\}$ and $c, s_i, s_j \in C$ such that $f(s_i) \notin \bigcup_{k\neq i} I_k$. If $\mu_\phi(s_i,c) < \mu_\phi(s_j,c)$, then $\mu_{\hat\phi_i}(s_i,c) < \mu_{\hat\phi_k}(s_j,c)$ for every $k \in \{1,\dots,n\}$.*

The role of a seed s_i is not only to indicate an approximate position of an object but also to indicate its approximate average intensity m_i. This is the only way to insure that s_i indicates the correct object. Thus, if one allows the situation in which $s_i \in I_j$ for some $j\neq i$, then s_i would really represent j's object and $\mu_\phi(s_i,c)$ would be represented by $\mu_{\hat\phi_j}(s_i,c)$ rather than by $\mu_{\hat\phi_i}(s_i,c)$. Not surprisingly, in such a situation, the conclusion of the theorem cannot be expected. This explains our assumption $f(s_i) \notin \bigcup_{k\neq i} I_k$. In fact, we could as well assume $f(s_i) \in I_i$, as otherwise (i.e., when $f(s_i)$ belongs to no I_k) $\mu_\phi(s_i,c) = \infty$ for every c, so $\mu_\phi(s_i,c)$ caries no valuable information, and the conclusion of the theorem is satisfied in void.

Clearly, truncating each $\bar\varphi_i$ to $\varphi_i = \varphi^{I_i}_i$ is causing the loss of some information. In fact, the most common definition of ϕ used in the literature till now, see e.g. [29], coincides with ours if one drops the matter of truncation: define $\bar\phi(c,d) = \min_{i=1,\dots,n}\bar\phi_i(c,d)$, where $\bar\phi_i(c,d) = \max\{\bar\varphi_i(c), \bar\varphi_i(d)\}/\alpha(c,d)$ for $c\neq d$. Then $\mu_{\bar\phi}$ is defined as usual. Clearly, at the first glance it seems that affinity $\bar\phi$ is superior to its truncated version ϕ defined above and that the information truncation makes the ability to distinguish among objects weaker. Although, to some extent, this is a legitimate concern, it should be noted that the objects obtained with the use of $\bar\phi$ may be bigger than those obtained with the use of ϕ. However, since $\bar\phi$ is not required to satisfy (8), it is possible that a spel c is assigned to object $P^\phi_{s_i\theta}$ while it truly belongs to another object. (See Example 7.) Thus, under the circumstances, we believe that it is better to leave c unassigned to any object, rather than to run into the risk of assigning it to an incorrect object. (Nevertheless, in practical applications there is always some noise in the image and the errors should be expected. Therefore, for noisy images the benefit of removing some incorrect assignments by using truncated φ's may be of only small practical benefit.)

Another possible way for defining object feature based connectivity, $\mu_{\bar\phi}$, is to put $\bar\varphi(c) = \min_{i=1,\dots,n}\bar\varphi_i(c)$ and define it as in (3) and (4). Although $\mu_{\bar\varphi}$ is equal to $\mu_{\bar\phi}$ when $n = 1$, in general this is not the case. This is best seen in Example 6 in Appendix, which fully discredits $\mu_{\bar\varphi}$ as a valid definition of an object feature based connectivity measure. Example 7 shows that the motivational implication (8) fails for $\mu_{\bar\phi}$.

In summary, the proper choice of the object feature based affinity is a delicate mater. The natural requirement of the path connectedness property of delineated objects dictates the use of a single affinity function. If it is also desirable to completely ensure property (8) (to guarantee that the RFC object $P^\phi_{s_i\{s_j\}}$ is contained in a set $O_{ij} = \{c \in C : \mu_{\hat\phi_i}(s_i,c) < \mu_{\hat\phi_j}(s_j,c)\}$), then the truncated ver-

sion of ϕ (or its Gaussian modification) must be used. Nevertheless, in practical image segmentation tasks, perfection is an unachievable goal, due to different imperfections at image acquisition. This means that, in practice, the irregularity that the truncation is theoretically preventing, may appear any way. Therefore, in some applications, the use of untruncated version $\bar{\phi}$ of the object feature based affinity may be beneficial, especially taking into consideration that the examples as presented in Example 6 are not likely to be found in real life images and that $\bar{\phi}$ is easier to implement.

2.3. Homogeneity versus object feature based affinity

First note that the homogeneity based connectivity measure μ_ψ and the object feature based connectivity measure μ_ϕ, although related (as function f is related to its derivative f'), behave quite differently. For example, μ_ψ, unlike μ_ϕ, is not very sensitive to the slow background intensity variation often found in medical images as an artifact. To see this, imagine that the image consists of a long straight tube (say an artery) with the intensity of each spel in a tube around 10, and the intensity of each spel outside the tube around 20. Now, assume that a slow (spatially) changing artifact is applied to the image. This artifact is often multiplicative in nature. For simplicity, assume that it is additive and that it changes along the length of the tube from 0 to 20. Then, the beginning of the tube will have intensity around 10, while its end will have a value around 30. Now, the artifact we added changes little the value of μ_ψ, so the entire tube can still be easily obtained as $O_{s\theta}$ or O_{st} if one uses μ_ψ as a connectivity measure. On the other hand, if s is a seed located at the beginning of the tube and $O_{s\theta} = \{c \in C : \mu_{\phi_{f(s)}}(s,c) \leqslant \theta\}$ contains a spel t from the end of the tube, then $\theta \geqslant \mu_{\phi_{f(s)}}(s,t) \approx 20$. Therefore, $O_{s\theta}$ must contain also many spels outside the tube, since for any spel c outside the tube close to the beginning, we have $f(c) \approx 20$, so $\mu_{\phi_{f(s)}}(s,c) \approx 10 < \theta$.

On the other hand, if a scene $\mathscr{C}$ contains seeds s and t with $|f(s) - f(t)|$ being large, it still may happen that there is a long path p from s to t along which the intensity changes very slowly. Then $\mu_\psi(s,t) \leqslant \mu_\psi(p)$ is very small, which makes it nearly impossible to distinguish s and t by means of homogeneity based connectivity measure alone. However, since $\mu_\phi(s,t)$ is large, we can easily distinguish s and t with the help of object feature based connectivity measure.

