Affinity functions in fuzzy connectedness based image segmentation I: Equivalence of affinities

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ABSTRACT

Fuzzy connectedness (FC) constitutes an important class of image segmentation schemas. Although affinity functions represent the core aspect (main variability parameter) of FC algorithms, they have not been studied systematically in the literature. In this paper, we began filling this gap by introducing and studying the notion of equivalent affinities: if any two equivalent affinities are used in the same FC schema to produce two versions of the algorithm, then these algorithms are equivalent in the sense that they lead to identical segmentations. We give a complete and elegant characterization of the affinity equivalence. We also demonstrate that any segmentation obtained via a relative fuzzy connectedness (RFC) algorithm can be viewed as segmentation obtained via absolute fuzzy connectedness (AFC) algorithm with an automatic and adaptive threshold detection. Since the main goal of the paper is to identify, by formal mathematical arguments, the affinity functions that are equivalent, extensive experimental confirmations are not needed — they show completely identical segmentations — and as such, only relevant examples of the theoretical results are provided.

1. Introduction

Image segmentation — the process of partitioning the image domain into meaningful object regions — is perhaps the most challenging and critical problem in image processing and analysis. Research in this area will probably continue indefinitely long because the solution space is infinite dimensional, and since any single solution framework is unlikely to produce an optimal solution (in the sense of the best possible precision, accuracy, and efficiency) for all possible application domains. It is important to distinguish between two types of activities in segmentation research — the first relating to the development of application domain-independent general solution frameworks, and the second pertaining to the construction of domain-specific solutions starting from a known general solution framework. The latter is not a trivial task most of the time. Both these activities are crucial, the former for advancing the theoretical aspects of, and shedding new light on, segmentation research, and the latter for bringing the theoretical advances to actual practice. The topic of this paper touches both

of these activities, but has more pertinence to the former than the latter.

General segmentation frameworks [1–12] may be broadly classified into three groups: boundary-based [1–5], region-based [6–10], and hybrid [11,12]. As the nomenclature indicates, in the first two groups, the focus is on recognizing and delineating the boundary or the region occupied by the object in the image. In the third group, the focus is on exploiting the complementary strengths of each of boundary-based and region-based strategies to overcome their individual shortcomings. The segmentation framework discussed in the present paper belongs to the region-based group and constitutes an extension of the fuzzy connectedness (abbreviated from now on as FC) methodology [8].

In the FC framework [8], a fuzzy topological construct, called fuzzy connectedness, characterizes how the spatial elements (abbreviated as *spels*) of an image hang together to form an object. This construct is arrived at roughly as follows. A function called *affinity* is defined on the set $C \times C$ of all pairs of spels from the image domain C; the strength of affinity between any two spels depends on how close the spels are spatially and how similar their intensity-based properties are in the image. Affinity is intended to be a local relation. A global fuzzy relation called *fuzzy connectedness* is induced on the image domain by affinity as follows. For any two spels c and d in the image domain, all possible paths connecting c and d are considered. Each path is assigned a strength of connectedness which is simply the minimum of the affinities of consecutive spels along the path. The level of fuzzy connectedness

between c and d is considered to be the maximum of the strengths of all paths between c and d. For segmentation purposes, FC is utilized in several ways as described below. (Compare also Section 2.3.) See [13] for a review of the different FC definitions and how they are employed in segmentation and applications.

In *absolute FC* (abbreviated AFC) [8], the support of a segmented object is considered to be the maximal set of spels, containing one or more seed spels, within which the level of FC is at or above a specific threshold. To obviate the need for a threshold, *relative FC* (or RFC) [14] was developed by letting all objects in the image to compete simultaneously via FC to claim membership of spels in their sets. Each co-object is identified by one or more seed spels. Any spel c in the image domain is claimed by that co-object with respect to whose seed spels c has the largest level of FC compared to the level of FC with the seed sets of all other objects.

To avoid treating the core aspects of an object (that are very strongly connected to its seeds) and the peripheral subtle aspects (that may be less strongly connected to the seeds) in the same footing, an iterative refinement strategy is devised in *iterative RFC* (or IRFC) [15–18]. This has been shown to lead to better object definition than RFC with a theoretical construct similar to that of RFC. The proper design of affinity is crucial to the effectiveness of the segmentations that ensue, no matter what type of FC is used. In *scale-based* [29] and *vectorial FC* [19], which are applicable to all of AFC, RFC, and IRFC, affinity definition is not based just on the scalar properties of the two spels under question but also on the vectorial properties of all spels in the local scale region around the two spels. The FC family of methods developed to date [13–24] consists of various combinations of absolute, relative, and iterative FC with scale-based and vectorial versions.

The fundamental construct and core in any FC method is the affinity function. Its choice determines the effectiveness of the particular FC method. In the published literature on FC, affinity functions have not been studied in depth, leaving open several fundamental questions relating to their form, parameters, and effectiveness. A side effect and a manifestation of this gap is that, sometimes, certain modifications of affinities and their parameters are construed to result in improved FC segmentations, while, in reality, they lead to theoretically equivalent segmentations. These cannot be identified as such empirically.

In the present paper, we make a fundamental contribution toward a solution to this problem by creating theoretical tools to address these issues. More precisely, we define the notion of equivalent affinities (Section 2.2), and prove that if any two equivalent affinities are used in the same FC schema to produce two versions of the algorithm, then these algorithms are equivalent in the sense that they lead to identical segmentations when applied to any digital image initialized with the same seeds (Section 2.3). The resulting characterization of equivalent affinities is used in the second part of this paper [25] to analyze two main affinity types, homogeneity based and object feature based, to study the way they can be combined, and to determine which of these combinations lead to truly distinct segmentations.

We also show (in Section 3) that the RFC segmentation can be viewed to some extent as an AFC segmentation with an automatic threshold selection.

The notion of equivalence of algorithms, that stands behind the notion of equivalent affinities, is at the foundation of our more general study of the equivalences among segmentation algorithms, the theory of which we initiated in [26].

2. Affinities equivalent in the FC sense

The main purpose of this section is to uncover the essence of the relationship between the local measure of connectedness of pairs of spels, the affinity function, and the resulting segmentations obtained via FC algorithms. In particular, we will introduce the notion of the *equivalence (in the sense of FC) of the affinities* and show that equivalent affinities are indistinguishable from the point of view of FC segmentations, no matter what the empirical results indicate.

To make this work complete and useful, our definition of the affinity function will be more general than the one commonly used in the literature. However, we will show that each class of equivalent affinities contains at least one standard (meaning commonly used) affinity.

2.1. Preliminary definitions

2.1.1. Fuzzy sets and relations

We will use the following interpretation of the notions of (hard) functions and relations, which is standard in set theory (see e.g. [27,28]) and is used in many calculus books. A binary relation R from a set X to a set Y is identified with its graph; that is, the relation R equals $\{\langle x, y\rangle \in X \times Y : xRy$ holds$\}$. Since a function $f : X \to Y$ is a (special) binary relation from X to Y, in particular we have $f = \{\langle x, f(x)\rangle : x \in X\}$. With this interpretation, fuzzy sets and fuzzy relations have the following representations. Let $\mathscr{Z}$ be a *fuzzy subset* of a hard set X with a membership function $\mu_{\mathscr{Z}} : X \to [0, 1]$. For each $x \in X$ we interpret $\mu_{\mathscr{Z}}(x)$ as the degree to which x belongs to $\mathscr{Z}$. Usually such a fuzzy set $\mathscr{Z}$ is defined as $\{\langle x, \mu_{\mathscr{Z}}(x)\rangle : x \in X\}$, which is the graph of $\mu_{\mathscr{Z}}$. Thus, according to our interpretation, $\mathscr{Z}$ actually equals $\mu_{\mathscr{Z}}$. Note that this interpretation agrees quite well with the situation when $\mathscr{Z}$ is a hard subset Z of X, as then $\mathscr{Z} = \mu_{\mathscr{Z}}$ is equal to the *characteristic function* χ_Z of Z (defined as $\chi_Z(x) = 1$ for $x \in Z$ and $\chi_Z(x) = 0$ for $x \in X \setminus Z$), and the identification of Z with χ_Z is quite common in analysis and set theory. Notice also that a fuzzy binary relation ρ from X to Y is just a fuzzy subset of $X \times Y$, so it is equal to its membership function $\mu_{\rho} : X \times Y \to [0, 1]$.