As pointed out in [33], these two concepts — one related to homogeneity (a derivative $f\prime(c)$ concept) and another related to the intensity $f(c)$ — are fundamental to any segmentation methods that are based purely on information derived from the given image. In FC in particular, as illustrated above, both components are needed for effective segmentation. This is one of the reasons why we dealt with the theory relating to the two components separately. This naturally leads us to the next section which will study how these components may be utilized in the same FC segmentation algorithm.

3. How to combine different affinities?

In this section, we will discuss the issue of how to combine two or more different affinities of the sort described in the previous section into one affinity. We will also examine which parameters in the definitions of the combined affinity are redundant, in the sense that their change leads to an equivalent affinity.

3.1. Affinity combining methods

Assume that for some $k \geqslant 2$ we have affinity functions $\kappa_i : C \times C \to \langle L_i, \preceq_i \rangle$ for $i = 1, \ldots, k$. For example, we can have $k = 2$, $\kappa_1 = \psi$, and $\kappa_2 = \phi$. The most flexible way of combining all these affinities into a single one is to put $\kappa(c,d) = \langle \kappa_1(c,d), \ldots, \kappa_k(c,d) \rangle$ and define an appropriate linear order $\preceq$ on $L = L_1 \times \cdots \times L_k$. To understand this formalism better, we will start with the following examples, which also constitute our practical approach to the affinity combining problem.

Example 4 (*Weighted averages*). Assume that all linear orderings L_i are equal to the same ordering $\langle L_0, \preceq_0 \rangle$ which is either $\langle [0,\infty], \geqslant \rangle$ or $\langle [0,1], \leqslant \rangle$ and fix a vector $\mathbf{w} = \langle w_1, \ldots, w_k \rangle$ of numbers from $[0,1]$ (weights) such that $w_1 + \cdots + w_k = 1$; we allow a weight w_i to be equal to 0 (meaning "ignore influence of κ_i") assuming that $0 \cdot \infty = 0$ and $0^0 = \infty^0 = 1$.

Additive average: Let $h^{add}_{\mathbf{w}}(\mathbf{a}) = w_1 a_1 + \cdots + w_k a_k$ for $\mathbf{a} = \langle a_1, \ldots, a_k \rangle \in (L_0)^k$. If $\mathbf{a} \leqslant^{add}_{\mathbf{w}} \mathbf{b} \iff h^{add}_{\mathbf{w}}(\mathbf{a}) \preceq_0 h^{add}_{\mathbf{w}}(\mathbf{b})$, then affinity $\kappa : C \times C \to \langle L, \leqslant^{add}_{\mathbf{w}} \rangle$ is equivalent to $\kappa^a_{\mathbf{w}} : C \times C \to \langle L_0, \preceq_0 \rangle$ defined as $\kappa^a_{\mathbf{w}}(c,d) = h^{add}_{\mathbf{w}}(\kappa_1(c,d), \ldots, \kappa_k(c,d))$. Note that, for $k = 2$, the affinity $\kappa^a_{\mathbf{w}} = w_1\kappa_1 + w_2\kappa_2$ has been already considered in [33].

Multiplicative average: Let $h^{mul}_{\mathbf{w}}(\mathbf{a}) = a_1^{w_1} \cdots a_k^{w_k}$ for $\mathbf{a} = \langle a_1, \ldots, a_k \rangle \in (L_0)^k$. If $\mathbf{a} \leqslant^{mul}_{\mathbf{w}} \mathbf{b} \iff h^{mul}_{\mathbf{w}}(\mathbf{a}) \preceq_0 h^{mul}_{\mathbf{w}}(\mathbf{b})$, then $\kappa : C \times C \to \langle L, \leqslant^{mul}_{\mathbf{w}} \rangle$ is equivalent to $\kappa^m_{\mathbf{w}} : C \times C \to \langle L_0, \preceq_0 \rangle$ defined as $\kappa^m_{\mathbf{w}}(c,d) = h^{mul}_{\mathbf{w}}(\kappa_1(c,d), \ldots, \kappa_k(c,d))$. For $k = 2$, the affinity $\kappa^m_{\mathbf{w}} = \kappa_1^{w_1}\kappa_2^{w_2}$ has been already considered in [33].

Recall that the *lexicographical order* $\leqslant_{lex}$ on $L = L_1 \times \cdots \times L_k$ is defined for distinct $\mathbf{a} = \langle a_1, \ldots, a_k \rangle, \mathbf{b} = \langle b_1, \ldots, b_k \rangle \in L$ as

$$\mathbf{a} <_{lex} \mathbf{b} \iff a_i \prec_i b_i, \quad \text{where} \quad i = \min\{j : a_j \neq b_j\}.$$

Example 5 (*Lexicographical order*). Affinity function affinity $\kappa_{lex} : C \times C \to \langle L, \leqslant_{lex} \rangle$ establishes the strongest possible hierarchy between the coordinate affinities κ_i: in establishing whether $\kappa_{lex}(a,b) \leqslant_{lex} \kappa_{lex}(c,d)$, the values $\kappa_i(a,b)$ and $\kappa_i(c,d)$ are completely irrelevant, unless $\kappa_j(a,b) = \kappa_j(c,d)$ for all $j < i$, in which case $\kappa_i(a,b) \prec_i \kappa_i(c,d)$ implies $\kappa_{lex}(a,b) <_{lex} \kappa_{lex}(c,d)$.

Notice that κ_{lex} cannot be expressed in the form of $h(\kappa_1, \ldots, \kappa_k)$ for any continuous function on $[0,1]^k$ or on $[0,\infty]^k$. In what follows, we will restrict our attention to the situation when $k = 2$. In this case the lexicographical order is defined as $\langle a_1, a_2 \rangle <_{lex} \langle b_1, b_2 \rangle$ provided either $a_1 \prec_1 b_1$ or $a_1 = b_1$ and $a_2 \prec_2 b_2$.