2.1.2. Adjacency and digital space

Let $\mathfrak{n} \geqslant 2$ and let $\mathbb{Z}^{\mathfrak{n}}$ stand for the set of all $\mathfrak{n}$-tuples of integer numbers. A binary fuzzy relation α on $\mathbb{Z}^{\mathfrak{n}}$ is said to be a *fuzzy adjacency* if α is *symmetric* (i.e., $\alpha(c, d) = \alpha(d, c)$) and *reflexive* (i.e., $\alpha(c, c) = 1$). The value of $\alpha(c, d)$ depends only on the relative spatial position of c and d. Usually $\alpha(c, d)$ is decreasing with respect to the distance function $\|c - d\|$. In most applications, α is just a hard case relation like 4-adjacency relation for $\mathfrak{n} = 2$ or 6-adjacency in the three-dimensional case, defined as $\alpha(c, d) = 1$ for $\|c - d\| \leqslant 1$ and $\alpha(c, d) = 0$ for $\|c - d\| > 1$. By an $\mathfrak{n}$-*dimensional fuzzy digital space* we will understand a pair $\langle \mathbb{Z}^{\mathfrak{n}}, \alpha \rangle$. The elements of the digital space are called *spels*. (For $\mathfrak{n} = 2$ also called *pixels*, while for $\mathfrak{n} = 3 -$ *voxels*.)

2.1.3. Digital scene

Let $k \geqslant 1$. A *scene* over a fuzzy digital space $\langle \mathbb{Z}^{\mathfrak{n}}, \alpha \rangle$ is a pair $\mathscr{C} = \langle C, f \rangle$, where $C = \prod_{j=1}^{\mathfrak{n}} [-b_j, b_j] \subset \mathbb{Z}^{\mathfrak{n}}$, each $b_j > 0$ being an integer, and $f : C \to \mathbb{R}^k$ is a *scene intensity function*. The value of f represents either the original acquired image intensity or an estimate of certain image properties (such as gradients and texture measures) obtained from the given image.

2.1.4. Standard affinity functions

An affinity function for a scene $\mathscr{C}$, defined in its general form in the next subsection, is usually denoted by κ and it assigns to any pair $\langle c, d \rangle \in C \times C$ of spels the strength $\kappa(c, d)$ of their local hanging togetherness in $\mathscr{C}$. Within this class, a special role is played by *standard affinities*, that is, mappings $\kappa : C \times C \to [0, 1]$ which, treated as fuzzy binary relations, are symmetric and reflexive. In all practical applications, the value of $\kappa(c, d)$ depends on the adjacency strength $\alpha(c, d)$ of c and d (i.e., on the spatial relative position of c and d) as well as on the intensity function f. So far, only standard affinities

have been used in applications in the literature.[3] Of those, the most prominent are [29]: (1) the *homogeneity based affinity*

$$\psi_\sigma(c,d) = \alpha(c,d) \ e^{-\|f(c)-f(d)\|^2/\sigma^2}, \quad \text{where } \sigma > 0, \ c,d \in C \tag{1}$$

with its value being close to 1 (meaning that c and d are well connected) when the spels are spatially close and have very similar intensity values; (2) the *object feature based affinity* (single object case, with an expected intensity m for the object)

$$\phi_\sigma(c,d) = \alpha(c,d) \ e^{-\max\{\|f(c)-m\|, \|f(d)-m\|\}^2/\sigma^2},$$
$$\text{where } \sigma > 0, \ c,d \in C \tag{2}$$

with its value being close to 1 when the spels are spatially close and both have intensity values close to m. The weighted averages of these two forms of standard affinity — either additive or multiplicative — have also been used.

It has been demonstrated [14] that, in the standard FC algorithms of AFC and RFC (defined below), to fulfill certain desirable properties of segmentations (such as robustness with respect to seed points), affinities must be symmetric. In this paper, therefore, we will restrict ourselves to symmetric affinities. However, we will go quite afar from previous publications otherwise in considering affinity in its very general form.

2.1.5. Affinity as an operator

The affinity function is usually associated with each scene $\mathscr{C}$ according to some specific rule, such as in the examples (1) and (2). In such case, we can treat the rule of such association as an operator $\langle \mathscr{C}, p \rangle \overset{\mathscr{K}}{\mapsto} \kappa = \mathscr{K}(\mathscr{C}, p)$, where p represents all additional parameters like a prior knowledge (e.g. m in (2)) or other parameters (e.g. σ).

2.2. Equivalent affinities

In this subsection, we define the notion of the affinity function in its general form, without just confining to the basis of standard affinities as in (1) and (2), and introduce the concept of equivalent affinities. The motivation for developing equivalent affinities comes from our desire to recognize those differences among affinities that are inessential, and, therefore, lead to the same FC segmentations, from those that are essential and may give rise to different segmentations.

We refrain from formally defining equivalent affinities as "leading to the same FC segmentations in all FC schemas" since the term "all FC schemas" may change in time, leading to a confusion. Nevertheless, Theorem 5 and Remark 6 show that this intuitive definition fully agrees with our formal definition given below.

Let $\preceq$ be a linear order relation [27] on a set L and let C be an arbitrary finite non-empty set. We say that a function $\kappa : C \times C \to L$ is an *affinity function* (from C into $\langle L, \preceq \rangle$) provided κ is symmetric (i.e., $\kappa(a,b) = \kappa(b,a)$ for every $a,b \in C$) and $\kappa(a,b) \preceq \kappa(c,c)$ for every $a,b,c \in C$. Clearly, any standard affinity, as defined above, is an affinity function with $\langle L, \preceq \rangle = \langle [0,1], \leqslant \rangle$. Note that $\kappa(d,d) \preceq \kappa(c,c)$ for every $c,d \in C$. So, there exists an element in L, which we will denote by a symbol $\mathbf{1}_\kappa$, such that $\kappa(c,c) = \mathbf{1}_\kappa$ for every $c \in C$. Notice that $\mathbf{1}_\kappa$ is the largest element of $L_\kappa = \{\kappa(a,b) : a,b \in C\}$, although it does not need to be the largest element of L. In what follows, the strict inequality related to $\preceq$ will be denoted by $\prec$, that is, $a \prec b$ if and only if $a \preceq b$ and $a \neq b$. Certainly, in image processing, C will be always the domain of the scene intensity function. In all specific cases examined so far (compare [25]), we take $\langle L, \preceq \rangle$ as either the standard range $\langle [0,1], \leqslant \rangle$ or $\langle [0,\infty], \geqslant \rangle$.

We say that the affinities $\kappa_1 : C \times C \to \langle L_1, \preceq_1 \rangle$ and $\kappa_2 : C \times C \to \langle L_2, \preceq_2 \rangle$ are *equivalent (in the FC sense)* provided, for every $a,b,c,d \in C$

$$\kappa_1(a,b) \preceq_1 \kappa_1(c,d) \quad \text{if and only if} \quad \kappa_2(a,b) \preceq_2 \kappa_2(c,d)$$

or, equivalently,

$$\kappa_1(a,b) \prec_1 \kappa_1(c,d) \quad \text{if and only if} \quad \kappa_2(a,b) \prec_2 \kappa_2(c,d).$$

For example, it can be easily seen that for any constants $\sigma, \tau > 0$ the homogeneity based affinities ψ_σ and ψ_τ, see (1), are equivalent, since for any pairs $\langle a,b \rangle$ and $\langle c,d \rangle$ of adjacent spels,

$$\psi_\sigma(a,b) < \psi_\sigma(c,d) \iff \|f(a) - f(b)\| > \|f(c) - f(d)\|$$
$$\iff \psi_\tau(a,b) < \psi_\tau(c,d). \tag{3}$$

(Symbol $\iff$ means "if and only if.")

We say that the affinity operators $\mathscr{K}_1$ and $\mathscr{K}_2$ are equivalent provided the associated affinities $\kappa_1 = \mathscr{K}_1(\mathscr{C}, p)$ and $\kappa_2 = \mathscr{K}_2(\mathscr{C}, p)$ are equivalent for all scenes $\mathscr{C}$ and appropriate parameters p.

Equivalent affinities can be characterized as follows, where $\circ$ stands for the composition of functions, that is, $(g \circ \kappa_1)(a,b) = g(\kappa_1(a,b))$.