The lexicographical order approach is quite appealing in case when $\kappa_1 = \psi$ and $\kappa_2 = \phi$ as the decision whether $\mu_\kappa(c,s) \leqslant_{lex} \mu_\kappa(c,t)$ becomes hierarchical in nature: if $\mu_\psi(c,s) < \mu_\psi(c,t)$, then $\mu_\kappa(c,s) \leqslant_{lex} \mu_\kappa(c,t)$ independent of the values of $\mu_\phi(c,s)$ and $\mu_\phi(c,t)$; only when the homogeneity based connectivity measure cannot decide the matter, that is, when $\mu_\psi(c,s) = \mu_\psi(c,t)$, we decide on the direction of $\leqslant_{lex}$ between $\mu_\kappa(c,s)$ and $\mu_\kappa(c,t)$ based on the direction of $\preceq_2$ between $\mu_\phi(c,s)$ and $\mu_\phi(c,t)$. Thus, we treat the homogeneity based connectivity measure as dominant over object feature based connectivity measure. (Note that this will become reversed if $\kappa_1 = \phi$ and $\kappa_2 = \psi$.) However, there is more to it. If $\mu_\psi(c,s) = \mu_\psi(c,t)$, then we decide about $\mu_\kappa(c,s) \leqslant_{lex} \mu_\kappa(c,t)$ only along the paths $p \in \mathscr{P}_{cs}$ and $q \in \mathscr{P}_{ct}$ with $\mu_\psi(p) = \mu_\psi(q) = \mu_\psi(c,s)$. Only to these paths we apply μ_ϕ measure. Thus, we use the object based feature measure in this schema in a considerably more sophisticated way than what is suggested by the threshold-like interpretation described in Section 2. It should be also clear that, if we agree that we should give priority to homogeneity based connectivity measure in the RFC approach, this is precisely the way we should proceed.

Next, consider the coordinate order preserving property of the combined affinity $\kappa(c,d) = \langle \kappa_0(c,d), \kappa_1(c,d) \rangle$:

(C) for every $i = 0, 1$ and c, d, c', d', if $\kappa_i(c,d) = \kappa_i(c',d')$, then $\kappa(c,d) \prec \kappa(c',d') \iff \kappa_{1-i}(c,d) \prec_{1-i} \kappa_{1-i}(c',d')$.

Property (C) says that if one of the coordinate affinities does not distinguish between two pairs of spels, then the combined affinity

decides on this pair according to the other coordinate affinity. This seems to be a very natural and desirable property. It is easy to see that, by design, the κ_{lex} affinity has this property. However, in general, (C) is not satisfied for the multiplicative average $\kappa^m_{\mathbf{w}}$: if $\kappa_i(c,d)=\kappa_i(c',d')=0$, then $\kappa^m_{\mathbf{w}}(c,d)=\kappa^m_{\mathbf{w}}(c',d')=0$ independently of the value of κ_{1-i} on these pairs. A similar problem arises for $\kappa_i(c,d)=\kappa_i(c',d')=\infty$, although for $\kappa_i(c,d)=\kappa_i(c',d')\in(0,\infty)$ the equivalence from (C) is satisfied. This creates a problem especially with the truncated version of the object-feature based affinity, since, in this case, affinity is equal to ∞ for many adjacent pairs of spels. Condition (C) also fails for $\kappa^{add}_{\mathbf{w}}$ when $\kappa^{add}_{\mathbf{w}}(c,d)=\kappa^{add}_{\mathbf{w}}(c',d')=\infty$, although for $\kappa^{add}_{\mathbf{w}}(c,d)=\kappa^{add}_{\mathbf{w}}(c',d')<\infty$ the equivalence is satisfied. In particular, (C) holds for $\kappa^{add}_{\mathbf{w}}$ formed with the coordinate affinities with range $\langle[0,1],\leqslant\rangle$.

Notice that the property (C) fails only if we allow values 0 or ∞ in the range of κ's. Therefore, if we like to insure (C), we can always replace κ_i's with their equivalent forms with the range in $(0,\infty)$ (e.g. by replacing ∞ with some large but finite number), which will insure (C) in the above described combining methods.

3.2. Counting essential parameters

Next, let us turn our attention to the determination of the number of parameters essential in defining the affinities presented in the previous section. We will consider here only the parameters explicitly mentioned there, since any implicit parameters (like the parameters for getting intensity function from the actual acquisition data) could not be handled by the methods we will employ. This exercise is useful in tuning the FC segmentation methods to different applications. It is also useful in comparing these methods with others. Recall that for a $\sigma\in(0,\infty)$ we defined $g_\sigma:[0,\infty]\to[0,1]$ by $g_\sigma(x)=e^{-x^2/\sigma^2}$.

Homogeneity based affinity, ψ, is defined as $\psi(c,d)=|f(c)-f(d)|$ for $\|c-d\|\leqslant 1$ and $\psi(c,d)=\infty$ otherwise. As such, there are no parameters in this definition. In its standard form, $g_\sigma\circ\psi$, the parameter σ is redundant, since, by Proposition 1, $g_\sigma\circ\psi$ is equivalent to ψ. This beautiful characteristic says that FC partitioning of a scene utilizing homogeneity based affinity is an inherent property of the scene and is independent of any parameters, beside a threshold in case of AFC.

Object feature based affinity for one object, ϕ, is defined by a formula $\phi(c,d)=\max\{|f(c)-m_1|,|f(d)-m_1|\}/\sigma_1$ for $\|c-d\|=1$, $\phi(c,d)=0$ for $c=d$, and $\phi(c,d)=\infty$ otherwise. From the two parameters, m_1 and σ_1, present in this definition, only m_1 is essential. Parameter σ_1 is redundant, since function $\sigma_1\cdot\phi$ is independent of its value and $\sigma_1\cdot\phi$ is equivalent to ϕ, as $\sigma_1\cdot\phi=h\circ\phi$ for an increasing function $h(x)=\sigma_1 x$. As before, the standard form $g_\sigma\circ\phi$ of ϕ is equivalent to it, so the only essential parameter in the definition of $g_\sigma\circ\phi$ is the number m_1.

Object feature based affinity for multiple objects. Suppose that the affinity is defined for $n>1$ different objects for which $\bar m=\langle m_1,\dots,m_n\rangle$ and $\bar\sigma=\langle\sigma_1,\dots,\sigma_n\rangle$ represent their average intensities and standard deviations, respectively. Let $\phi_{\bar m,\bar\sigma}$ represent the object feature affinity in its main truncated form and let $\bar\phi_{\bar m,\bar\sigma}$ stand for its untruncated version. (See Section 2.2.2.) Then $\sigma_1\cdot\phi_{\bar m,\bar\sigma}=\phi_{\bar m,\bar\delta}$ and $\sigma_1\cdot\bar\phi_{\bar m,\bar\sigma}=\bar\phi_{\bar m,\bar\delta}$, where $\bar\delta=\langle 1,\delta_2,\dots,\delta_n\rangle$ and $\delta_i=\sigma_i/\sigma_1$. Since $\sigma_1\cdot\phi_{\bar m,\bar\sigma}$ is equivalent to $\phi_{\bar m,\bar\sigma}$, affinity $\phi_{\bar m,\bar\sigma}$ depends essentially only on $2n-1$ parameters $m_1,\dots,m_n,\delta_2,\dots,\delta_n$. The same is true for its standard form $g_\sigma\circ\phi_{\bar m,\bar\sigma}$ as well as for their untruncated counterparts $\bar\phi_{\bar m,\bar\sigma}$ and $g_\sigma\circ\bar\phi_{\bar m,\bar\sigma}$.

In what follows, we will assume that $w,\sigma,\tau\in(0,1)$ and that ϕ is equal to either $\phi_{\bar m,\bar\delta}$ or to $\bar\phi_{\bar m,\bar\delta}$, so it has $2n-1$ essential parameters. Then we have the following methods of combining, denoted m1–m5, for homogeneity and object feature based affinities.

- m1 The additive average $\kappa=(1-w)\psi+w\phi$ of ψ and ϕ has $2n$ parameters. It is equivalent to $\psi+x\phi$, where $x=\frac{w}{1-w}\in(0,\infty)$. Note that if ϕ is replaced by an equivalent affinity $\sigma_1\phi$, then the resulting average affinity $(1-w)\psi+w\sigma_1\phi$ is also equivalent to $\psi+x\phi$ with $x\in(0,\infty)$. Note also that κ does not satisfy property (C), unless we insure that ψ and ϕ admit no ∞ value.
- m2 The additive average $\kappa=(1-w)g_\sigma\circ\psi+wg_\tau\circ\phi$ of $g_\sigma\circ\psi$ and $g_\tau\circ\phi$ has $2n+2$ essential parameters. Since $\kappa=e^{\ln(1-w)-\psi^2/\sigma^2}+e^{\ln w-\phi^2/\tau^2}$, this operation strangely mixes additive and multiplicative modifications of ψ and ϕ. The additional two parameters, σ and τ, are of importance in this mix. This affinity does satisfy property (C).
- m3 The multiplicative average $\kappa=\psi^{(1-w)}\phi^w$ of ψ and ϕ has $2n$ parameters and it is equivalent to $\psi\phi^x$, where $x=\frac{w}{1-w}\in(0,\infty)$, as $\kappa=(\psi\phi^x)^{1-w}$. If ϕ is replaced by an equivalent affinity $\sigma_1\phi$, then the resulting average $(\psi\sigma_1^x\phi^x)^{1-w}$ is also equivalent to $\psi\phi^x$ with $x\in(0,\infty)$, since function $h(t)=(\sigma_1^x t)^{1-w}$ is increasing as a composition of two increasing functions. This κ does not satisfy property (C), unless we insure that ψ and ϕ admit no 0 and ∞ values.
- m4 The multiplicative average $\kappa=(g_\sigma\circ\psi)^{(1-w)}(g_\tau\circ\phi)^w$ of $g_\sigma\circ\psi$ and $g_\tau\circ\phi$ has $2n+2$ parameters, but only $2n$ of them are essential. This is so since $\kappa=(e^{-\psi^2/\tau^2})^{1-w}(e^{-\phi^2/\sigma^2})^w=(e^{-\psi^2-x\phi^2})^{(1-w)/\tau^2}$, where $x=\frac{\tau^2}{\sigma^2}\frac{w}{1-w}\in(0,\infty)$, is equivalent to $\psi^2+x\phi^2$. The same is true if ϕ is replaced by $\sigma_1\phi$. This κ does not satisfy property (C), unless we insure that ψ and ϕ admit no ∞ value.
- m5 There are only two essential possibilities for lexicographical order of ψ and ϕ: $\langle\psi,\phi\rangle$ and $\langle\phi,\psi\rangle$, even if we allow replacement of each of the coordinate affinities by any of their equivalent forms, including but not restricted to $g_\sigma\circ\psi$ and $\sigma_1\phi$, $g_\tau\circ\phi$, or $g_\tau\circ(\sigma_1\phi)$. This follows from Proposition 1, since for any pair $\langle\psi^*,\phi^*\rangle$ such that ψ^* is equivalent to ψ and ϕ^* is equivalent to ϕ, there are strictly monotone functions g and h such that $\psi^*=g\circ\psi$ and $\phi^*=h\circ\phi$, and then $\langle\psi^*,\phi^*\rangle=\langle g,h\rangle\circ\langle\psi,\phi\rangle$, so $\langle g,h\rangle$ establishes the equivalence of $\langle\psi,\phi\rangle$ and $\langle\psi^*,\phi^*\rangle$.

4. Affinity functions used in the literature

In this section, we present a short review of the literature pertinent to the above discussion. In particular, we will emphasize different affinity functions used in the published papers as well as any generalizations of the FC algorithms.

Papers coming from our group (e.g. [38,29,33]) utilized the affinities described in (m2) and (m4), although the object feature based affinity component used in [33] was defined in format discussed in Section 2.2.2, which is slightly different from our recommended format. In addition, the affinity components used in Section 2.2.2 were scale-based, which is essentially (but not precisely) equal to a non-scale-based affinity applied to an appropriately filtered version of the intensity function.

Paper [44] interestingly utilizes the atlas based prior knowledge on the image to influence the FC image segmentation. To achieve this, the object feature and homogeneity feature affinity components are modified with the probability distribution P_k and the difference probability distribution DP_k functors, respectively. The resulting components were combined into final affinity via the (m3) method with weight $w=1/2$. Prior knowledge is also incorporated to FC segmentations as presented in [19].