Proposition 1. *Affinities $\kappa_1 : C \times C \to \langle L_1, \preceq_1 \rangle$ and $\kappa_2 : C \times C \to \langle L_2, \preceq_2 \rangle$ are equivalent if and only if there exists a strictly increasing function g from $\langle L_{\kappa_1}, \preceq_1 \rangle$ onto $\langle L_{\kappa_2}, \preceq_2 \rangle$ such that $\kappa_2 = g \circ \kappa_1$.*

Proof. If κ_1 and κ_2 are equivalent, define g by putting $g(\kappa_1(a,b)) = \kappa_2(a,b)$ for every $a,b \in C$. Note that g is well defined, since $\kappa_1(a,b) = \kappa_1(c,d)$ implies that $\kappa_2(a,b) = \kappa_2(c,d)$. Also, inequality $\kappa_1(a,b) \preceq_1 \kappa_1(c,d)$ implies that $\kappa_2(a,b) \preceq_2 \kappa_2(c,d)$, so g is a strictly increasing map from L_{κ_1} onto L_{κ_2}. Conversely, if $\kappa_2 = g \circ \kappa_1$, where g is strictly increasing, then κ_1 is equivalent to κ_2 since for every $a,b,c,d \in C$ we have:

$$\kappa_2(a,b) \preceq_2 \kappa_2(c,d) \iff g(\kappa_1(a,b)) \preceq_2 g(\kappa_1(c,d))$$
$$\iff \kappa_1(a,b) \preceq_1 \kappa_1(c,d). \quad \square$$

Notice that when two affinity operators $\mathscr{K}_1$ and $\mathscr{K}_2$ are equivalent, then, for all appropriate pairs $\langle \mathscr{C}, p \rangle$, the affinities $\mathscr{K}_1(\mathscr{C}, p)$ and $\mathscr{K}_2(\mathscr{C}, p)$ are equivalent and, by Proposition 1, there exists an increasing function $g_{\mathscr{C},p}$ for which $\mathscr{K}_2(\mathscr{C}, p) = g_{\mathscr{C},p} \circ \mathscr{K}_1(\mathscr{C}, p)$. However, in general, there is no single increasing function g, independent of $\langle \mathscr{C}, p \rangle$, for which

$$\mathscr{K}_2(\mathscr{C}, p) = g \circ \mathscr{K}_1(\mathscr{C}, p) \text{ for all appropriate pairs } \langle \mathscr{C}, p \rangle. \tag{4}$$

(An example can be constructed from the affinity operator obtained by combining two affinities via lexicographical order, see [25, Example 5].) Nevertheless, an increasing function g, independent of $\langle \mathscr{C}, p \rangle$ and satisfying (4), can often be found for equivalent affinity operators $\mathscr{K}_1$ and $\mathscr{K}_2$, as seen in Example 4 and in [25].

One of the specific conclusions from Proposition 1 is the following fact.

Corollary 2. *If $\kappa : C \times C \to \langle [0,\infty], \geqslant \rangle$ is an affinity, then, for every strictly decreasing function g from $[0,\infty]$ onto $[0,1]$, a map $g \circ \kappa : C \times C \to \langle [0,1], \leqslant \rangle$ is an affinity equivalent to κ.*

Our interest in equivalent affinities comes from the fact (see Theorem 5) that any FC segmentation of a scene $\mathscr{C}$ remains unchanged if an affinity on C used to get the segmentation is replaced by an equivalent affinity. Keeping this in mind, it makes sense to find for each affinity function an equivalent affinity in a nice form:

Theorem 3. *Every affinity function is equivalent (in the FC sense) to a standard affinity.*

Proof. Let $\kappa : C \times C \to \langle L, \preceq \rangle$ be an arbitrary affinity. Note that there is a strictly increasing function $g : L_\kappa \to [0,1]$ with $g(\mathbf{1}_\kappa) = 1$. (If $L_\kappa = \{l_1, \ldots, l_m\}$ with $l_1 = \mathbf{1}_\kappa$, then such a g can be constructed by an easy induction on m.) Let $\kappa_2(c,d) = g(\kappa(c,d))$ for every $c,d \in C$. Then, by Proposition 1, κ is equivalent to the standard affinity $\kappa_2 : C \times C \to \langle [0,1], \leqslant \rangle$. $\square$

Once we agree that equivalent affinities lead to the same segmentations, Theorem 3 says that we can restrict our attention to standard affinities without losing any generality of our method. Then, one may wonder why study other affinities at all. The answer to this question is simple — in most cases, it is more natural to define an affinity function with a more abstract range, and any translation of such affinity to the standard one is a redundant step adding only unnecessary computational burden, although some researchers may believe, that it helps intuitive understanding. Moreover, in some of these cases there is no simple (i.e., continuous) translation of the natural affinity to the standard one. (See [25, Example 5].) On the other hand, Theorems 3 and 5 tell us that all the theoretical results that are true for the standard affinities hold also for the affinities as we defined them. Thus, there is no particular reason to restrict our attention to the affinities in the standard form.

The following constitutes an example of two equivalent forms of the homogeneity based affinity (1), each form treated as an affinity operator. (See [25] for more examples.)

Example 4. For a scene $\mathscr{C} = \langle C, f \rangle$, a natural form of the *homogeneity based affinity* is a function $\psi : C \times C \to \langle [0, \infty], \geqslant \rangle$ given by $\psi(c,d) = \|f(c) - f(d)\|$ for adjacent spels $c,d \in C$ and $\psi(c,d) = \infty$ otherwise. (See also [25].) The more commonly used version of the homogeneity based affinity is the standard affinity $\psi_\sigma(c,d) = e^{-\psi(c,d)^2/\sigma^2}$, which is the composition of ψ with the Gaussian function $g_\sigma(x) = e^{-x^2/\sigma^2}$. Note that, by Corollary 2, ψ and ψ_σ are equivalent, independently of the value of the parameter σ, since g_σ is strictly decreasing from $[0, \infty]$ onto $[0,1]$. (Compare also with (3), which constitutes a direct argument.)

In particular, the parameter σ in the definition of ψ_σ is totally non-essential from the FC segmentation point of view (see Theorem 5), as varying σ results in a different (non-linear) scaling of the strength of connectedness. Therefore, for example, the same segmentation of a given image is obtained by using AFC algorithm with (a) affinity ψ and threshold θ; (b) affinity ψ_σ and threshold $g_\sigma(\theta)$, independently of the value of σ. This phenomenon is illustrated in Fig. 1 on a 2D scene — a CT slice of a human knee, Fig. 1(a). In Fig. 1(d) and (e) segmented binary scenes are shown, resulting from the use of ψ_σ with $\sigma = 1$ and $\sigma = 10.8$, respectively, and the corresponding thresholds $g_\sigma(\theta)$. The results are identical. Fig. 1(b) and (c) show the corresponding connectivity scenes, in which the intensity of each spel c represents the ψ_σ-connectivity strength between the seed and c (i.e., the strength of the strongest path joining the seed and c).