In [25] the authors use the AFC algorithm with the following modification of κ defined according to the (m2) schema. First, for every principal direction $\vec r$ of the scene the authors choose some

modification coefficient $m(\vec{r})$ and they modify the standard homogeneity based affinity $\psi(c,d)$ according to the coefficient $m(\vec{cd})$. Although the precise modification is not specified in [25], it seems that they use $\psi^*(c,d) = m(\vec{cd}) \cdot \psi(c,d)$ as a modified homogeneity based affinity. Note that ψ^* is not equivalent to ψ, unless all coefficients $m(\vec{r})$ are equal. This may be a good approach for images with constant slow varying intensity change in one direction. Further on, they apply the weighted average to $g_{\sigma_1} \circ \psi^*$ and $g_{\sigma_2} \circ \varphi$ with weights w_1 and w_2 varying, depending on the intensity values at the spels c and d. Once again, the obtained modification is, in general, significant. However, the justification for the specific formula for w_i's is not provided in the paper.

In paper [1], the fuzzy connectedness approach, used with the affinity defined via (m2) format, is combined with the artificial neural network approach.

In paper [18] (see p. 465), the authors employ different affinity for each object to be delineated — a modified version of the single object feature based affinity, in its Gaussian modified form. These affinities are not combined into a single affinity and the resulting segmentations do have the path connectedness property discussed in Section 2.2. If used with the same affinity for all objects, the result of the algorithm from [18] is identical with that from the IRFC algorithm [31,32,13] (after reassignment of not-uniquely-assigned spels to non-assigned status). The same approach and affinity functions (see p. 67) are used in [9], see p. 67. (Note, that the comparison with RFC and IRFC presented in [9, section 6] is incorrect, since the paper uses incorrect definition of RFC objects: the inequality in formula (17) should have been strict!) Paper [8] uses affinity defined by formula (m2) with $w = 1/2$.

Paper [35] uses the following affinities: (1) In the background, a shifted version of homogeneity based affinity. The shift is redundant, according to our theory. (2) In the foreground, a directional version of the object-feature-based affinity, which is only a small adjustment of the standard affinity as we use. The authors write about affinities: "*The exact values turned out to be not very critical — the segmentation result is nearly identical over a relatively wide range of μ_0 and σ*," which, in case of σ is clear, as it is a redundant parameter according to our results. Interestingly, the paper does not explain as to how to decide the areas in which to use foreground affinity and background affinity.

Paper [4] proves that, in a general setting, the watershed and FC segmentation algorithms are equivalent. They do not restrict affinity functions to any specific format. A discussion of FC methods, used with affinity defined via (m2), and its practical comparison with watershed method is also present in [15].

Other applications of fuzzy connectedness can be found in: [17] (no specification of affinity), [5] (a combination of ψ and ϕ is used, the details are missing), [43] (used with standard (m2) defined affinity), [41,40,22] (use multiple affinities for each segmentation), [42] (uses (m4)-defined affinity), [21,2] (the homogeneity based affinity component is shifted by its mean value), [3] (uses vectorial affinity).

5. Concluding remarks

Theorem 2 and Proposition 1 imply that, from the perspective of FC methodology, the only essential attribute of an affinity function is its order. In particular, many transformations (like Gaussian) of the natural affinity definitions (like derivative-driven homogeneity based affinity) are of esthetic value only and do not influence the FC segmentation outcomes. Nevertheless, such transformations may play a role in combining different affinities, as can be seen in methods m1 and m2, since only one of them has the property (C).

The investigation presented in Section 3 shows which parameters in the definitions of homogeneity and object-feature based affinities, as well as their combinations, are of importance. In particular, we uncovered that many of the parameters in these definitions are of no consequence. Thus, for the tasks of application-driven optimization of the parameters, the number of parameters to be optimized is reduced. This aspect of setting values of parameters for segmentation methods is ridden with confusion. There are no scientific and systematic solutions for this problem. We indicated a solution in [39] which consisted of simultaneously minimizing false positive and false negative regions as a function of the parameter values. It makes sense, therefore, to first identify what the essential parameters of a segmentation method are, since such an attempt does not seem to have been made in the literature. This especially is relevant if we choose optimal parameter settings as mentioned by an optimization process.

In Section 2, we discussed two commonly used affinities, homogeneity and object-feature based, and interpreted them, respectively, as approximations of the directional derivatives and the distance from the object's average intensities. We also pointed out some theoretical deficiencies with the standard format of the object-feature based affinity in the case of multiple objects and proposed a truncated version of such affinity, which avoids theoretical difficulties, but loses some information along the way.

In Section 3, combining the results from the previous sections, we discussed five distinct ways of constructing full affinity functions (m1–m5). Our analysis of FC literature in Section 4 shows that, while forms denoted m1–m4 or their slight variations have been used in segmentation, form m5 is a novel strategy which remains to be evaluated.

We did not undertake any empirical evaluation studies in this paper. A theoretical study preceding such an evaluation becomes essential to understand what affinity forms are distinct, what are redundant, and what parameters are essential/redundant. This paper constitutes a first such step. Analysis similar to the one conducted in this paper for FC can be carried out for other frameworks, such as level sets [34], watersheds [6], and graph cuts [7].

Other possible ways of defining affinities. Note that in the definition of the "object feature based affinity," described in Section 2, the only prior knowledge of the object we used was the image intensity distribution of the object. More elaborate object feature affinity can use some other prior knowledge on the object(s) to be delineated. For example, the general shape of the object(s) can constitute such prior knowledge. If shape for the family of the object under consideration is modeled in a statistical manner [14], then we can consider a model based component of affinity $\beta(c,d)$ between c and d to be higher only if c and d are inside or close to the boundary of the mean shape, and smaller otherwise. A simple strategy based on the distance from mean shape boundary has been employed in [23] in an attempt to bring in prior shape information into FC. This discussion of how to properly define β and how to combine this with ψ and ϕ, however, requires fundamental investigation along the lines of this paper.

Also, as mentioned in Section 2.1, in the definition of the homogeneity based affinity it makes sense to use the notion of the gradient as a base for its definition, instead of the notion of the directional derivative. The discussion of the gradient induced homogeneity based affinity is a part of our forthcoming paper.

Appendix A

The following example fully discredits $\mu_{\bar{\varphi}}$ as a valid definition of an object feature based connectivity measure, while Example 7 shows that the motivational implication (8) fails for $\mu_{\bar{\varphi}}$.

Example 6. Let $\mathscr{C}$ be a binary scene with two intensities $m_2 > m_1 = 0$ and $\sigma_1 = \sigma_2 = 1$. We will consider $\mathscr{C}$ as a two object scene: for $i = 1,2$ object O_i consists of all spels with the intensity

m_i. Then for every $c,d \in \mathscr{C}$ we have $\mu_{\bar{\varphi}}(c,d) = 0$, while $\mu_{\bar{\phi}}(c,d) = m_2 > 0$ provided $f(c) \neq f(d)$.

Example 7. Let $p = \langle s_1, a, c, b, s_2\rangle$ be a sequence of spels in scene $\mathscr{C}$ in which only consecutive spels are adjacent and assume that $\langle 0, 8, 14, 20 \pm 13, 20\rangle$ represents their intensities, respectively. We also assume that any other spel in $\mathscr{C}$ adjacent to one listed in p has the intensity at least 80. Consider s_1 and s_2 as the seeds of objects O_1 and O_2 with averages $m_1 = f(s_1) = 0$ and $m_2 = f(s_2) = 20$ and standard deviations $\sigma_1 = \sigma_2 = 1$, respectively. Then $\mu_{\bar{\phi}}(s_1, c) = 12 < 13 = \mu_{\bar{\phi}}(s_2, c)$. However, $\mu_{\hat{\phi}_1}(s_1, c) = 14 > 13 = \mu_{\hat{\phi}_2}(s_2, c)$.

Proof. For adjacent s and t we have

$\bar{\phi}(s,t) = \min\{\max\{|f(s)|, |f(t)|\}, \max\{|20 - f(s)|, |20 - f(t)|\}\}$. So, $\bar{\phi}(s_1, a) = \min\{8, 20\} = 8$, $\bar{\phi}(a, c) = \min\{14, 12\} = 12$, and $\mu_{\bar{\phi}}(s_1, c) = \mu_{\bar{\phi}}(\langle s_1, a, c\rangle) = \max\{8, 12\} = 12$. Similarly $\bar{\phi}(s_2, b) = \min\{\max\{20, 20 \pm 13\}, \max\{0, |\pm 13|\}\} = 13$ and $\bar{\phi}(b, c) = \min\{\max\{20 \pm 13, 14\}, \max\{|\pm 13|, 6\}\} = 13$, which leads to $\mu_{\bar{\phi}}(s_2, c) = \mu_{\bar{\phi}}(\langle s_2, b, c\rangle) = \max\{13, 13\} = 13$. On the other hand, by property (6), we have $\mu_{\hat{\phi}_1}(s_1, c) = \mu_{\hat{\phi}_1}(\langle s_1, a, c\rangle) = \max\{0, 8, 14\} = 14$, while $\mu_{\hat{\phi}_2}(s_2, c) = \mu_{\hat{\phi}_2}(\langle s_2, b, c\rangle) = \max\{0, 13, 6\} = 13$. □

To understand this example better, let x_1^2 be as in (9); that is, such that $\frac{|x_1^2 - m_1|}{\sigma_1} = \frac{|x_1^2 - m_2|}{\sigma_2}$. So, in the setting of Example 7, we have $\delta_1 = \delta_2 = x_1^2 = 10$. The key characteristics of this example, that allows us to negate property (8), is that the intensities present in the path $q = \langle s_2, b, c\rangle$ (i.e., $\{f(s_2), f(b), f(c)\}$) are not in $I_2 = (m_2 - \epsilon_2, m_2 + \epsilon_2)$, despite the fact that $f(s_2), f(c) \in I_2$. Indeed, if the equation $\mu_{\hat{\phi}_2}(s_2, c) = \mu_{\hat{\phi}_2}(q)$ was satisfied with the intensities of all spels in q belonging to J_2, then (by Lemmas 8 and 9) we would have had $\mu_{\phi}(s_2, c) = \mu_{\hat{\phi}_2}(s_2, c) < \mu_{\phi}(s_1, c)$ and $\mu_{\phi}(s_2, c) = \mu_{\hat{\phi}_2}(s_2, c) < \mu_{\hat{\phi}_1}(s_1, c)$, which is in agreement with (8).

In case when $f(b) = 20 - 13 = 7$, all the intensities under question are between m_1 and m_2. Moreover, $f(b)$ is just barely below the threshold $m_2 - \delta_2$. (A slight modification of the example can make it arbitrarily close to $m_2 - \delta_2$.) The case when $f(b) = 20 + 13 = 33$ shows that it is not enough to stay within the interval $I = (m_2 - \delta_2, \infty)$, for which we have $\frac{|x - m_1|}{\sigma_1} > \frac{|x - m_2|}{\sigma_2}$ for every $x \in I$. Thus, the symmetry of I_i's around m_i's is essential in proving (8). In other words, the above discussion shows that, if ϕ is defined via the truncation technique, then the intervals I_i are the largest with which we can still prove property (8).

For the rest of the discussion, we will assume that $f(s_i) \in I_i$ for every i. What is the format of the objects generated with μ_ϕ under such assumption? First notice that in the case of the absolute connectedness definition we get

$$P^{\phi}_{s_i\theta} = \begin{cases} \{c \in C : \theta \geqslant \mu_{\phi_i}(s_i, c)\} & \text{for } \theta < \frac{\epsilon_i}{\sigma_i}, \\ \left\{c \in C : \frac{\epsilon_i}{\sigma_i} > \mu_{\phi_i}(s_i, c)\right\} & \text{for } \frac{\epsilon_i}{\sigma_i} \leqslant \theta. \end{cases}$$

In other words, $P^{\phi}_{s_i\theta}$ can be expressed in terms of the objects defined via AFC with respect to the affinity ϕ_i: $P^{\phi_i}_{s_i\theta} = \{c \in C : \theta \geqslant \mu_{\phi_i}(s_i, c)\}$. It is also easy to see that the ith object defined via RFC is the largest among the above objects: $\bigcap_{j\neq i} P^{\phi_i}_{s_i\{s_j\}} = \{c \in C : \frac{\epsilon_i}{\sigma_i} > \mu_{\phi_i}(s_i, c)\} = \bigcup_{\theta < \frac{\epsilon_i}{\sigma_i}} P^{\phi_i}_{s_i\theta}$. The same remains true for the IRFC case.

Since the above reduces RFC and IRFC objects defined with respect to ϕ to the unions of absolute connectedness objects $P^{\phi_i}_{s_i\theta}$ with respect to ϕ_i, one might wonder whether there is any sense at all in defining object feature based affinity ϕ. However, the full definition of ϕ is necessary in order to amalgamate ϕ with any other affinity, as discussed in Section 3.