2.3. FC segmentations for equivalent affinities

Fix an affinity $\kappa : C \times C \to \langle L, \preceq \rangle$. To define fuzzy connectedness segmentation of $\mathscr{C}$, we need first to translate the local measure of connectedness given by κ into the global strength of connected-
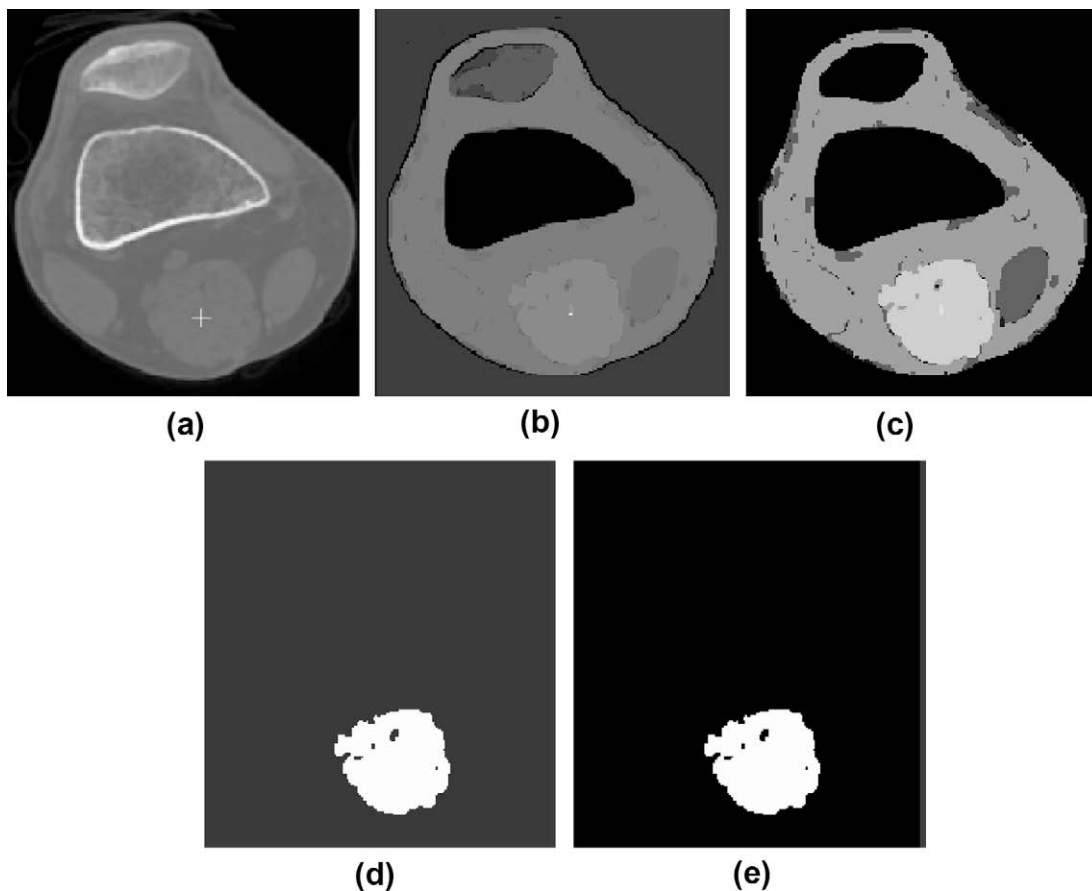
Fig. 1. Illustration of equivalent affinities. (a) A 2D scene — a CT slice of a human knee. (b and c) Connectivity scenes corresponding to affinities ψ_σ with $\sigma = 1$ and $\sigma = 10.8$, respectively, and the same seed spel (indicated by + in (a)) specified in a soft tissue region of the scene in (a). (d) and (e) Identical AFC objects obtained from the scenes in (b) and (c), respectively.

ness. For this, we will need the notions of a path and its strength. A *path* in $A \subseteq C$ is any sequence[4] $p = \langle c_1, \ldots, c_l \rangle$, where $l > 1$ and $c_i \in A$ for every $i = 1, \ldots, l$. (Notice that there is no assumption on any adjacency of the consecutive spels in a path.) The family of all paths in A is denoted by $\mathscr{P}^A$. If $c, d \in A$, then the family of all paths $\langle c_1, \ldots, c_l \rangle$ in A from c to d (i.e., such that $c_1 = c$ and $c_l = d$) is denoted by $\mathscr{P}^A_{cd}$.

The *strength* $\mu_\kappa(p)$ of a path $p = \langle c_1, \ldots, c_l \rangle \in \mathscr{P}^C$ is defined as the strength of its κ-weakest link; that is, $\mu_\kappa(p) \stackrel{\text{def}}{=} \min\{\kappa(c_{i-1}, c_i) : 1 < i \leqslant l\}$. (Note that, if one follows the common practice of defining $\kappa(c, d)$ to be the minimal element of L_κ for any non-adjacent c and d, then only paths with adjacent consecutive spels can have non-minimal strength.) For $c, d \in A \subseteq C$, the (global) κ-connectedness strength in A between c and d is defined as the strength of a strongest path in A between c and d; that is,

$$\mu^A_\kappa(c, d) \stackrel{\text{def}}{=} \max\left\{\mu_\kappa(p) : p \in \mathscr{P}^A_{cd}\right\}. \tag{5}$$

Notice that $\mu^A_\kappa(c, c) = \mu_\kappa(\langle c, c \rangle) = \mathbf{1}_\kappa$. We will often refer to the function $\mu^A_\kappa : C \times C \to L$ as a *connectivity measure* (on A) induced by κ. For $c \in A \subset C$ and a non-empty $D \subset A$, we also define $\mu^A_\kappa(c, D) \stackrel{\text{def}}{=} \max_{d \in D} \mu^A_\kappa(c, d)$. We will write μ for μ_κ and μ^A for μ^A_κ when κ is clear from the context. The issue of why μ^A_κ should be defined from κ by the procedure described above is discussed in detail in [30]. Note that if κ is a hard binary relation (i.e., when $L = \{0, 1\}$), then μ^C_κ is a relation (or, more precisely, its characteristic function) known as a *transitive closure of* κ, which is defined as the set of all pairs $\langle c, d \rangle \in C \times C$ for which there exists a sequence $c = c_0, c_1, \ldots, c_m = d$ such that $\kappa(c_i, c_{i+1}) = 1$ for every $i < m$.

To define fuzzy objects delineated by FC segmentations, we start with a family $\mathbb{S}$ of non-empty pairwise disjoint subsets of C, where each $S \in \mathbb{S}$ represents a set of spels, known as *seeds*, which will belong to the object generated by it. Also, fix a threshold $\theta \in L, \theta \leqslant \mathbf{1}_\kappa$. For every $S \in \mathbb{S}$, put $W = \bigcup(\mathbb{S} \setminus \{S\})$ and, similarly as in [18] (see also [31]), define

- $P^\kappa_{S\theta} = \{c \in C : \theta \preceq \mu^C_\kappa(c, S)\}$;
- $P^\kappa_{S\mathbb{S}} = \{c \in C : \mu^C_\kappa(c, W) \prec \mu^C_\kappa(c, S)\}$;
- $P^{I\kappa}_{S\mathbb{S}} = \bigcup_{i=0}^\infty P^{i,\kappa}_{S\mathbb{S}}$, where sets $P^{i,\kappa}_{S\mathbb{S}}$ are defined inductively by $P^{0,\kappa}_{S\mathbb{S}} = \emptyset$ and $P^{i+1,\kappa}_{S\mathbb{S}} = P^{i,\kappa}_{S\mathbb{S}} \cup \{c \in C \setminus P^{i,\kappa}_{S\mathbb{S}} : \mu^{C \setminus P^{i,\kappa}_{S\mathbb{S}}}_\kappa(c, W) \prec \mu^C_\kappa(c, S)\}$.

Then AFC, RFC, and IRFC segmentations of C are defined, respectively, as $\mathbb{P}^\theta_\kappa(\mathbb{S}) = \{P^\kappa_{S\theta} : S \in \mathbb{S}\}$, $\mathbb{P}_\kappa(\mathbb{S}) = \{P^\kappa_{S\mathbb{S}} : S \in \mathbb{S}\}$, and $\mathbb{P}^I_\kappa(\mathbb{S}) = \{P^{I\kappa}_{S\mathbb{S}} : S \in \mathbb{S}\}$. Notice that an AFC object $P^\kappa_{S\theta}$ consists of all spels connected with at least one seed s in S with the κ-connectivity strength at least θ. An RFC object is created via competition of seeds for each spel: a spel c belongs to $P^\kappa_{S\mathbb{S}}$ provided there is a seed s in S for which the κ-connectivity between c and s exceeds the κ-connectivity between c and any other seed indicating another object. Finally, IRFC objects are obtained by refining the RFC competition: a spel c is unassigned to any RFC object provided there is a tie between two seeds s and t from different objects, e.g., $\mu^C_\kappa(c, w) \preceq \mu^C_\kappa(c, s) = \mu^C_\kappa(c, t)$ for any seed w. However, such a tie can be resolved if the strongest paths justifying $\mu^C_\kappa(c, s)$ and $\mu^C_\kappa(c, t)$ cannot pass through the spels already assigned to another object. Upon such resolution, the spel under question is assigned to the winning object in the next iteration of IRFC.

Now we can formalize our (previous) main claim, and a central result of this paper, that the fuzzy connectedness segmentations (i.e., those obtained via AFC, RFC, and IRFC algorithms) are unchanged if an affinity function is replaced by an equivalent one.