The remainder of this paper is devoted to the proof of Theorem 3.

Lemma 8. *Let $p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathscr{P}}_{cs}$ and $i \in \{1, \ldots, n\}$. If $f(c_k) \in I_i$ for every $k \in \{1, \ldots, l\}$, then $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\epsilon_i}{\sigma_i}$.*

Proof. Notice that for every distinct $i, j \in \{1, \ldots, n\}$ and for every index $k \in \{1, \ldots, l-1\}$ we have $\phi_j(c_k, c_{k+1}) = \max\{\varphi_j(c_k), \varphi_j(c_{k+1})\} \geqslant \varphi_j(c_k) = \infty$, since $f(c_k) \notin I_j$. So, $\phi(c_k, c_{k+1}) = \min_{j=1,\ldots,n}\phi_j(c_k, c_{k+1}) = \phi_i(c_k, c_{k+1}) = \max\{\varphi_i(c_k), \varphi_i(c_{k+1})\} = \max\{\bar{\varphi}_i(c_k), \bar{\varphi}_i(c_{k+1})\} = \hat{\phi}_i(c_k, c_{k+1})$ and thus $\mu_\phi(p) = \max_{k=1,\ldots,l-1}\phi(c_k, c_{k+1}) = \max_{k=1,\ldots,l-1}\hat{\phi}_i(c_k, c_{k+1}) = \mu_{\hat{\phi}_i}(p)$. In addition, by (6), we have $\mu_{\hat{\phi}_i}(p) = \max_{k=1,\ldots,l}\bar{\varphi}_i(c_k)$. So, there is a $k \in \{1, \ldots, l\}$ for which $\mu_{\hat{\phi}_i}(p) = \bar{\varphi}_i(c_k) = \frac{|f(c_k) - m_i|}{\sigma_i} < \frac{\epsilon_i}{\sigma_i}$, since $f(c_k) \in I_i = (m_i - \epsilon_i, m_i + \epsilon_i)$. □

Lemma 9. *Let $p = \langle c_1, \ldots, c_l\rangle \in \bar{\mathscr{P}}_{cs}$ be such that $\mu_\phi(p) < \infty$. Then, for every $i \in \{1, \ldots, n\}$, the following conditions are equivalent.*

(a) $f(c) \in I_i$.
(b) $f(c_k) \in I_i$ for every $k \in \{1, \ldots, l\}$.
(c) $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\epsilon_i}{\sigma_i} < \bar{\varphi}_j(c)$ for every $j \neq i$.
(d) $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p)$.

Moreover, there is an $i \in \{1, \ldots, n\}$ for which each of these conditions holds.

Proof. Note that $\alpha(c_k, c_{k+1}) = 1$ for every $k = 1, \ldots, l-1$, since $p \in \bar{\mathscr{P}}_{cs}$. To see that $\mu_\phi(p) < \infty$ implies that $f(c) \in I_i$ for some i, note that $\infty > \mu_\phi(p) = \max_{k=1,\ldots,l-1}\phi(c_k, c_{k+1}) \geqslant \phi(c_1, c_2) = \min_{i=1,\ldots,n}\phi_i(c_1, c_2)$. So, there exists an $i \in \{1, \ldots, n\}$ with $\infty > \phi_i(c_1, c_2) = \frac{\max\{\varphi_i(c_1), \varphi_i(c_2)\}}{\alpha(c_1, c_2)}$. Thus, $\infty > \varphi_i(c_1)$ and $f(c) = f(c_1) \in I_i$.

"(a)⇒(b)" Let $Z = \{k \in \{1, \ldots, l\} : f(c_k) \in I_i\}$. Then (a) says that $1 \in Z$. We need to prove that $Z = \{1, \ldots, l\}$. By way of contradiction, assume that this is not the case and let $m \in \{1, \ldots, l\}$ be the smallest such that $m \notin Z$. Then $m > 1$, as $1 \in Z$, so $k = m - 1 \in Z$. In particular, $f(c_k) \in I_i$, so, for $j \in \{1, \ldots, n\}$,

$$\varphi_j(c_k) < \infty \Longleftrightarrow j = i.$$

Since $\infty > \mu_\phi(p) \geqslant \phi(c_k, c_{k+1}) = \min_{j=1,\ldots,n}\phi_j(c_k, c_{k+1})$, there exists a j with $\infty > \phi(c_k, c_{k+1}) = \phi_j(c_k, c_{k+1}) = \max\{\varphi_j(c_k), \varphi_j(c_{k+1})\}/\alpha(c_k, c_{k+1})$. In particular, $\infty > \varphi_j(c_k)$ and $\infty > \varphi_j(c_{k+1})$. Hence, from the first of these inequalities, we get $j = i$. Therefore, the second inequality becomes $\infty > \varphi_i(c_{k+1}) = \varphi_i(c_m)$, implying that $m \in Z$, contrary to our assumption. Thus, $Z = \{1, \ldots, l\}$ and (b) holds.

"(b)⇒(c)" By Lemma 8, we have $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\epsilon_i}{\sigma_i}$. Also, since $f(c) \in I_i = (m_i - \epsilon_i, m_i + \epsilon_i) \subseteq (m_i - \delta_i^j, m_i + \delta_i^j)$, condition (9) implies that $\frac{\epsilon_i}{\sigma_i} \leqslant \frac{\delta_i^j}{\sigma_i} < \bar{\varphi}_j(c)$.

Implication "(c)⇒(d)" is obvious.