Theorem 5. *Let $\kappa_1 : C \times C \to \langle L_1, \preceq_1 \rangle$ and $\kappa_2 : C \times C \to \langle L_2, \preceq_2 \rangle$ be equivalent affinity functions and let $\mathbb{S}$ be a family of non-empty pairwise disjoint subsets of C. Then for every $\theta_1 \preceq_1 \mathbf{1}_{\kappa_1}$ in L_1, there exists a $\theta_2 \preceq_2 \mathbf{1}_{\kappa_2}$ in L_2 such that, for every $S \in \mathbb{S}$ and $i \in \{0, 1, 2, \ldots\}$, we have $P^{\kappa_1}_{S\theta_1} = P^{\kappa_2}_{S\theta_2}, P^{\kappa_1}_{S\mathbb{S}} = P^{\kappa_2}_{S\mathbb{S}}$, and $P^{i,\kappa_1}_{S\mathbb{S}} = P^{i,\kappa_2}_{S\mathbb{S}}$. In particular, $\mathbb{P}^{\theta_1}_{\kappa_1}(\mathbb{S}) = \mathbb{P}^{\theta_2}_{\kappa_2}(\mathbb{S}), \mathbb{P}_{\kappa_1}(\mathbb{S}) = \mathbb{P}_{\kappa_2}(\mathbb{S})$, and $\mathbb{P}^I_{\kappa_1}(\mathbb{S}) = \mathbb{P}^I_{\kappa_2}(\mathbb{S})$.*

Moreover, if $g : C \to C$ is a strictly monotone function such that $\kappa_2 = g \circ \kappa_1$ (which exists by Proposition 1), then we can take $\theta_2 = g(\theta_1)$.

Proof. First note that, for any paths $p = \langle c_1, \ldots, c_l \rangle$ and $q = \langle d_1, \ldots, d_m \rangle$ from $\mathscr{P}^C$, we have[5]

$$\mu_{\kappa_1}(p) \preceq_1 \mu_{\kappa_1}(q) \Longleftrightarrow (\forall\, 1 < j \leqslant m)\, (\exists 1 < i \leqslant l)\, \kappa_1(c_{i-1}, c_i) \preceq_1 \kappa_1(d_{j-1}, d_j)$$
$$\Longleftrightarrow (\forall\, 1 < j \leqslant m)\, (\exists 1 < i \leqslant l)\, \kappa_2(c_{i-1}, c_i) \preceq_2 \kappa_2(d_{j-1}, d_j)$$
$$\Longleftrightarrow \mu_{\kappa_2}(p) \preceq_2 \mu_{\kappa_2}(q).$$

Similarly, for every $a, c \in A \subseteq C$ and $b, d \in B \subseteq C$, we have

$$\mu^A_{\kappa_1}(a, c) \preceq_1 \mu^B_{\kappa_1}(b, d) \Longleftrightarrow (\forall\, p \in \mathscr{P}^A_{ac})\, (\exists q \in \mathscr{P}^B_{bd})\, \mu_{\kappa_1}(p) \preceq_1 \mu_{\kappa_1}(q)$$
$$\Longleftrightarrow (\forall\, p \in \mathscr{P}^A_{ac})\, (\exists q \in \mathscr{P}^B_{bd})\, \mu_{\kappa_2}(p) \preceq_2 \mu_{\kappa_2}(q)$$
$$\Longleftrightarrow \mu^A_{\kappa_2}(a, c) \preceq_2 \mu^B_{\kappa_2}(b, d).$$

If, in addition, $\emptyset \neq W \subseteq A$ and $S \subseteq B$, then also

$$\mu^A_{\kappa_1}(a, W) \preceq_1 \mu^B_{\kappa_1}(b, S) \Longleftrightarrow (\forall\, c \in W)\, (\exists d \in S)\, \mu^A_{\kappa_1}(a, c) \preceq_1 \mu^B_{\kappa_1}(b, d)$$
$$\Longleftrightarrow (\forall\, c \in W)\, (\exists d \in S)\, \mu^A_{\kappa_2}(a, c) \preceq_2 \mu^B_{\kappa_2}(b, d)$$
$$\Longleftrightarrow \mu^A_{\kappa_2}(a, W) \preceq_2 \mu^B_{\kappa_2}(b, S). \tag{6}$$

Let $a, b \in C$ be such that $\kappa_1(a, b) = \min\{\kappa_1(x, y) : x, y \in C$ and $\theta_1 \preceq_1 \kappa_1(x, y)\}$ and put $\theta_2 = \kappa_2(a, b)$. Note that $\theta_2 = g(\theta_1)$ whenever $\kappa_2 = g \circ \kappa_1$. Then

$$P^{\kappa_1}_{S\theta_1} = \left\{c \in C : \theta_1 \preceq_1 \mu^C_{\kappa_1}(c, S)\right\} = \left\{c \in C : \kappa_1(a, b) \preceq_1 \mu^C_{\kappa_1}(c, S)\right\}$$
$$= \left\{c \in C : \kappa_2(a, b) \preceq_2 \mu^C_{\kappa_2}(c, S)\right\} = P^{\kappa_2}_{S\theta_2}.$$

Similarly, we have

$$P^{\kappa_1}_{S\mathbb{S}} = \left\{c \in C : \mu^C_{\kappa_1}(c, W) \prec_1 \mu^C_{\kappa_1}(c, S)\right\}$$
$$= \left\{c \in C : \mu^C_{\kappa_2}(c, W) \prec_2 \mu^C_{\kappa_2}(c, S)\right\} = P^{\kappa_2}_{S\mathbb{S}}.$$

The final equation we need to prove is $P^{i,\kappa_1}_{S\mathbb{S}} = P^{i,\kappa_2}_{S\mathbb{S}}$. This will be proved by induction on $i \geqslant 0$. For $i = 0$ this is true, since by definition both sets are empty. So assume that for some i we have $P^{i,\kappa_1}_{S\mathbb{S}} = P^{i,\kappa_2}_{S\mathbb{S}}$. Then

$$P^{i+1,\kappa_1}_{S\mathbb{S}} = P^{i,\kappa_1}_{S\mathbb{S}} \cup \left\{c \in C \setminus P^{i,\kappa_1}_{S\mathbb{S}} : \mu^{C \setminus P^{i,\kappa_1}_{S\mathbb{S}}}_{\kappa_1}(c, W) \prec_1 \mu^C_{\kappa_1}(c, S)\right\}$$
$$= P^{i,\kappa_2}_{S\mathbb{S}} \cup \left\{c \in C \setminus P^{i,\kappa_2}_{S\mathbb{S}} : \mu^{C \setminus P^{i,\kappa_2}_{S\mathbb{S}}}_{\kappa_1}(c, W) \prec_1 \mu^C_{\kappa_1}(c, S)\right\}$$
$$= P^{i,\kappa_2}_{S\mathbb{S}} \cup \left\{c \in C \setminus P^{i,\kappa_2}_{S\mathbb{S}} : \mu^{C \setminus P^{i,\kappa_2}_{S\mathbb{S}}}_{\kappa_2}(c, W) \prec_2 \mu^C_{\kappa_2}(c, S)\right\} = P^{i+1,\kappa_2}_{S\mathbb{S}},$$

where the second equation follows from the inductive assumptions and the third one is implied by (6). The equality of the segmentations associated with κ_1 and κ_2 follows directly from the definitions of $\mathbb{P}^\theta_\kappa(\mathbb{S}), \mathbb{P}_\kappa(\mathbb{S})$, and $\mathbb{P}^I_\kappa(\mathbb{S})$. $\square$

2.4. Remarks on and consequences of Theorems 3 and 5

In summary, Theorem 3 says that for every affinity function there is a standard affinity equivalent to it, while Theorem 5 says that for any two equivalent affinities we get the same FC

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segmentations in each of AFC, RFC, and IRFC. To further illustrate this, we examine the previous example in Fig. 1 for AFC by using two affinities ψ_σ, with $\sigma = 1$ and $\sigma = 10.8$. Fig. 1(b) and (c) display the connectivity scenes $\mathscr{C}_\kappa = \langle C, f_\kappa \rangle$ for the 2D scene of Fig. 1(a), where for any $c \in C$ and the same fixed spel $s \in C$, $f_\kappa(c) = \mu_\kappa^C(c, s)$, where κ is either ψ_1 or $\psi_{10.8}$. The resulting identical AFC objects are displayed in (d) and (e) as binary scenes. Of course, different thresholds were used in producing scenes (d) and (e) from those in (b) and (c), respectively, which precisely makes our point that *segmented object information* in Fig. 1(b) and (c) is identical.

Remark 6. It can be proved, under some natural assumptions on affinity operators $\mathscr{K}_1$ and $\mathscr{K}_2$ satisfied by all currently used affinities, that the equivalence of $\kappa_1 = \mathscr{K}_1(\mathscr{C}_0, p)$ and $\kappa_2 = \mathscr{K}_2(\mathscr{C}_0, p)$ is not only sufficient, but also necessary for the conclusion of Theorem 5. More specifically, for such affinities, if κ_1 and κ_2 are not equivalent and $\mathscr{C}_0 = \langle C, f_0 \rangle$, then there exist a scene $\mathscr{C} = \langle C, f \rangle$, a non-empty set of seeds $S \subset C$, and $\theta_1 \preceq_1 \mathbf{1}_{\kappa_1}$ such that $P_{S\theta_1}^{\kappa_1} \neq P_{S\theta_2}^{\kappa_2}$ for every $\theta_2 \preceq_2 \mathbf{1}_{\kappa_2}$.

Note also that, in general, conclusion of Theorem 5 may hold also for the affinities that are not equivalent. Indeed, it is easy to find a standard affinity κ_1 on a scene $\mathscr{C} = \langle C, f \rangle$ for which: (a) between any two spels there is a path of maximal strength 1, while (b) there are many pairs $\langle a, b \rangle$ of adjacent spels in C with $\kappa_1(a, b) < 1$. Now, if κ_2 is obtained from κ_1 by changing only its values on adjacent pairs $\langle a, b \rangle$ with $\kappa_1(a, b) < 1$, then the conclusion of Theorem 5 will still hold, while we can insure that κ_1 and κ_2 are not equivalent.

2.4.1. Practical considerations

The equivalence theorems say that, if a function g is strictly monotone, then the affinities κ and $g \circ \kappa$ are equivalent and they lead to identical segmentations. However, the segmentations are insured to be identical only when there are no rounding errors. In actual implementations, it is feasible that for distinct numbers x and y in the range of κ, the actual values $g(x)$ and $g(y)$ are so close that the implemented algorithm identifies $g(x)$ with $g(y)$. In such implementations some information is lost when passing from κ to $g \circ \kappa$, which may lead to different segmentations. This problem must be considered, when performing any experimental comparisons. Note also that, even when there is no rounding error in the algorithm that influences our theoretical results, a human operator may have an impression that some information is lost when passing from κ to $g \circ \kappa$, due to the limited resolution perception of the human eye. This phenomenon can be noticed in Fig. 1(b) and (c): it is easier for human eyes to identify the object in Fig. 1(c) than it is in Fig. 1(b).

Notice, that all the results presented in this section are applicable to the vectorial images (compare [19]), since we allow the image intensity value to be vectors from $\mathbb{R}^k$. In addition, with minor modifications to the definition of general affinities, the scale-based version of standard affinity [29] can also be covered under Theorems 3 and 5. (Any scale-based affinity is essentially equal to a non-scale-based affinity applied to an appropriately filtered version of the intensity function.) This implies that those results are applicable to all currently known FC schemas involving different combinations of scale-based and/or vectorial AFC, RFC, and IRFC.

Theorems 3 and 5 also imply that any result proved for the FC segmentations in the context of standard affinities remains valid for the affinities in our general setting, that is, the FC algorithms used with our general affinities have all nice properties that the FC algorithms have when used with the standard affinities. For example, the properties listed in Corollary 7 below are the translation of some of the results from [18]. Property from (a) is technical, and will be used in the proof of one of the following results. The robustness property (b) says, that the output of the AFC algorithm

remains unchanged when some seeds are replaced by other seeds within the same object. This is of considerable practical importance, since seeds, whether chosen by a human operator or automatically by an algorithm, are likely to be different in different instances of running the algorithm. Nevertheless, according to the robustness property, the segmentation results remain identical, as long as the indicated seeds will be chosen within the respective objects. The last property (c) insures that the distinct objects delineated by IRFC and RFC are disjoint.

In what follows, if affinity κ is clear from the context, we will drop the symbol κ from the object symbols $P_{S\theta}^\kappa, P_{S\mathbb{S}}^\kappa$, and $P_{S\mathbb{S}}^{I\kappa}$.

Corollary 7. *Let $\kappa : C \times C \to \langle L, \preceq \rangle$ be an arbitrary affinity function.*
(a) *For any $a, b, c \in A \subseteq C$, if $\mu^A(b, c) \prec \mu^A(a, b)$, then $\mu^A(a, c) = \mu^A(b, c)$.*
(b) *(Robustness) Let $\mathbb{S} = \{S_1, \ldots, S_m\}$ be a family of singletons, and for every $i \in \{1, \ldots, m\}$, let $T_i \subset P_{S_i\mathbb{S}}$ be a singleton. If $\mathscr{T} = \{T_1, \ldots, T_m\}$, then $P_{T_i\mathscr{T}} = P_{S_i\mathbb{S}}$ for every $i \in \{1, \ldots, m\}$.*
(c) *For any family $\mathbb{S}$ of pairwise disjoint non-empty subsets of C, we have $P_{S\mathbb{S}}^I \cap P_{U\mathbb{S}}^I = \emptyset$ for every distinct $S, U \in \mathbb{S}$.*

Proof. This follows directly from our remark above and, respectively, from [18, Proposition 2.1, Proposition 2.2, Theorem 2.4]. □

3. Relative fuzzy connectedness segmentation as absolute fuzzy connectedness segmentation

In AFC, to obtain the FC object $P_{S\theta}^\kappa$, a threshold θ for the strength of connectedness must be specified. This threshold is obviated in defining RFC objects $P_{S\mathbb{S}}^\kappa$ (see definition above) simply by determining the membership of a spel c in an object by its largest strength of connectedness with respect to the seed sets assigned to the different objects. In this section, we will show that the RFC segmentation can be viewed to some extent as an AFC segmentation wherein the required threshold is determined automatically.

Theorem 8. *Let $\kappa : C \times C \to \langle L, \preceq \rangle$ be an arbitrary affinity function and $\mathbb{S}$ be a non-empty family of pairwise disjoint, non-empty sets of seeds in C. Fix an $S \in \mathbb{S}$ and let $W = \bigcup(\mathbb{S} \setminus \{S\})$. For every $s \in S$ let $\theta_s = \mu_\kappa^C(s, W)$. Then $P_{S\mathbb{S}} = \bigcup_{s \in S} \bigcup_{\theta_s \prec \theta} P_{\{s\}\theta}$.*

Proof. Note that, by Corollary 7(a), for every $c, s, w \in C$,

$$\mu_\kappa^C(c, w) \prec \mu_\kappa^C(c, s) \quad \text{if and only if} \quad \mu_\kappa^C(s, w) \prec \mu_\kappa^C(c, s).$$

Thus

$$
\begin{aligned}
P_{S\mathbb{S}} &= \{c \in C : \mu_\kappa^C(c, W) \prec \mu_\kappa^C(c, S)\} \\
&= \{c \in C : (\exists s \in S)\, (\forall w \in W)\, \mu_\kappa^C(c, w) \prec \mu_\kappa^C(c, s)\} \\
&= \{c \in C : (\exists s \in S)\, (\forall w \in W)\, \mu_\kappa^C(s, w) \prec \mu_\kappa^C(c, s)\} \\
&= \{c \in C : (\exists s \in S)\, \mu_\kappa^C(s, W) \prec \mu_\kappa^C(c, s)\} \\
&= \bigcup_{s \in S} \{c \in C : \theta_s \prec \mu_\kappa^C(c, s)\} \\
&= \bigcup_{s \in S} \bigcup_{\theta_s \prec \theta} \{c \in C : \theta \preceq \mu_\kappa^C(c, s) = \bigcup_{s \in S} \bigcup_{\theta_s \prec \theta} P_{\{s\}\theta}. \quad \square
\end{aligned}
$$

For an affinity $\kappa : C \times C \to \langle L, \preceq \rangle$ and $\theta < \mathbf{1}_\kappa$, let θ^+ be the smallest element of $L_\kappa = \{\kappa(a, b) : a, b \in C\}$ greater than θ; that is, $\theta^+ \overset{\text{def}}{=} \min\{\rho \in L_\kappa : \theta \prec \rho\}$.