"(d)⇒(a)" Condition (d) implies that $\mu_\phi(p) < \infty$. So, by the first remark, $f(c) \in I_j$ for some j. If $j = i$, we are done. So, by way of contradiction, assume that $j \neq i$. Then, using the implication "(a)⇒(c), " we have $\mu_\phi(p) = \mu_{\hat{\phi}_j}(p) < \frac{|x_j^k - m_j|}{\sigma_j} < \bar{\varphi}_k(c)$ for every $k \neq j$. In particular, for $k = i$ we get $\mu_\phi(p) < \bar{\varphi}_i(c) \leqslant \max_{k=1,\ldots,l}\bar{\varphi}_i(c_k) = \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$, a contradiction. □

Lemma 10. *Let $p, q \in \bar{\mathscr{P}}_{cs}$ and $i \in \{1, \ldots, n\}$. If $\mu_{\hat{\phi}_i}(q) \leqslant \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$, then $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q)$. In particular, if $\mu_{\hat{\phi}_i}(p) = \mu_\phi(p) = \mu_\phi(s, c)$, then also $\mu_{\hat{\phi}_i}(s, c) = \mu_{\hat{\phi}_i}(p) = \mu_\phi(s, c)$.*

Proof. Let us choose two paths, $p = \langle c_1, \ldots, c_l\rangle$ and $q = \langle d_1, \ldots, d_m\rangle$. Since we have $\mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \infty$ (remember that $\hat{\phi}_i$ is a non-truncated version of the object feature base affinity for the ith object) we can apply Lemma 9. Since Lemma 9(d) holds, so must also Lemma 9(c). Hence, by (6), for every index $k \in \{1, \ldots, m\}$ we

have $\frac{|f(d_k)-m_i|}{\sigma_i} = \bar{\varphi}_i(d_k) \leqslant \max_{j=1,\ldots,l}\bar{\varphi}_i(d_j) = \mu_{\hat{\phi}_i}(q) \leqslant \mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\epsilon_i}{\sigma_i}$. Thus, $f(d_k) \in I_i$ for every $k \in \{1,\ldots,m\}$. So, again by Lemma 9, we have $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q)$.

The additional part is obvious when $s = c$, since then $\mu_{\hat{\phi}_i}(p) = \mu_\phi(s,c) = 0 = \mu_{\hat{\phi}_i}(s,c)$. So, assume that $s \neq c$ and that $\mu_{\hat{\phi}_i}(p) = \mu_\phi(p) = \mu_\phi(s,c)$. Then, by (7), there exists a path $q \in \bar{\mathscr{P}}_{cs}$ with $\mu_{\hat{\phi}_i}(s,c) = \mu_{\hat{\phi}_i}(q)$. Then $\mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_i}(s,c) \leqslant \mu_{\hat{\phi}_i}(p) = \mu_\phi(p)$. So, by the first part, $\mu_\phi(q) = \mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_i}(s,c) \leqslant \mu_{\hat{\phi}_i}(p) = \mu_\phi(s,c) \leqslant \mu_\phi(q)$, proving that $\mu_{\hat{\phi}_i}(s,c) = \mu_{\hat{\phi}_i}(p)$. □

A.1. Proof of Theorem 3

Assume that $c, s_i, s_j \in C$ are as in the theorem, that is, such that $f(s_i) \notin \bigcup_{k\neq i} I_k$ and $\mu_\phi(s_i,c) < \mu_\phi(s_j,c)$. Fix a $k \in \{1,\ldots,n\}$. We need to show that $\mu_{\hat{\phi}_i}(s_i,c) < \mu_{\hat{\phi}_k}(s_j,c)$.

Note that $s_j \neq c$, since otherwise $\mu_\phi(s_i,c) < \mu_\phi(s_j,c) = 0$, which is impossible. Thus, by (7), there exists a $q = \langle d_1,\ldots,d_m\rangle \in \bar{\mathscr{P}}_{cs_j}$ such that $\mu_{\hat{\phi}_k}(s_j,c) = \mu_{\hat{\phi}_k}(q)$. Also, if $s_i = c$ then, by the definition (10) of μ_ϕ, we have $\mu_{\hat{\phi}_i}(s_i,c) = 0 = \mu_\phi(s_i,c) < \mu_\phi(s_j,c) \leqslant \mu_{\hat{\phi}_k}(s_j,c)$. Thus, we can assume that $s_i \neq c$. In particular, using the argument utilized in the proof of (7), we can show that there exists a $p = \langle c_1,\ldots,c_l\rangle \in \bar{\mathscr{P}}_{cs_i}$ such that $\mu_\phi(p) = \mu_\phi(c,s_i)$.

We have $\mu_\phi(p) = \mu_\phi(s_i,c) < \mu_\phi(s_j,c)$, so $\mu_\phi(p) < \infty$. Thus, by Lemma 9, there exists an i' for which $f(s_i) = f(c_l) \in I_{i'}$. Therefore $i' = i$, since $f(s_i) \notin \bigcup_{k\neq i} I_k$. So, by Lemma 9(c), $\mu_\phi(s_i,c) = \mu_\phi(p) = \mu_{\hat{\phi}_i}(p) < \frac{\epsilon_i}{\delta_i} < \bar{\varphi}_j(c)$ for every $j \neq i$. Also, by Lemma 10, we have $\mu_{\hat{\phi}_i}(s_i,c) = \mu_\phi(s_i,c)$.

Now, if $k \neq i$, then, by (9) and the above, $\mu_{\hat{\phi}_i}(s_i,c) \leqslant \mu_{\hat{\phi}_i}(p) < \bar{\varphi}_k(c) \leqslant \max_{r=1,\ldots,m}\bar{\varphi}_k(d_r) = \mu_{\hat{\phi}_k}(q) = \mu_{\hat{\phi}_k}(s_j,c)$. So, assume that $k = i$. If there is an $r \in \{1,\ldots,m\}$ for which $f(d_r) \notin I_i$, then, by Lemma 9(c), $\mu_{\hat{\phi}_i}(s_i,c) \leqslant \mu_{\hat{\phi}_i}(p) < \frac{\epsilon_i}{\sigma_i} < \frac{|f(d_r)-m_i|}{\sigma_i} = \bar{\varphi}_i(d_r)$, so we have $\mu_{\hat{\phi}_i}(s_i,c) < \bar{\varphi}_i(d_r) = \bar{\varphi}_k(d_r) \leqslant \max_{r=1,\ldots,m}\bar{\varphi}_k(d_r) = \mu_{\hat{\phi}_k}(q) = \mu_{\hat{\phi}_k}(s_j,c)$. So, assume that $f(d_r) \in I_i$ for every $r \in \{1,\ldots,m\}$. Then, by Lemma 8, $\mu_{\hat{\phi}_i}(q) = \mu_\phi(q)$. So, $\mu_{\hat{\phi}_i}(s_i,c) = \mu_\phi(s_i,c) < \mu_\phi(s_j,c) \leqslant \mu_\phi(q) = \mu_{\hat{\phi}_i}(q) = \mu_{\hat{\phi}_k}(q) = \mu_{\hat{\phi}_k}(s_j,c)$ finishing the proof. □

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