Theorem 8 has the nicest form when each object is initiated by just one single seed spel.

Corollary 9. *Let $\langle C, \kappa, \preceq \rangle$ be an arbitrary affinity structure and $\mathbb{S}$ be a non-empty family of singletons in C such that $\mu_\kappa^C(s, t) \neq \mathbf{1}_\kappa$ for every distinct $S = \{s\}$ and $T = \{t\}$ from $\mathbb{S}$. For $S = \{s\} \in \mathbb{S}$, define $\theta_S = \mu_\kappa^C(s, \bigcup(\mathbb{S} \setminus \{S\}))$. Then $P_{S\mathbb{S}} = P_{S\theta_S^+}$ for every $S \in \mathbb{S}$. In particular, $\mathbb{P}_\kappa(\mathbb{S}) = \{P_{S\theta_S^+} : S \in \mathbb{S}\}$.*

Proof. Let $S = \{s\} \in \mathbb{S}$. Then $\theta_S = \theta_s$ and, by Theorem 8, we have $P_{S\mathbb{S}} = \bigcup_{\theta_S \prec \theta} P_{S\theta} = \{c \in C : \theta_S \prec \mu_\kappa^C(c, S)\} = \{c \in C : \theta_S^+ \preceq \mu_\kappa^C(c, S)\} = P_{S\theta_S^+}$. $\square$

Notice that if for a family $\mathbb{S}$ containing only singletons there exist distinct $S, T \in \mathbb{S}$ such that $\mu_\kappa^C(S, T) \overset{\text{def}}{=} \max_{s \in S} \mu_\kappa^C(s, T) = \mathbf{1}_\kappa$, then $P_{S\mathbb{S}} = P_{T\mathbb{S}} = \emptyset$. That is, in this case, S and T are in the same object, and therefore, the sets that contain S and T and that separate them in the FC sense are obviously empty. In all practical cases we are interested only in the families $\mathbb{S}$ of seeds for which $\mu_\kappa^C(S, T) \neq \mathbf{1}_\kappa$ for any distinct $S, T \in \mathbb{S}$. In fact, if S and T are in different object regions in a scene, then we expect the strength of connectedness $\mu_\kappa^C(S, T)$ between them to be low. Thus, this assumption in Corollary 9 does not really restrict its usefulness, but actually warrants it from practical requirements.

If $\mathbb{S}$ from Corollary 9 has just two elements, say $\mathbb{S} = \{\{s\}, \{t\}\}$, then $\theta_{\{s\}} = \theta_{\{t\}}$ and for $\theta = \theta_{\{s\}}^+$ we have $\mathbb{P}_\kappa(\mathbb{S}) = \{P_{S\theta} : S \in \mathbb{S}\} = \mathbb{P}_\kappa^\theta(\mathbb{S})$. Thus, in this case, the RFC segmentation $\mathbb{P}_\kappa(\mathbb{S})$ is just an AFC segmentation $\mathbb{P}_\kappa^\theta(\mathbb{S})$, where θ was automatically set by the RFC procedure. However, when there are more than two objects involved in RFC and $\mathbb{S}$ contains three or more singletons, the thresholds θ_S^+, $S \in \mathbb{S}$, need not be equal. In this case, each $P_{S\mathbb{S}}$ from $\mathbb{P}_\kappa(\mathbb{S})$ is an AFC object $P_{S\theta_S^+}$, where the different thresholds are automatically tailored to the different objects under consideration. In other words, in general, it is not possible to derive RFC objects via AFC segmentation $\mathbb{P}_\kappa^\theta(\mathbb{S})$. That is the beauty of RFC compared to AFC.

We illustrate this property of RFC versus AFC in a schematic (Fig. 2), as well as in an actual medical image (Fig. 3). In both figures, we consider three objects, indicated by seeds s, t, and u. In Fig. 2, region W is more strongly connected to seed u than to either s or t. As such, RFC correctly assigns it to the region $P_{u,\{s,t,u\}}$ indicated by u, as shown in Fig. 2(b). However, there is no single threshold that could lead to an AFC segmentation coinciding with the RFC segmentation: a threshold θ below $(.6)^+$ causes objects $P_{s,\theta}$ and $P_{t,\theta}$ to be equal and too big, as shown in Fig. 2(d), while $\theta \geqslant (.6)^+$ cuts region W from $P_{u,\theta}$, see Fig. 2(c). Nevertheless, every
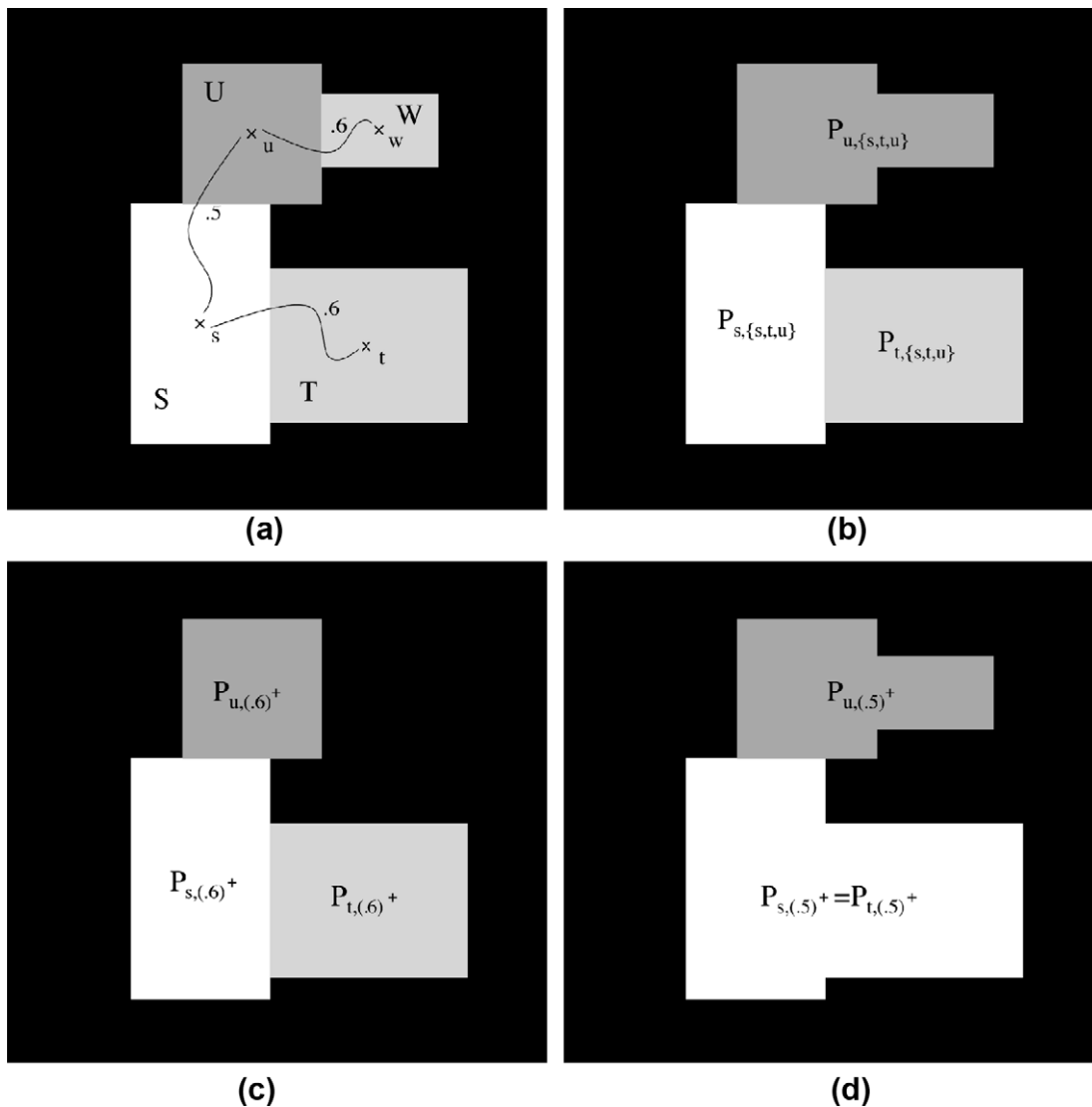
Fig. 2. (a) A schematic scene with a uniform background and four distinct areas denoted by S, T, U, W, and indicated by seeds marked by $\times$. It is assumed that each of these areas is uniform in intensity and the connectivity strength within each of these areas has the maximal value of 1, the connectivity between the background and any other spel is $\leqslant .2$, while the connectivity between the adjacent regions is as indicated in the figure: $\mu(s, t) = .6$, $\mu(s, u) = .5$, and $\mu(u, w) = .6$. (b) The RFC segmentation of three objects indicated by seeds s, t, and u, respectively. (c) Three AFC objects indicated by the seeds s, t, u and delineated with threshold $\theta = (.6)^+$. Notice that while $P_{s,\{s,t,u\}} = P_{s,(.6)^+}$ and $P_{t,\{s,t,u\}} = P_{t,(.6)^+}$, object $P_{u,(.6)^+}$ is smaller than RFC indicated $P_{u,\{s,t,u\}}$. (d) Same as in (c) but with $\theta = (.5)^+$. Note that while $P_{u,\{s,t,u\}} = P_{u,(.5)^+}$, objects $P_{s,(.5)^+}$ and $P_{t,(.5)^+}$ coincide and lead to an object bigger than $P_{s,\{s,t,u\}}$ and $P_{t,\{s,t,u\}}$.
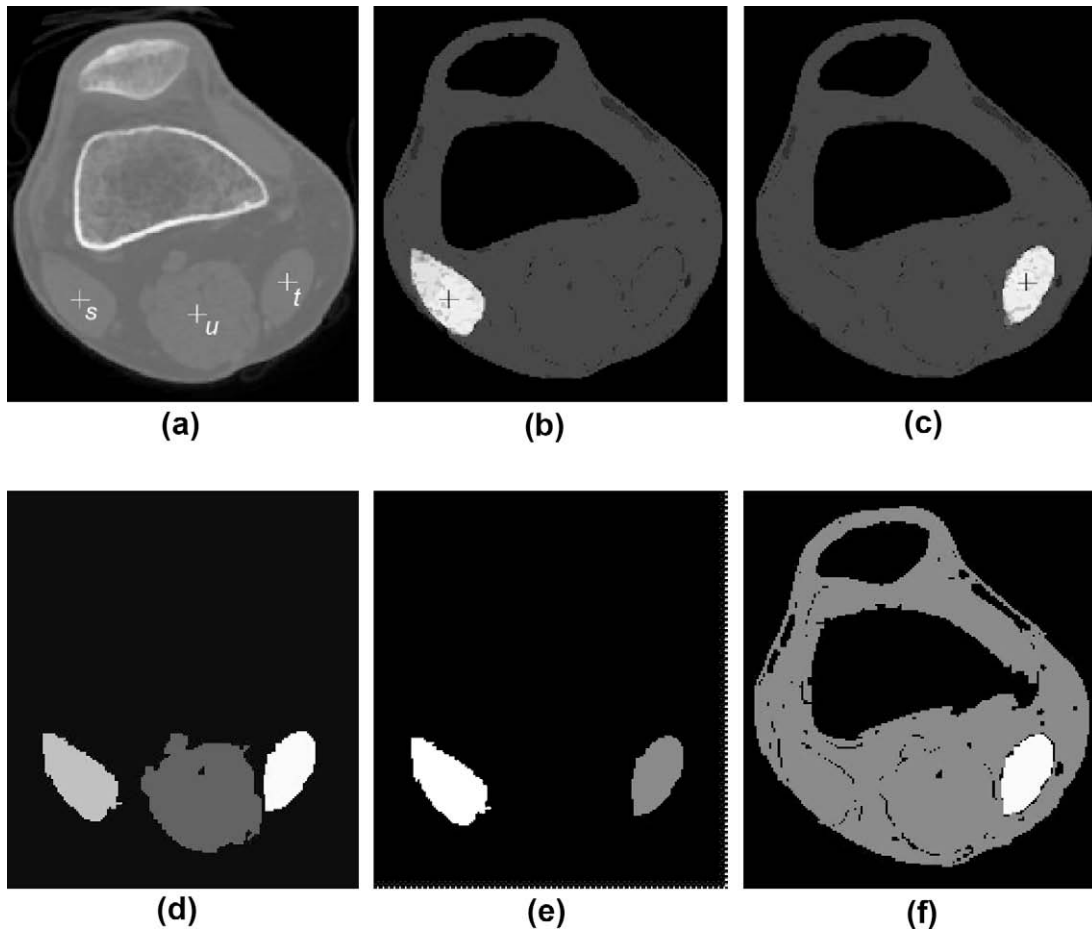
Fig. 3. (a) A 2D scene, same as in Fig. 1(a), with three indicated seeds. (b) and (c) Connectivity scenes corresponding to the two AFC objects indicated by *s* and *t*. (d) The RFC segmentation for the three indicated objects. (e) The AFC objects initiated with seeds *s* and *t* obtained with the threshold $\theta_{\{s\}} < \theta_{\{t\}}$ determined automatically by RFC. Although the result is a binary image, the two objects are shown at two gray levels. The object indicated by seed *s* agrees with its counterpart in (d). The smaller threshold caused the *t*-indicated object to be slightly smaller than in (d). (f) Same as (e) but with threshold $\theta_{\{t\}}$. The object indicated by seed *t* agrees with its counterpart in (d). However, the larger threshold caused the *s*-indicated object (gray) to leak to a big part of the scene.

RFC delineated object is also equal to appropriate AFC object: $P_{s,\{s,t,u\}} = P_{s,(.6)^+}, P_{t,\{s,t,u\}} = P_{t,(.6)^+}$, and $P_{u,\{s,t,u\}} = P_{u,(.5)^+}$.

In Fig. 3, we concentrate on the objects indicated by seeds *s* and *t*, corresponding to soft tissue regions. The third object is the rest of the background and is denoted by seed *u*. The 2D scene is the one employed in Fig. 1. Identical seed spels denoted by +'s in Fig. 3(a) were specified for AFC and RFC. The two connectivity scenes corresponding to the two AFC objects are displayed in Fig. 3(b) and (c), and the resulting AFC objects obtained with two different thresholds θ_S^+ from the scenes in (b) and (c) are shown in Fig. 3(e) and (f). The RFC objects obtained appear in Fig. 3(d), wherein the two objects of interest are identical to the AFC objects in (e) and (f) obtained via different thresholds.

Note also that the main reason we could represent RFC objects in terms of AFC objects was that two appearances of *c* in the inequality $\mu_\kappa^C(c,w) \prec \mu_\kappa^C(c,s)$ could be reduced to one: $\mu_\kappa^C(s,w) \prec \mu_\kappa^C(c,s)$, as both these inequalities are equivalent. In the case of IRFC, the defining inequality is $\mu_\kappa^A(c,w) \prec \mu_\kappa^C(c,s)$ for an appropriate $A \subset C$, and there is no equivalent form of this inequality with just one appearance of *c*. Thus, no natural AFC representation of IRFC object seems possible. Although increasing sophistication from AFC to RFC to IRFC has been previously demonstrated via segmentation experiments [14,16,18], in this section we have now given a mathematical justification of that behavior.

4. Concluding remarks

The presented analysis shows that, from the perspective of FC methodology, the only essential attribute of an affinity function is its order. In particular, many transformations (like Gaussian) of the natural affinity definitions (like derivative-driven homogeneity based affinity) are of esthetic value only and do not influence the FC segmentation outcomes.

The analysis forms also the foundation of the investigation of the second part of this paper [25], which discusses homogeneity and object-feature based affinities, as well as their combinations. In particular, we show there that many of the parameters in these definitions are of no consequence.

We did not undertake any empirical evaluation studies in this paper. A theoretical study preceding such an evaluation becomes essential to understand what affinity forms are distinct, what are redundant, and what parameters are essential/redundant. This paper constitutes a first such step. Analysis similar to the one conducted in this paper for FC can also be carried out for other frameworks, such as level sets [9], watersheds [7], and graph cuts [10].

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